Polytopes derived from

cubic tessellations

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including joint work with

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TESSELLATIONS

A Euclidean tessellation is a collection of n - polytopes, called cells, which cover E^n and tile it in face-to-face manner.

A Euclidean tessellation $\mathcal{U}$ is said to be regular if its group of symmetries (isometries preserving $\mathcal{U}$) is transitive on the flags of $\mathcal{U}$**.** The cells of a regular tessellation are convex, isomorphic regular polytopes.

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REGULAR TESSELLATIONS E^n :

$$\{4,3^{n-2},4\} \ , \ n \geq 2$$

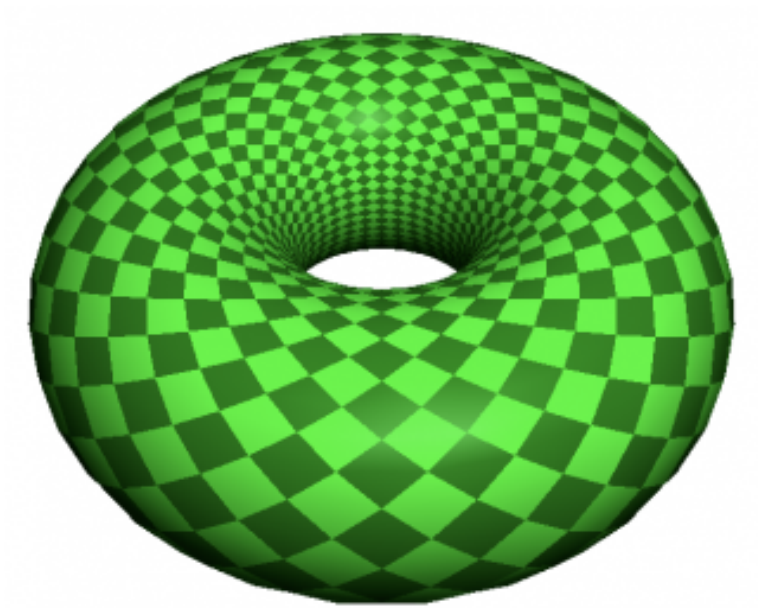
$$\{3,6\} \ , \ \{6,3\}$$

$$\{3,3,4,3\} \ , \ \{3,4,3,3\}$$

- Abstract polytope

- Abstract polytope
- Equivelar abstract polytope $\Leftrightarrow$ Schläfli symbol

- Abstract polytope
- Equivelar abstract polytope ⇔ Schläfli symbol
- Classification of equivelar abstract polytopes of type {4,4} and {4,3,4}



The group of symmetries $\Gamma(\mathscr{U})$ of the tessellation $\mathscr{U}$ is a Coxeter group. In this talk we will mostly be concerned with cubic tessellations in dimension 2 and 3 so that $\Gamma(\mathscr{U}) = [4,4]$ or $[4,3,4]$.

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$$\Gamma(\mathscr{U}) \cong T \rtimes S$$

where T is the translation subgroup and S is the stabilizer of origin (point group of $\mathscr{U}$).

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where T is the translation subgroup and S is the stabilizer of origin (point group of $\mathcal{U}$).

When G is a fixed-point free subgroup of $\Gamma(\mathcal{U})$ the quotient

$$\mathcal{T} = \mathcal{U} \,/\, G$$

is called a (cubic) twistoid.

Twistoid $\mathscr{T}$ is an abstract polytope whose faces are orbits of faces of $\mathcal{U}$ under the action of G.

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$$Sym(\mathscr{T}) := \{ \phi \in \Gamma(\mathcal{U}) \mid \phi^{-1}\alpha\phi \in G \text{ for all } \alpha \in G \}$$

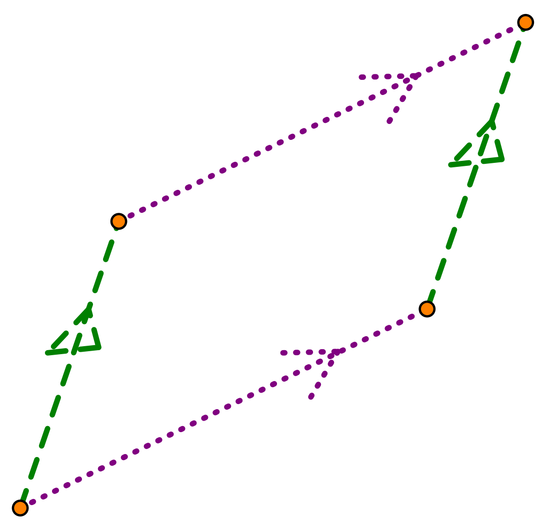
$$Aut(\mathscr{T}) := Sym(\mathscr{T}) / G$$

RANK 3

Fixed-point free crystallographic groups in Euclidean plane:

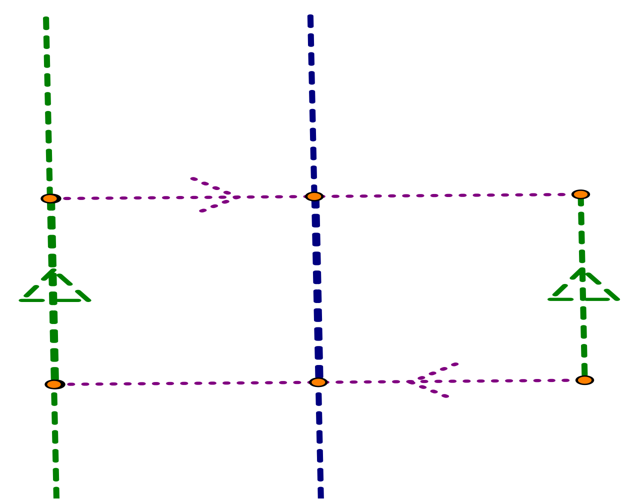
Generated by:

two independent translations



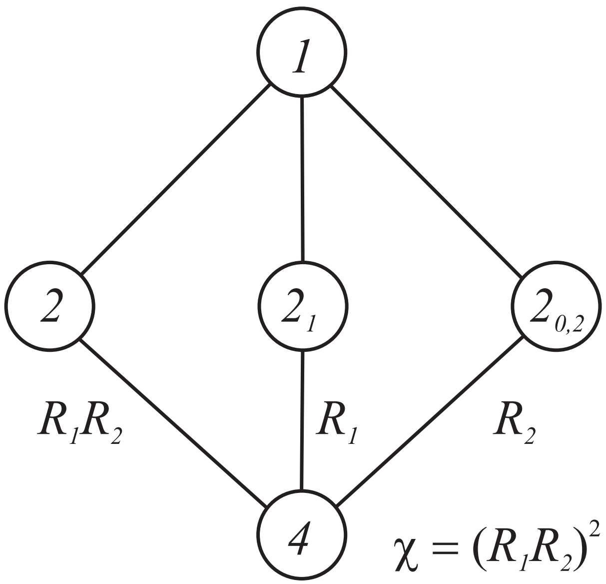
torus

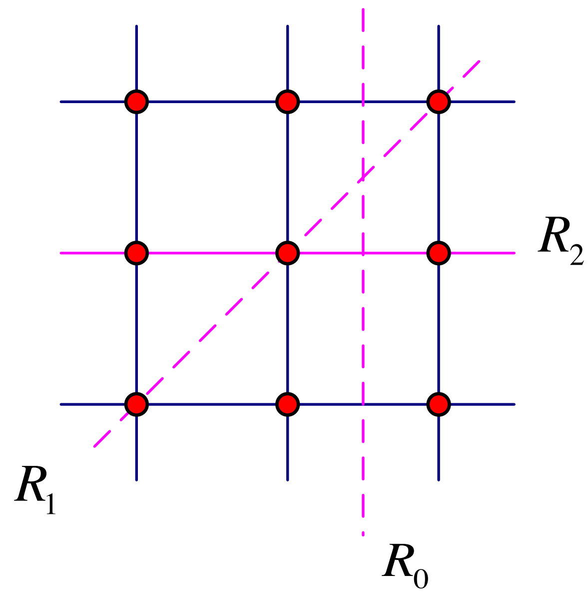
two parallel glide reflections
(same translation vectors)



Klein bottle

Equivelar Toroids of type {4, 4}:

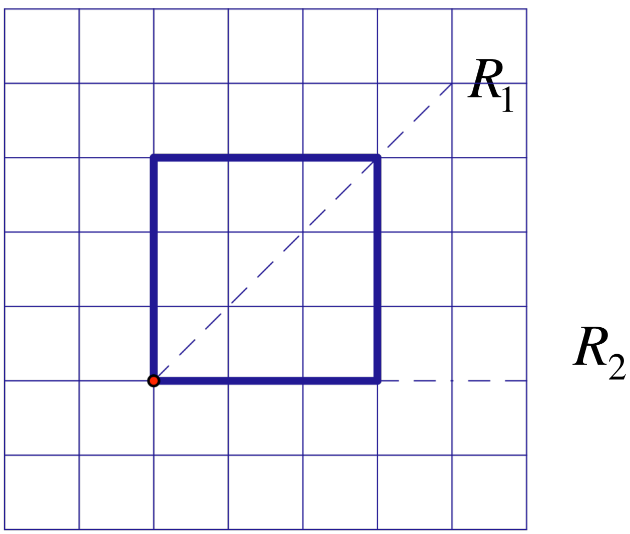




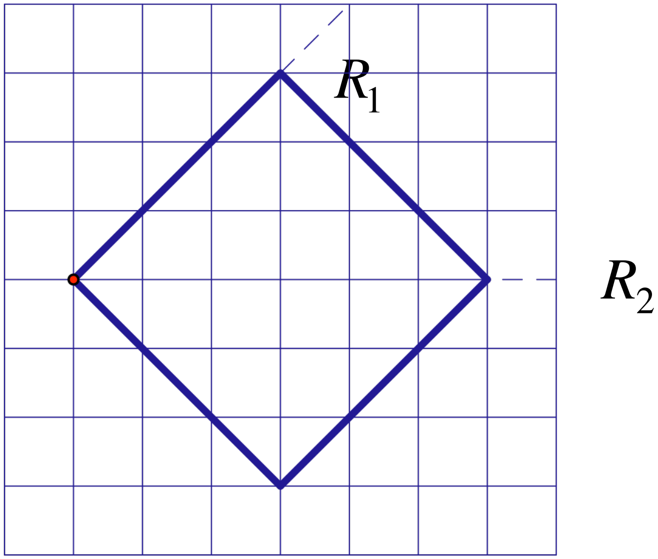
Conjugacy classes
of vertex stabilizers
for {4,4}

Class 1: regular {4,4} maps on torus

(Coxeter 1948)



$\{4,4\}_{(a,0)(0,a)}, \quad a > 0$



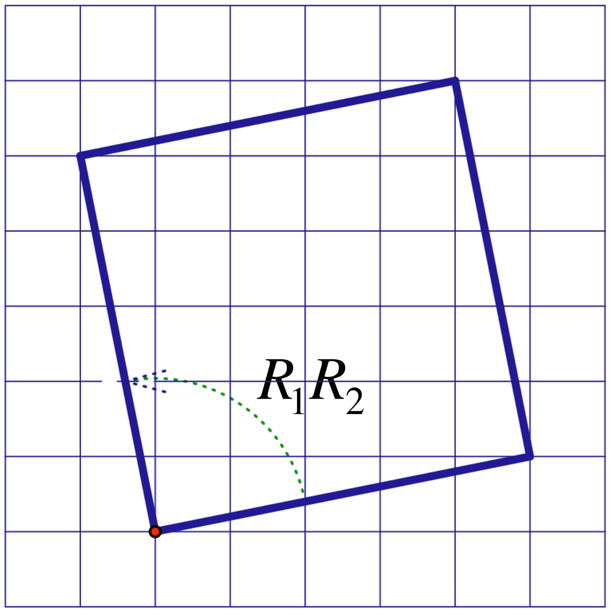
$\{4,4\}_{(a,a)(a,-a)}, \quad a > 0$

Class 2: chiral {4,4} maps on torus

(Coxeter 1948)

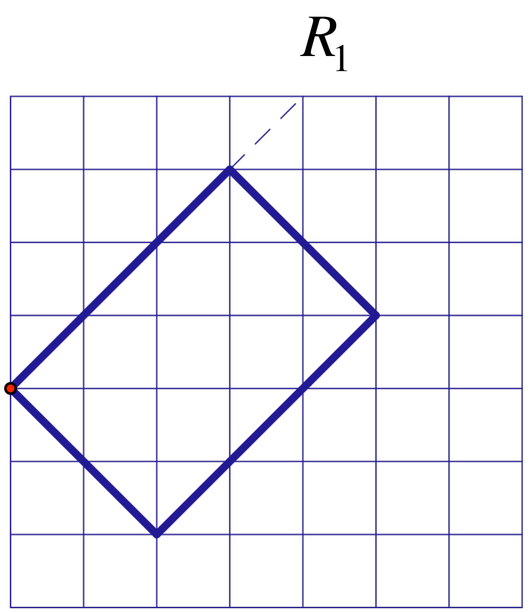
$$\{4,4\}_{(a,b)(-b,a)}$$

a>b>0

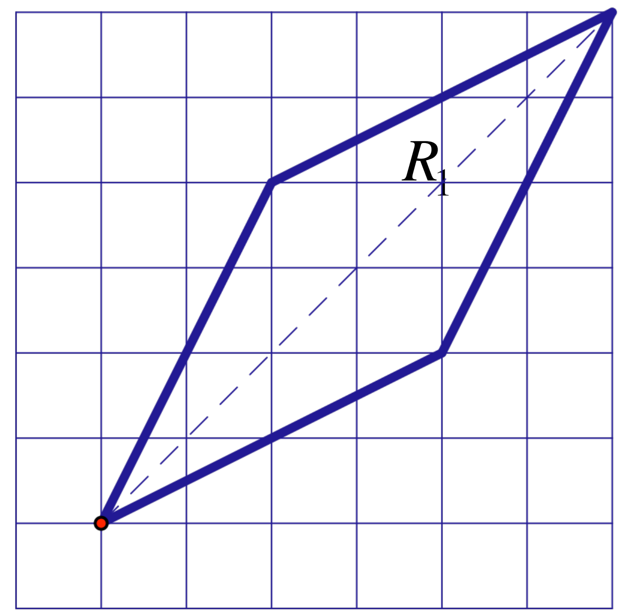


Class 2_1: vertex, edge and face transitive {4,4} maps on torus

(Širán, Tucker, Watkins, 2001)



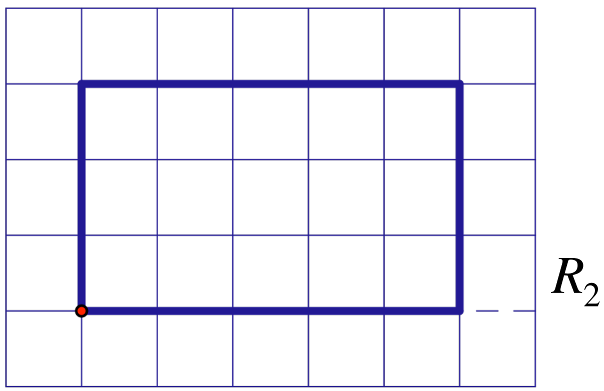
$\{4,4\}_{(a,a)(b,-b)}, \quad a > b > 0$

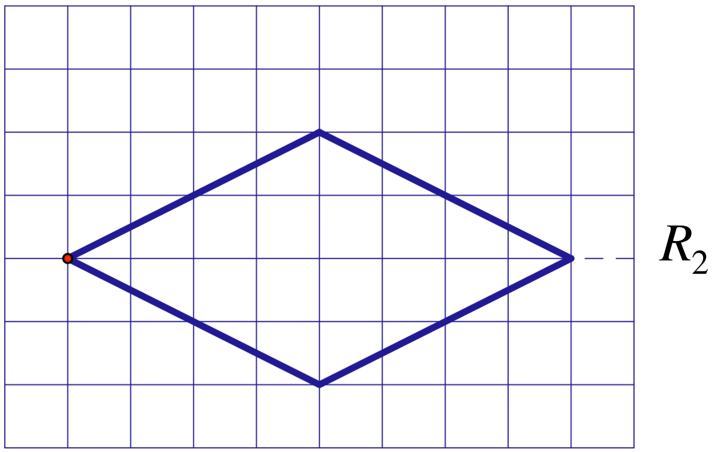


$\{4,4\}_{(a,b)(b,a)}, \quad a > b > 0$

Class 2_{02}: vertex and face transitive {4,4} maps on torus

(Hubard 2007; Duarte 2007)



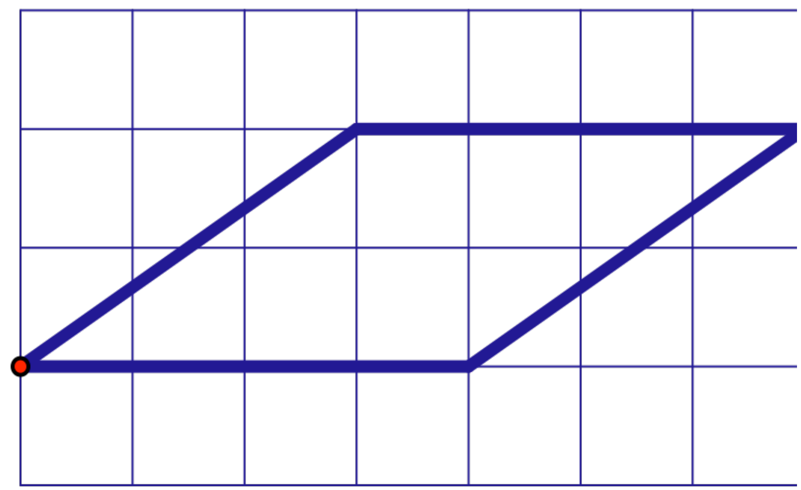


$\{4,4\}_{(a,0)(0,b)}, \quad a > b > 0$

$\{4,4\}_{(a,b)(a,-b)}, \quad a > b > 0$

Class 4: vertex and face transitive {4,4} maps on torus

(Brehm, Khünel 2008; Hubard, Orbanić, Pellicer, Asia 2007)



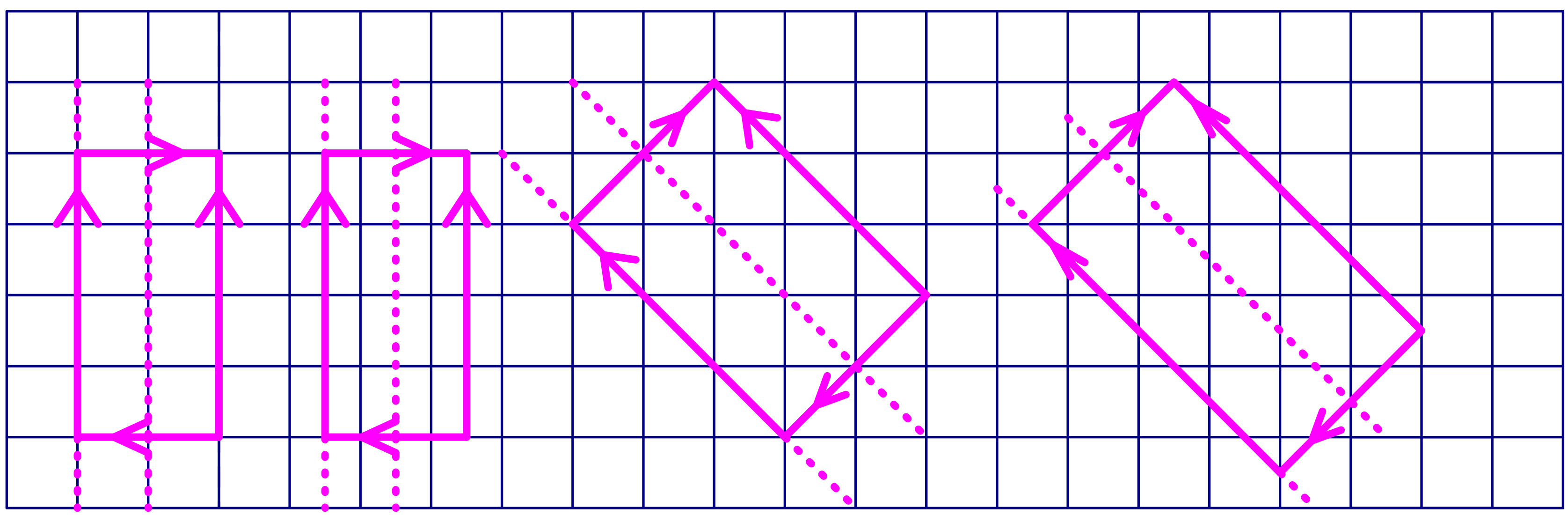
$$a > b > 0, \quad c \geq a - b, \quad c \neq 2a \neq 4c$$

and if $b \mid a, c$, then $\frac{c}{b} \nmid 1 \pm \frac{a^2}{b^2}$

$$\{4,4\}_{(a,b)(c,0)}$$

Equivelar maps of type {4,4} on Klein bottle

(Wilson 2006)



$\{4,4\}_{|4,2|}$ $\{4,4\}^{*}_{|4,2|}$ $\{4,4\}_{\backslash 6,2\backslash}$ $\{4,4\}_{\backslash 7,2\backslash}$

RANK 4

Fixed-point free crystallographic groups in Euclidean space:

Six generated by orientation preserving isometries (twists)

Four have orientation reversing generators (glide reflections)

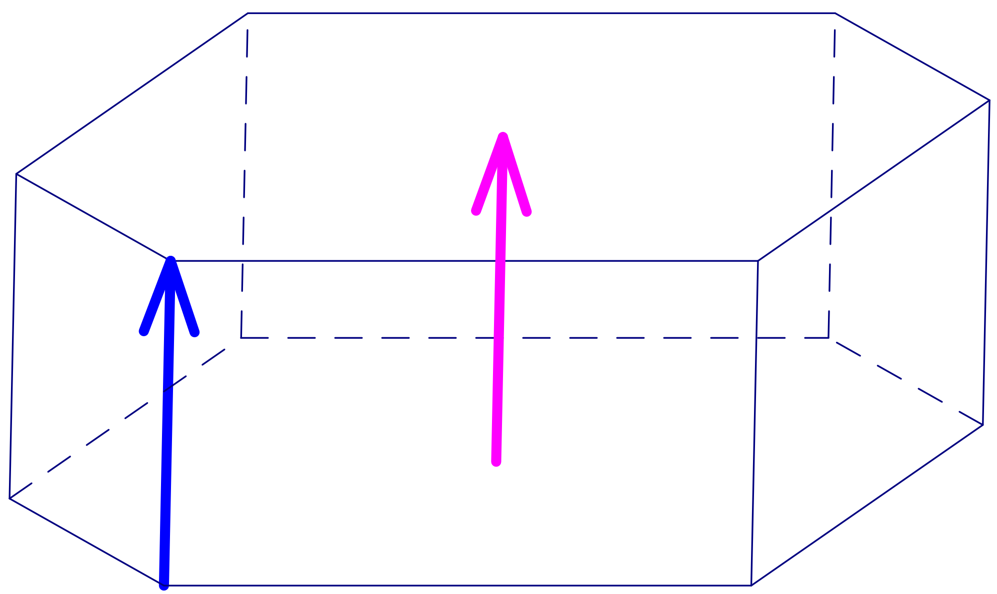
Platycosms are the corresponding 3-manifolds.

Classsification of twistoids on platycosms is mostly completed (Hubard, Mixer, Orbanić, Pellicer, Asia) and partially published in two papers.

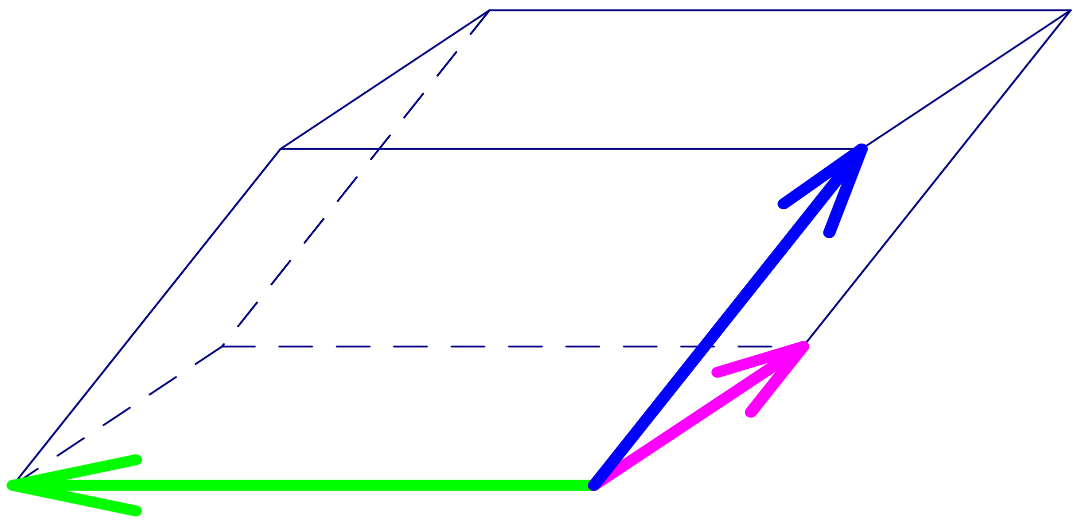
Platycosm arising from the group generated by

a six-fold twist and a three-fold twist

with parallel axes and congruent translation component is the only platycosm admitting no twistoids.



3-torus is the platycosm arising from the group G generated by three independent translations:



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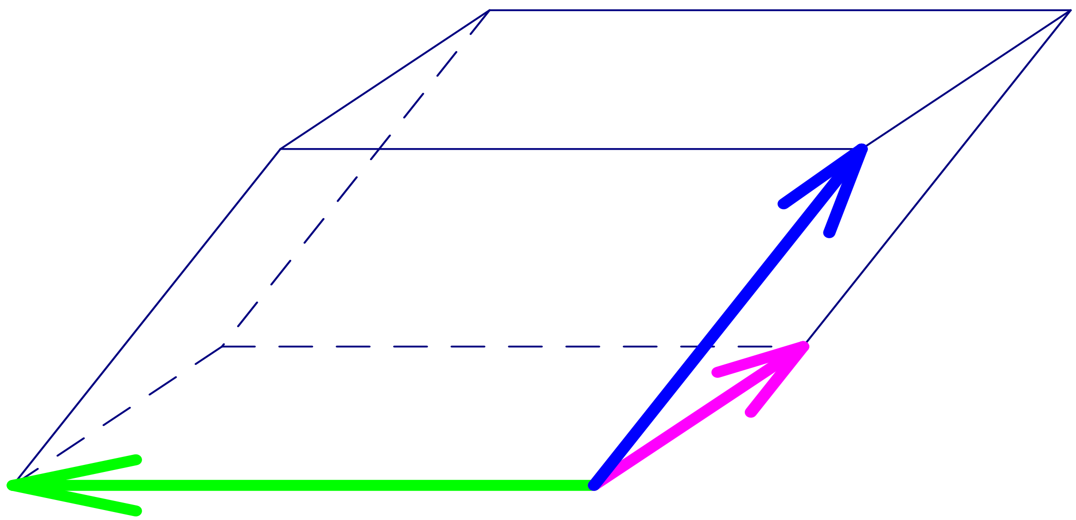
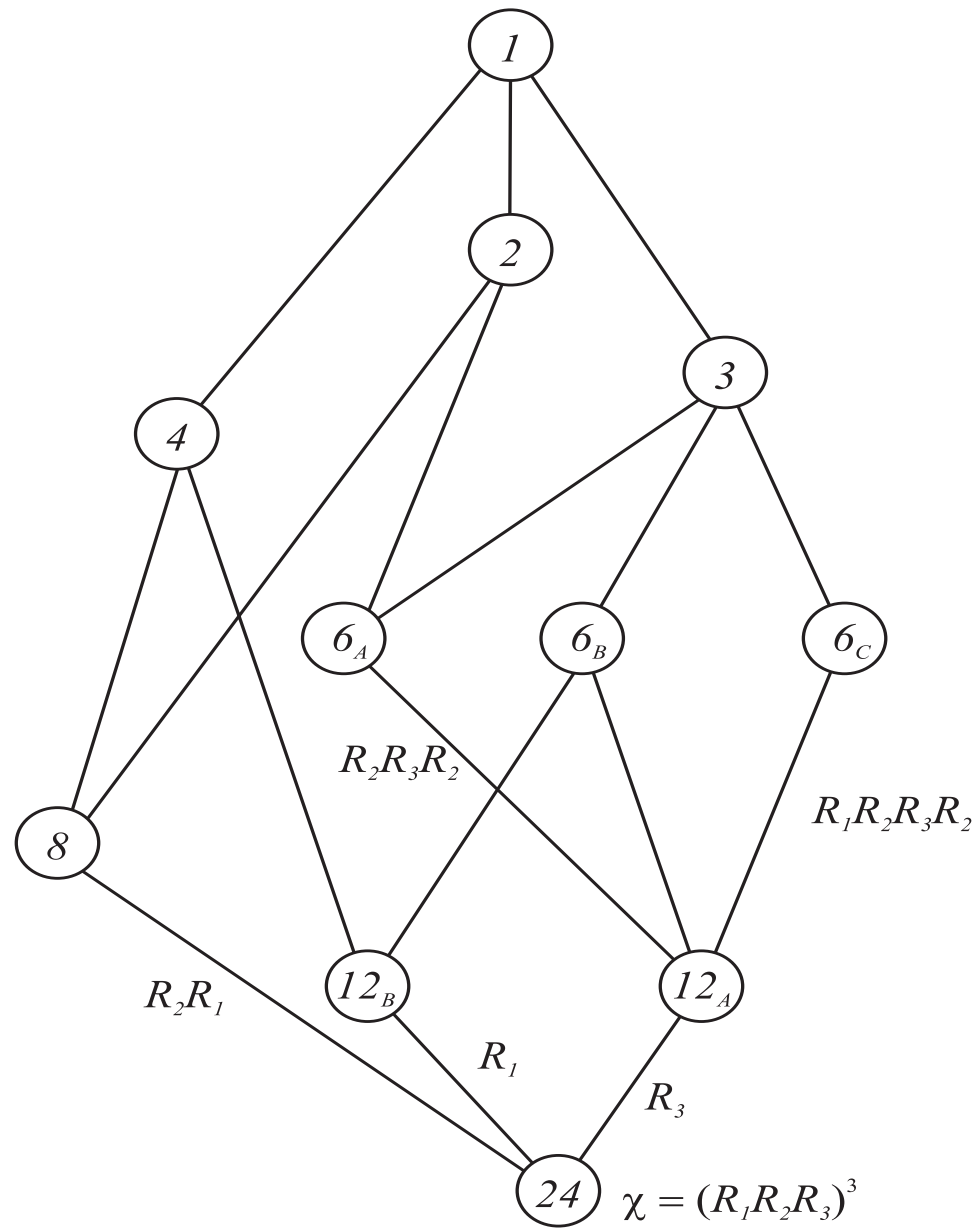


How can we place this fundamental region into a fixed cubical lattice {4,3,4} so that G is a subgroup of the lattice symmetries?

Twistoid on 3-torus is commonly referred to as 3-toroid.

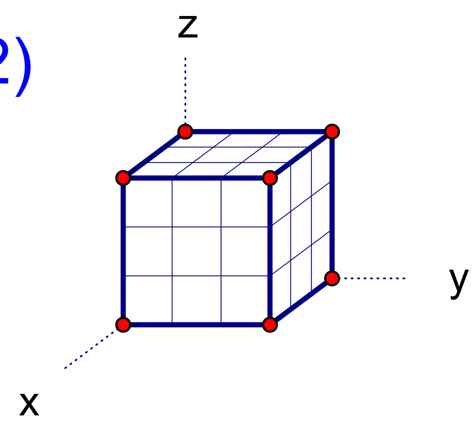
Conjugacy classes of vertex stabilizers of equivelar 3-toroids of type {4, 3, 4}:



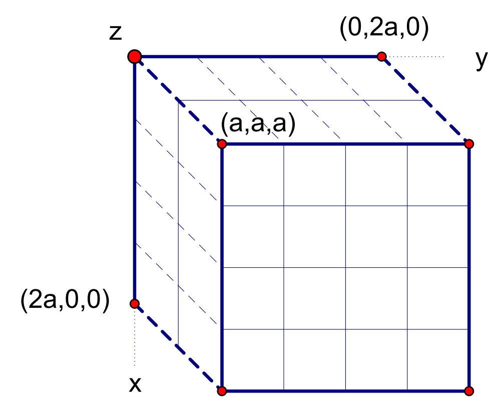
Class 1:

Theorem: Each regular rank 4 toroid belongs to one of the three families.
(McMullen & Schulte, 2002)

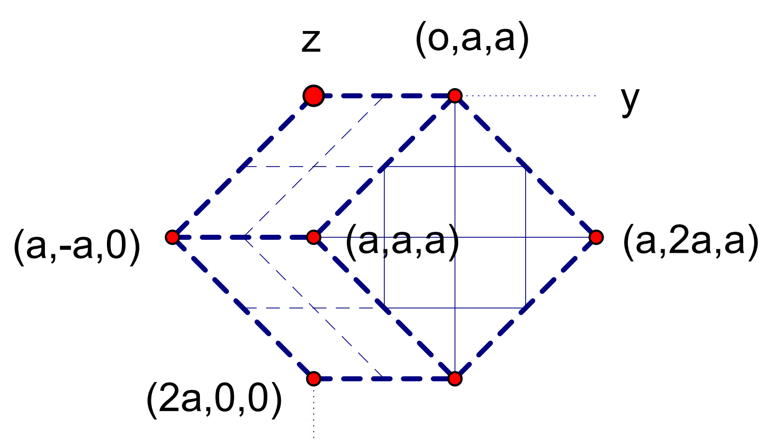
$\{4,3,4\}_{(a,0,0)(0,a,0)(0,0,a)}$



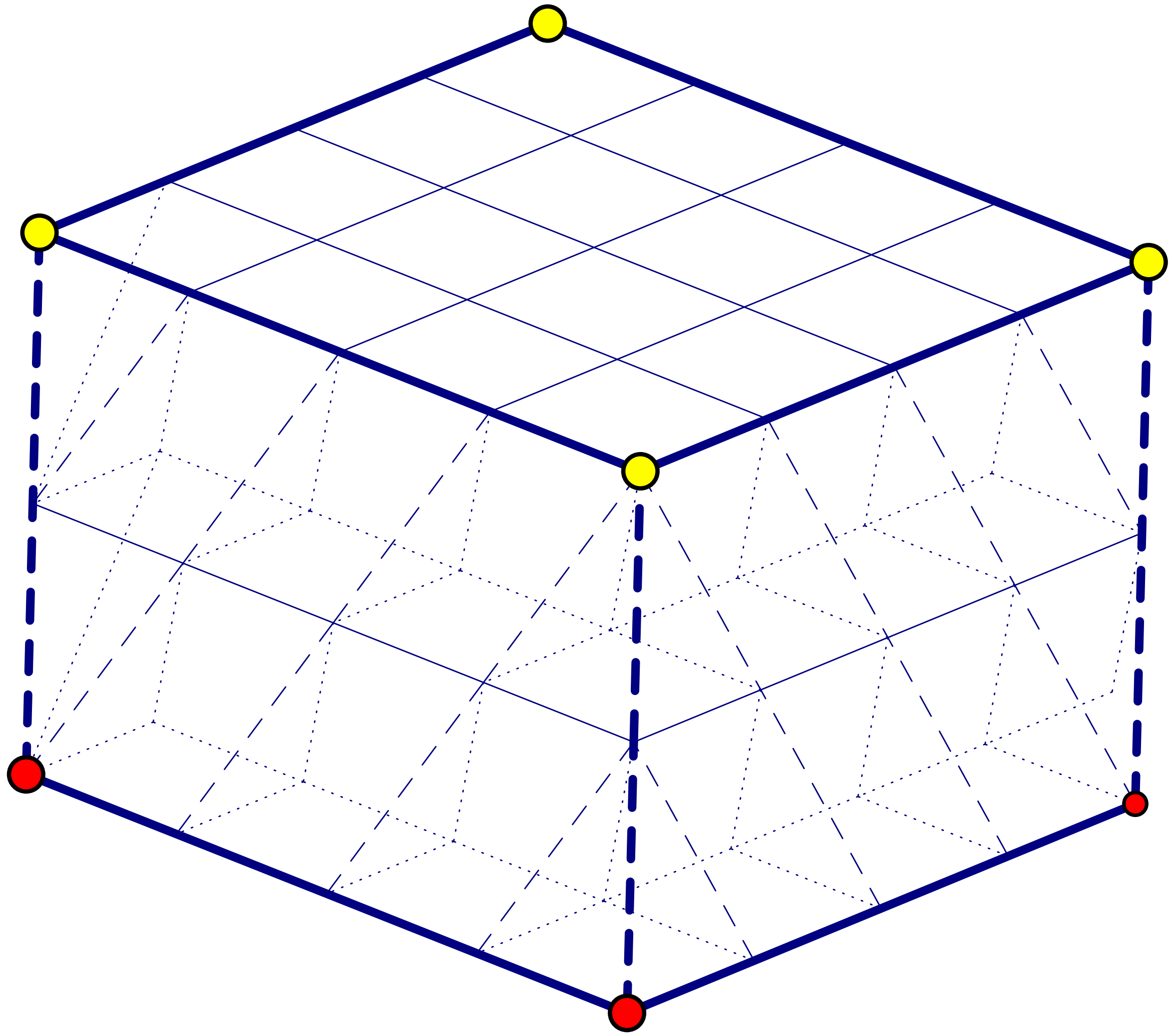
$\{4,3,4\}_{(2a,0,0)(0,2a,0)(a,a,a)}$



$\{4,3,4\}_{(a,a,0)(a,-a,0)(0,a,a)}$



A “closer” view of $\{4,3,4\}_{(2a,0,0)(0,2a,0)(a,a,a)}$



Class 2:

Theorem: There are no chiral toroids of rank > 3.
(McMullen & Schulte, 2002)

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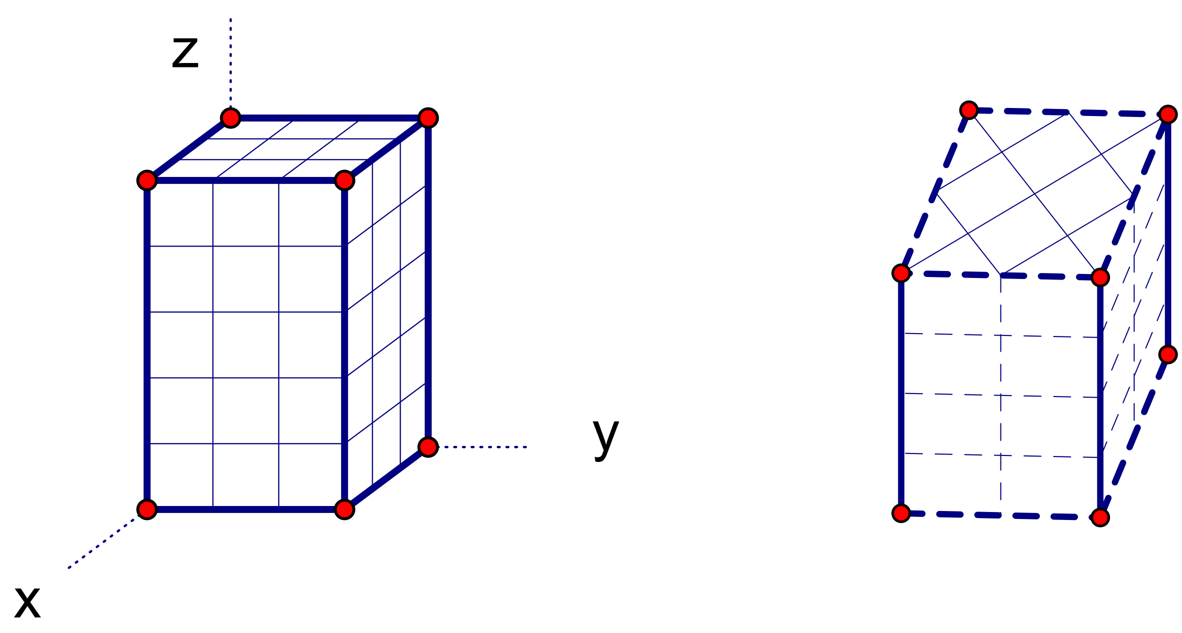
Theorem: There are no rank 4 toroids with two flag orbits
(in Class 2).

Examples in Class 3:

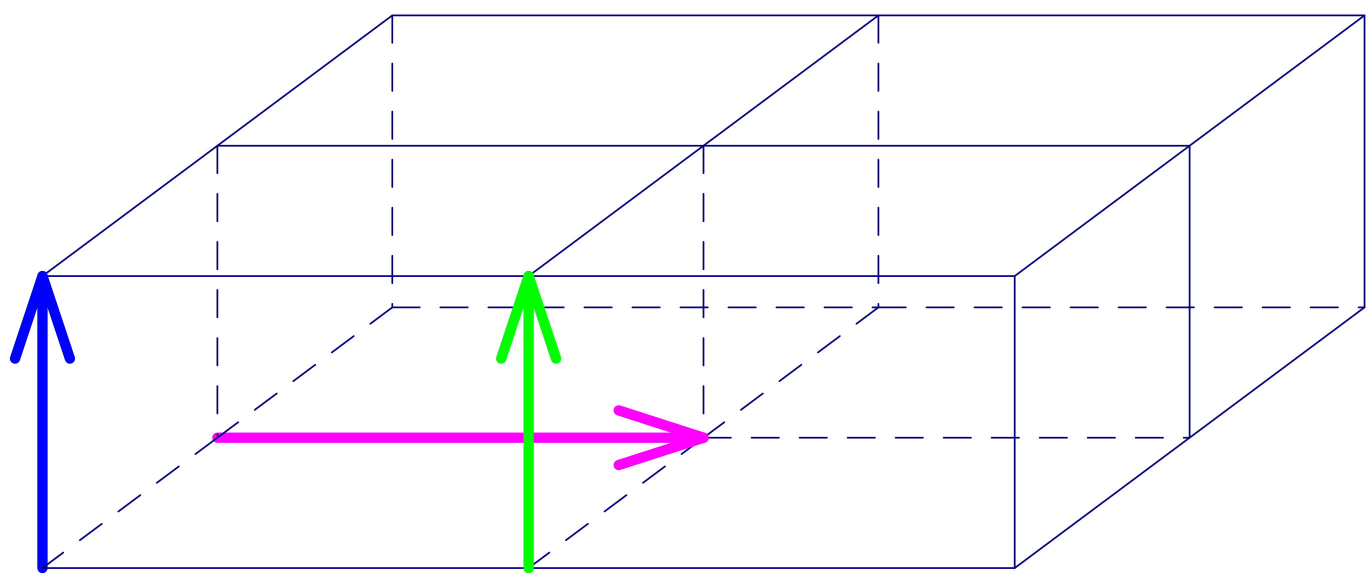


Projection of v	Class of $\mathcal{P}_\Pi$	Generators of $\mathcal{P}$	Parameters	Class of $\mathcal{P}$
o	1	$(a,0,0)$,$(0,a,0)$,$(0,0,d)$	$a,d>0$, $d\neq a$	3
o	1	$(a,a,0)$,$(a,-a,0)$,$(0,0,d)$	$a,d>0$	
$(v_1+v_2)/2$	1	$(a,0,0)$,$(0,a,0)$,$(a/2,a/2,d)$	$a,d>0$, $d\neq a/2$	
$(v_1+v_2)/2$	1	$(a,a,0)$,$(a,-a,0)$,$(a,0,d)$	$a,d>0$, $d\neq a$	
o	2_{02}	$(a,0,0)$,$(0,b,0)$,$(0,0,d)$	$a>b>d>0$	6_A
o	2_{02}	$(a,b,0)$,$(a,-b,0)$,$(0,0,d)$	$a>b>0$, $d>0$	
$(v_1+v_2)/2$	2_{02}	$(a,0,0)$,$(0,b,0)$,$(a/2,b/2,d)$	$a>b>2d>0$	
$(v_1+v_2)/2$	2_{02}	$(a,b,0)$,$(a,-b,0)$,$(a,0,d)$	$a>b>0$, $d>0$, $d\neq a,b$	
o	2_1	$(a,a,0)$,$(b,-b,0)$,$(0,0,d)$	$a>b>0$, $d>0$	6_B
o	2_1	$(a,b,0)$,$(b,a,0)$,$(0,0,d)$	$a>b>0$, $d>0$	
$(v_1+v_2)/2$	2_1	$(a,a,0)$,$(b,-b,0)$,$((a+b)/2,(a-b)/2,d)$	$a>b>0$, $d>0$	
$(v_1+v_2)/2$	2_1	$(a,b,0)$,$(b,a,0)$,$((a+b)/2,(a+b)/2,d)$	$a>b>0$, $d>0$	
$v_1/2$	1	$(a,a,0)$,$(a,-a,0)$,$(a/2,a/2,d)$	$a,d>0$	
$v_1/2$	2_1	$(a,a,0)$,$(b,-b,0)$,$(a/2,a/2,d)$	$a>b>0$, $d>0$	
$v_2/2$	2_1	$(a,a,0)$,$(b,-b,0)$,$(b/2,-b/2,d)$	$a>b>0$, $d>0$	
o	2	$(a,b,0)$,$(-b,a,0)$,$(0,0,d)$	$a>b>0$, $d>0$	6_C
$(v_1+v_2)/2$	2	$(a,b,0)$,$(-b,a,0)$,$((a-b)/2,(a+b)/2,d)$	$a>b>0$, $d>0$	
o	4	$(a,b,0)$, $(c,0,0)$, $(0,0,d)$	*	12_A
$(v_1+v_2)/2$	4	$(a,b,0)$, $(c,0,0)$, $((a+c)/2,b/2,d)$	**	
$v_1/2$	2	$(a,b,0)$, $(-b,a,0)$, $(a/2,b/2,d)$	$a>b>0$, $d>0$	
$v_1/2$	2_{02}	$(a,b,0)$, $(a,-b,0)$, $(a/2,b/2,d)$	$a>b>0$, $d>0$	
$v_1/2$	2_1	$(a,b,0)$, $(b,a,0)$, $(a/2,b/2,d)$	$a>b>0$, $d>0$	
$v_1/2$	4	$(a,b,0)$, $(c,0,0)$, $(a/2,b/2,d)$	***	
$v_2/2$	4	$(a,b,0)$, $(c,0,0)$, $(c/2,0,d)$	****	

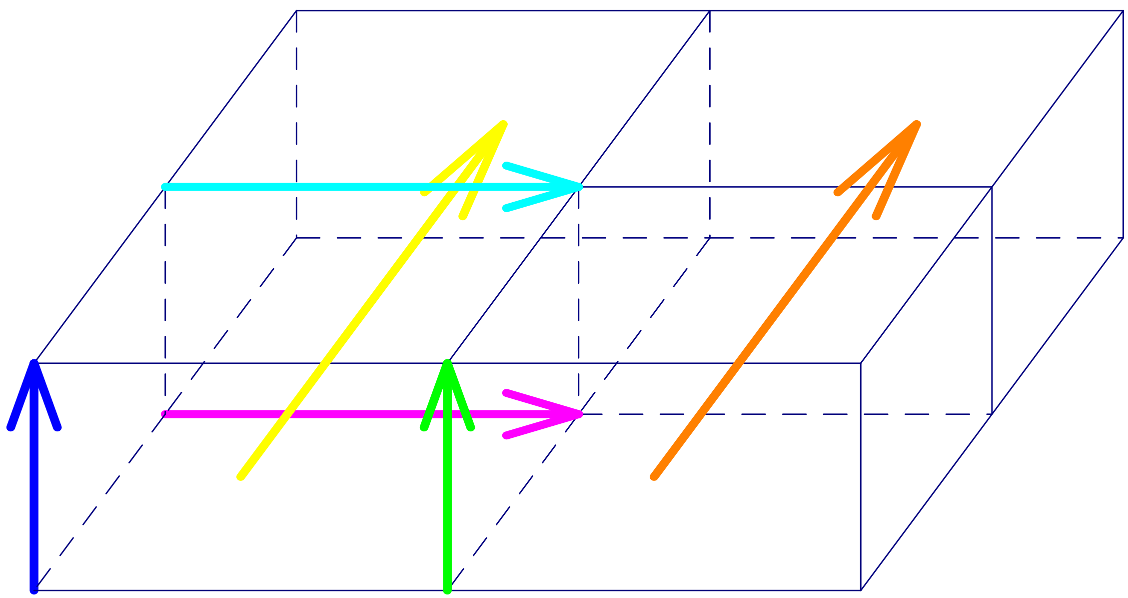
Projection of v	Class of $\mathcal{P}_{\Pi_e}$	Generators of $\mathcal{P}$	Parameters	Class of $\mathcal{P}$
o	1	$(a,0,-a)$,$(0,a,-a)$,(c,c,c)	$a,c>0$	4
$(v_1+v_2)/3$	1	$(a,0,-a)$,$(0,a,-a)$,$(\frac{a+c}{3},\frac{a+c}{3},\frac{-2a+c}{3})$	$a,c>0$, $3 \mid (a+c)$	4
o	1	$(a,a,-2a)$,$(2a,-a,-a)$,(c,c,c)	$a,c>0$	4
$(v_1+v_2)/3$	1	$(a,a,-2a)$,$(2a,-a,-a)$,$(a+c,c,-a+c)$	$a>b>0$, $c>0$	8
o	2	$(a,b,-a-b)$,$(-b,a+b,-a)$,(c,c,c)	$a>b>0$, $c>0$	8
$(v_1+v_2)/3$	2	$(a,b,-a-b)$,$(-b,a+b,-a)$, $(\frac{a-b+c}{3},\frac{a+2b+c}{3},\frac{-2a-b+c}{3})$	$a>b>0$, $c>0$ $3 \mid (a+2b+c)$	8

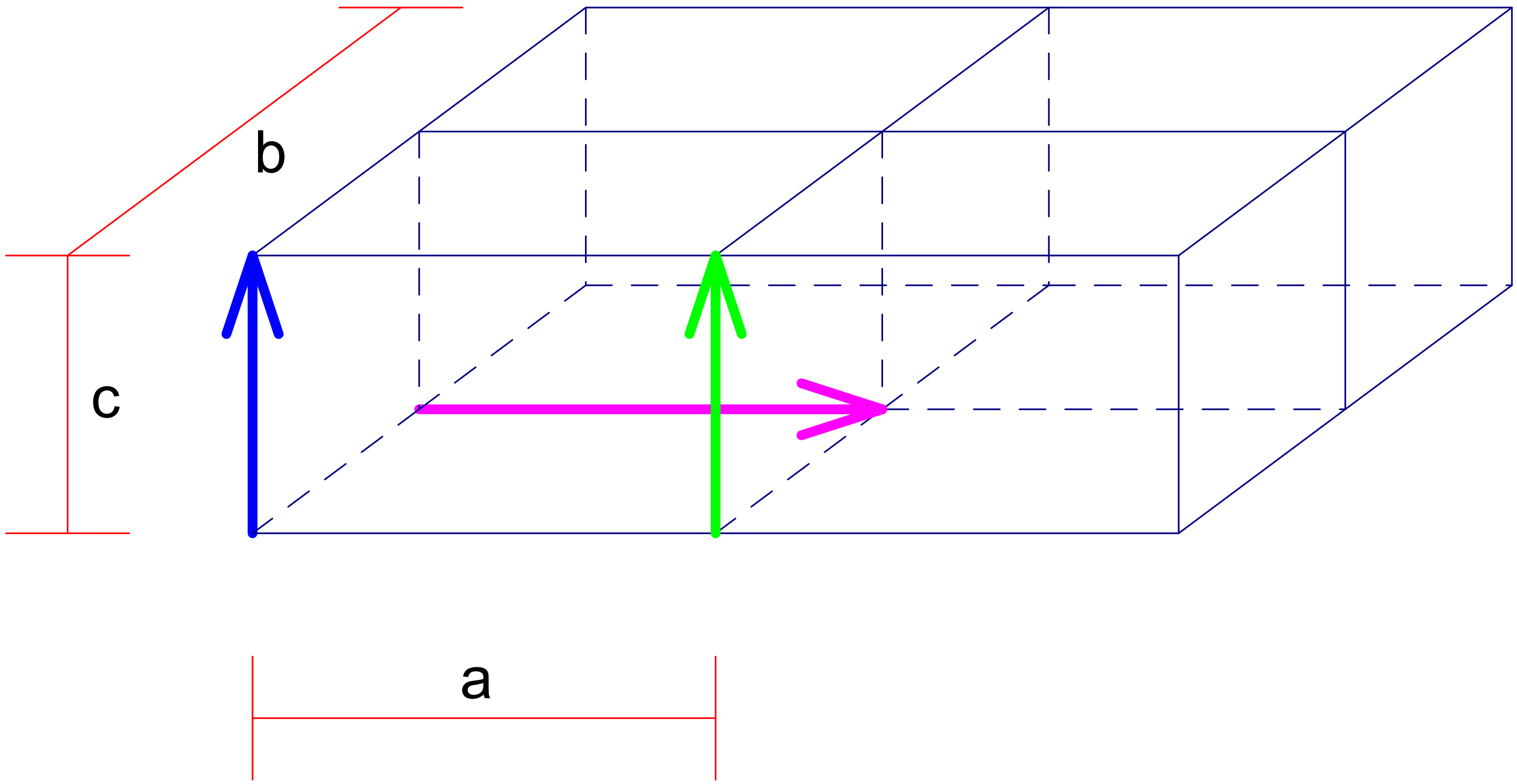
Didicosm is the platycosm arising from the group G generated by

- two half-turn twists with parallel axes and congruent translation component, and
- a twist whose axis does not intersect and is perpendicular to the axes of the other two twists and has the translation component equal to a vector between the other two axes:



Identification of points of the boundary of the fundamental region:





How can we place this fundamental region into a fixed cubical lattice {4,3,4} so that G is a subgroup of the lattice symmetries?

Classification of cubic tessellations on didicosm according to their automorphism groups:

Group	Size	Generators	Conditions	Toroidal Covers
1	96	$\tau_1, \tau_2, \tau_3, \chi, \chi_1, \chi_2, \chi_3, \rho$	$a = 2b = c$ even, $p = q = 0$	1
2	48	$\tau_1, \tau_2, \tau_3, \chi, \chi_1\chi_2, \rho$	$a = 2b = c$ odd, $p = \frac{1}{2}$, $q = 0$	1
3_1	32	$\tau_1, \tau_2, \tau_3, \chi, \chi_1, \rho$	$a = 2b$, c even, ($p = 0$ or $a \notin \mathbb{Z}$)	1, 3
3_2	32	$\tau_1, \tau_2, \tau_3, \chi, \chi_2, \rho$	$c = 2b$, a even, ($q = 0$, c even or $q = \frac{1}{2}$, c odd)	1, 3
3_3	32	$\tau_1, \tau_2, \tau_3, \chi, \chi_3, \rho$	$a = c$, $2b$ even, $p = q$	1, 3
6	16	$\tau_1, \tau_2, \tau_3, \chi, \rho$	$a, 2b \in \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$	1, 3, 6_A, 6_B
6_1	16	$\tau_1, \chi, \chi_1, \rho$	$a = 2b \notin \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$, c even	3
12_2	8	τ_1, τ_2, ρ	$a \in \sqrt{2}\mathbb{Z}$, $2b \notin \sqrt{2}\mathbb{Z}$	6_B
12_3	8	τ_1, τ_3, ρ	$2b \in \sqrt{2}\mathbb{Z}$, $a \notin \sqrt{2}\mathbb{Z}$	6_B
12	8	τ_1, χ, ρ	$(a + 2b) \notin \frac{\sqrt{2}}{2}\mathbb{Z} \setminus \sqrt{2}\mathbb{Z}$	3, 6_B