UNIVERSITY OF CALGARY

Department of Mathematics & Statistics

# Towards efficient approximation of $p$-cones

Pooyan Ghomi
Yuriy Zinchenko

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# Content

- Second order and *p*-norm cones
  - definitions
  - applications
  - some facts
- "Greedy" polyhedral approximation
  - complexity analysis
- Moving further: SOC approximation and beyond

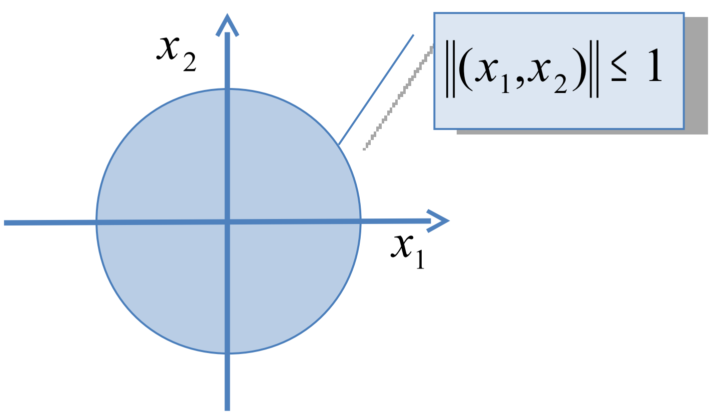
# Definitions

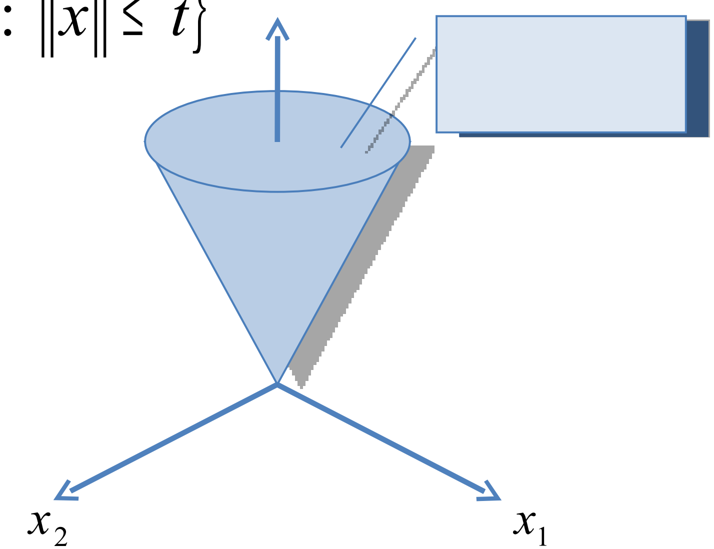
- Euclidean norm

$$\vec{x} \in \Re^{n}, \|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x^{2}}$$

- second order cone
  - homogenization of a ball

$$SOC = \left\{ (\vec{x}, t) \in \Re^{n} \times \Re \;:\; \|\vec{x}\| \leq t \right\}$$



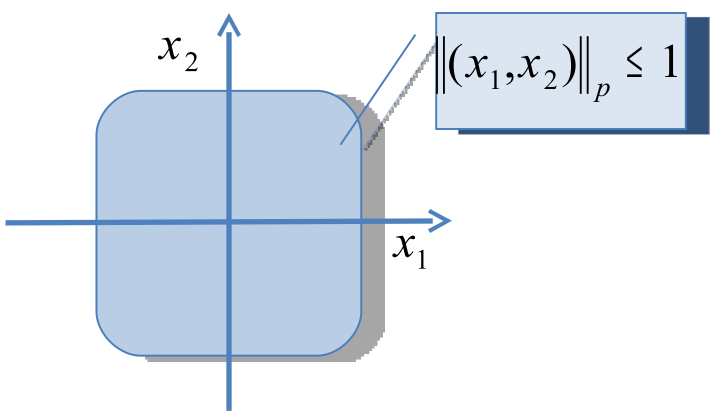
# Definitions

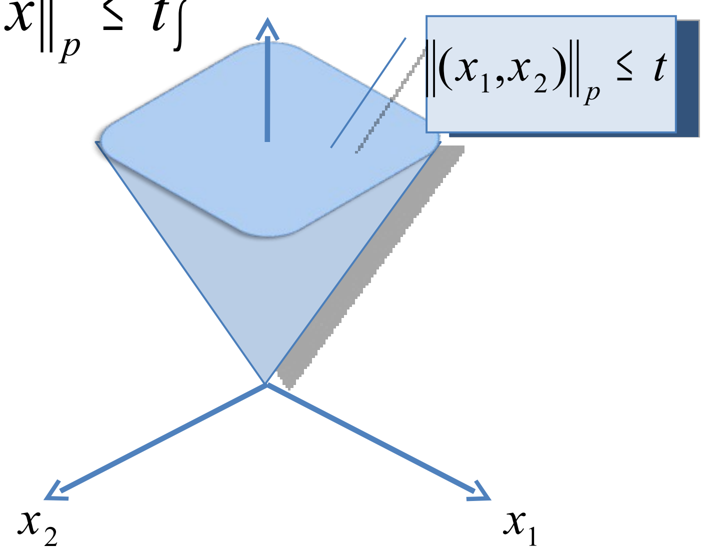
- *p*-norm

$$\vec{x} \in \Re^{n}, \|\vec{x}\|_{p} = \sqrt[p]{\sum_{i=1}^{n} x^{p}}$$

- *p*-cone
  - homogenization of a *p*-ball

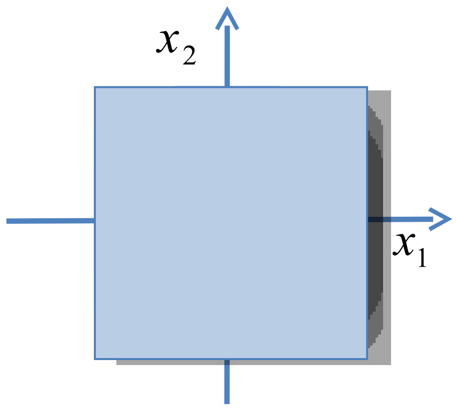
$$C_{p} = \left\{ (\vec{x}, t) \in \Re^{n} \times \Re : \|\vec{x}\|_{p} \leq t \right\}$$



# Facts

- *p*-norm geometry
  - given by $p \geq 1$
  - *p*-balls are convex, so are the cones
  - $p = 1, \infty$ are polyhedral
  - inclusion $C_1 \subseteq C_p \subseteq C_\infty, \forall p \geq 1$




- Duality
  - dual
  - given by conjugate $C_p^* = C_q$, where $\frac{1}{p} + \frac{1}{q} = 1$

# Facts

- First primitive
  - recursive definition via "tower of variables"
    - let $\vec{x} \in \Re^n, n = 2^k$ :

$$\|\vec{x}\| = \sqrt[p]{x_1^p + x_2^p + \ldots + x_n^p} \leq 1$$

$$\sqrt[p]{x_1^p + x_2^p} \leq x_{1,1}, \quad \sqrt[p]{x_3^p + x_4^p} \leq x_{1,2}, \quad \ldots, \quad \sqrt[p]{x_{n-1}^p + x_n^p} \leq x_{1,\frac{n}{2}},$$

$$\sqrt[p]{x_{1,1}^p + x_{1,2}^p} \leq x_{2,1}, \quad \ldots, \quad \sqrt[p]{x_{1,\frac{n}{2}-1}^p + x_{1,\frac{n}{2}}^p} \leq x_{1,\frac{n}{4}},$$

$$\vdots$$

$$\sqrt[p]{x_{k-1,1}^p + x_{k-1,2}^p} \leq 1$$

$n/2$
+ $n/4$
+ …
+ $1$

$(n-1)$ 3-dimensional cones

# Applications

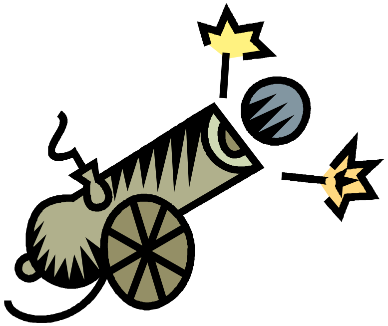
- Linear conic programming

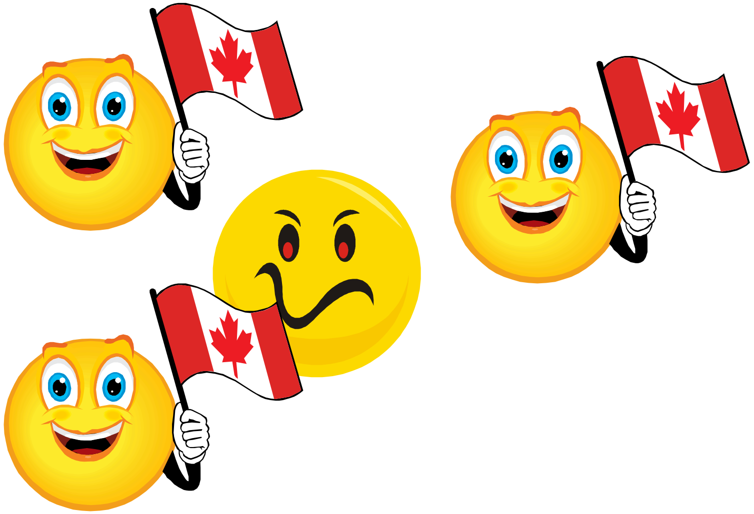
$$\inf_{\vec{x}} \vec{c} \bullet \vec{x} : A\vec{x} = \vec{b}, \vec{x} \in C$$

- SOCP – $C$ is a product of second order cones
  - superseeds convex quadratic programming,
  - has numerous applications,
    - sensor location,
    - mean-variance investment portfolio optimization,
    - robust linear programming, etc.
- *p*-cone programming –
  - has fewer known applications (?),
  - may be used to shape distributions,
    - radiotherapy planning

# Applications

- Radiotherapy planning basics





  - choose “intensity” so that
    - tumor gets killed,
    - healthy tissues are spared

- Radiotherapy planning basics
  - organ survival is ensured by "certain % of the organ receives no more than a certain dose",
    - e.g., no more than 30% of the liver receives 20Gy,
  - equivalent to specifying distribution for a (pseudo) random variable,
    - if compactly supported (true), equivalent to prescribing moments,
    - *p*-moments can be described using *p*-norms

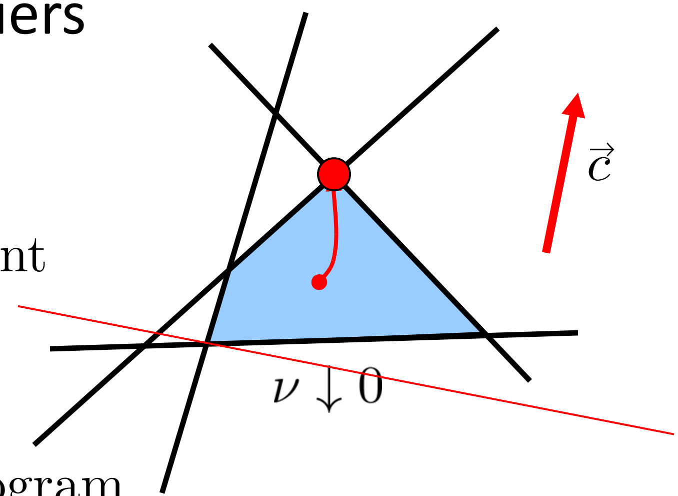
$$\inf_{\vec{x}} \vec{c} \bullet \vec{x} : A\vec{x} = \vec{b}, \vec{x} \in C$$

# Facts

- Solving *p*-cone programs $\inf_{\vec{x}} \vec{c} \bullet \vec{x} : A\vec{x} = \vec{b}, \vec{x} \in C$
  - interior-point methods
    - using suitable “barriers”
  - “efficient” approximation
    - by better understood class of optimization models

# Facts

- Solving *p*-cone programs
  - interior-point methods and barriers
    - $d$ – dimension
    - $n$ – # of constraints $(n > d)$
    - $A \in \Re^{n\times d}, \vec{b} \in \Re^{n}$ – **A**rrangement
    - $A\vec{x} \geq \vec{b}$ – **P**olytope
    - $\vec{c} \in \Re^{d}$ – objective
    - $\max\ \vec{c}^T\vec{x} :\ A\vec{x} \geq \vec{b}$ – Linear Program
    - solve by following “central path”




$$\mathcal{P} = \{\vec{x} \in \Re^{d} :\ \vec{x} = \arg\max_{\vec{z}\in\Re^{d}}\ \vec{c}^T\vec{z} + \nu \sum_{i=1}^{n} \ln(A\vec{z} - \vec{b})_i,\ \nu \in (0,\infty)\}$$

“barrier”

i.e., the set of solutions to

$$\inf_{\vec{x}} \vec{c}\cdot\vec{x} + b\cdot f(\vec{x}) : A\vec{x} = \vec{b}, \vec{x} \in C$$

# Facts

- Solving *p*-cone programs
  - interior-point methods and barriers
    - complexity of solving

$$\inf_{\vec{x}} \vec{c} \bullet \vec{x} : A\vec{x} = \vec{b}, \vec{x} \in C$$

by following the solutions of

$$\inf_{\vec{x}} \vec{c} \bullet \vec{x} + \upsilon \cdot f(\vec{x}) : A\vec{x} = \vec{b}, \vec{x} \in C$$

is driven by barrier parameter $\theta_f$

(length of a barrier gradient in a certain norm),

with number of iterations $O\left(\sqrt{\theta_f}\right)$

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# Facts

- Solving *p*-cone programs
  - interior-point methods and barriers
  - "efficient" approximation

| *p* | Barrier $\theta_f$ native | Barrier $\theta_f$ SOC | LP approximation |
|---|---|---|---|
| 1 | $2n+1$ | $2n+1$ | |
| $\infty$ | $2n+1$ | $2n+1$ | |
| 2 | 2 | 2 | |
| $2^k$ | $4(2^k)$ | $>2n\,k$ | |
| $m/q$ | $4n$ | $>2n\,(m+q)$ | |
| real | $4n$ | – | |

Special/polyhedral

Pathological / misunderstood ?

# Facts

- Solving *p*-cone programs
  - interior-point methods and barriers
    - reason for native barriers associated with

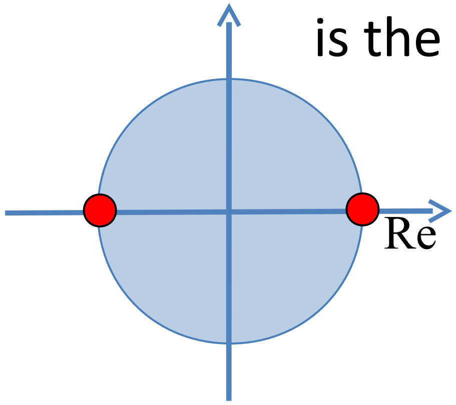
$$C_p = \{ (\vec{x}, t) \in \Re^n \times \Re \ : \ \|\vec{x}\|_p \le t \}$$

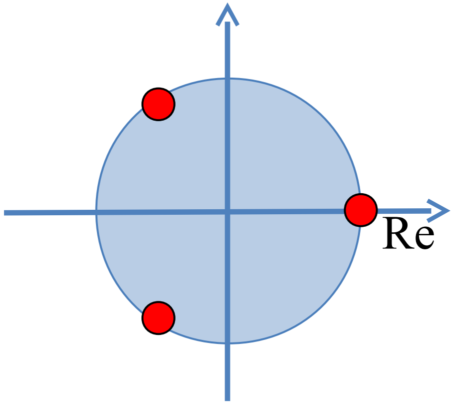
being so different for

$$p = 2, \qquad p = 3, 4, 5, \ldots$$

is the number of real roots of

$$z^p = 1$$



# Facts

- Solving *p*-cone programs
  - interior-point methods
    - using suitable "barriers"
  - "efficient" approximation
    - by better understood class of optimization models
      - specifically Linear Programming (LP),
      - polyhedral approximation to $C_p$ ?

Given $\varepsilon > 0$, determine $A \cdot (\vec{x}, t) + D \cdot \vec{y} \geq \vec{b}$ :

$i)\ (\vec{x}, t) \in C_p \Rightarrow \exists \vec{y}$ - feasible,

$ii)\ (\vec{x}, t, \vec{y})$ - feasible $\Rightarrow \frac{1}{1+\varepsilon}(\vec{x}, t) \in C_p$

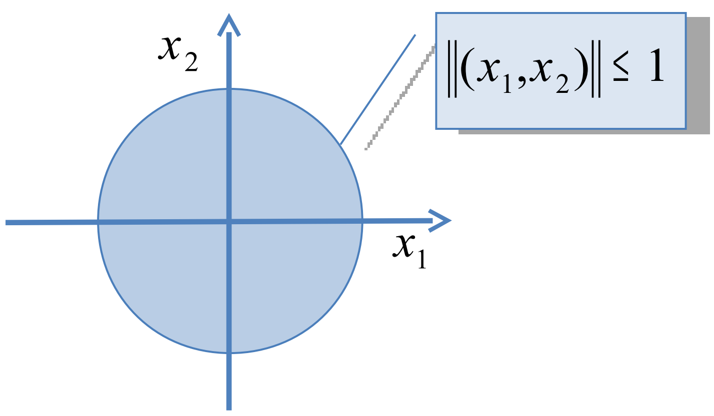
# Definitions

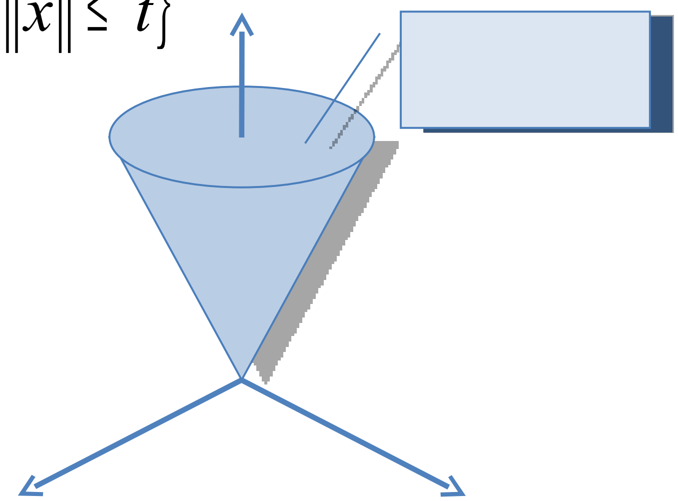
- Euclidean norm

$$\vec{x} \in \Re^n, \|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x^2}$$

- second order cone
  - homogenization of a ball

$$SOC = \{ (\vec{x}, t) \in \Re^n \times \Re : \|\vec{x}\| \le t \}$$

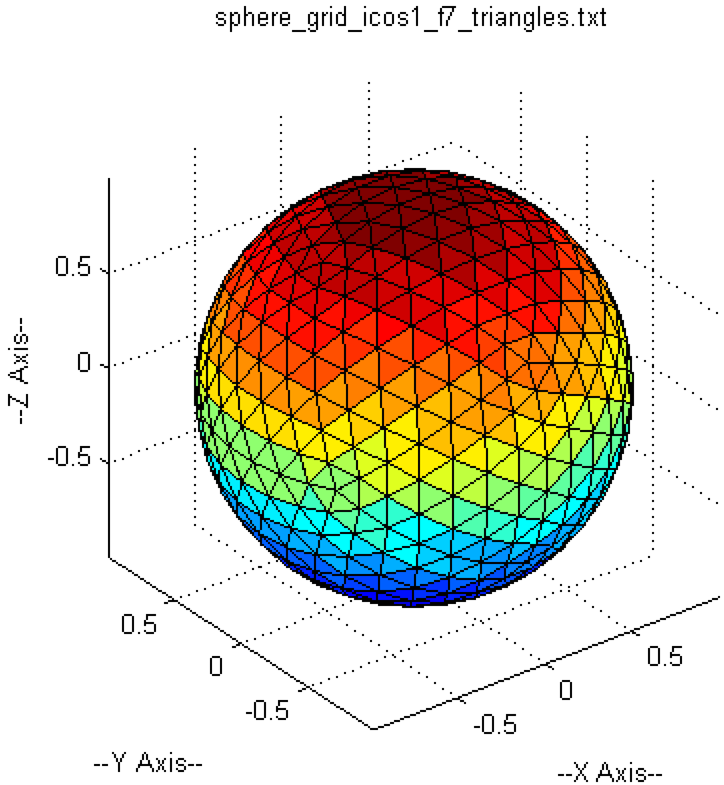


# Facts

- Solving *p*-cone programs
  - "efficient" approximation of SOC$\left(C_2\right)$
    - naïve
      - exponential number of inequalities

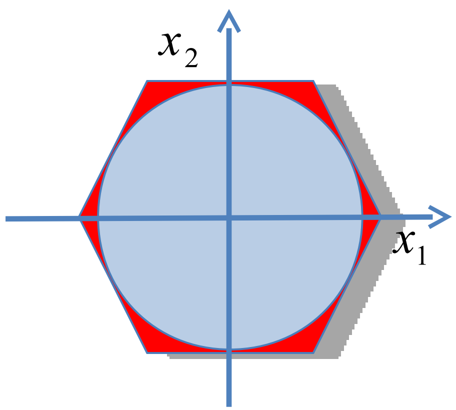
$$\sim\left(\frac{1}{\sqrt{\varepsilon}}\right)^n$$

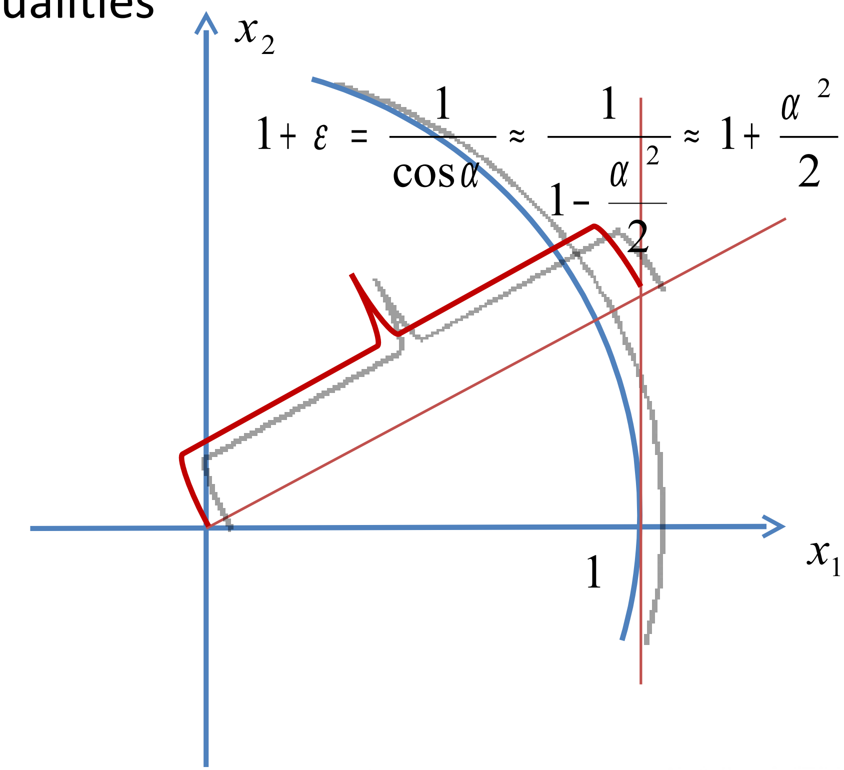


# Facts

- Solving *p*-cone programs
  - "efficient" approximation of SOC$\left(C_2\right)$
    - simple
      - using tower of variables, suffices to describe 3D cone,
      - number of inequalities

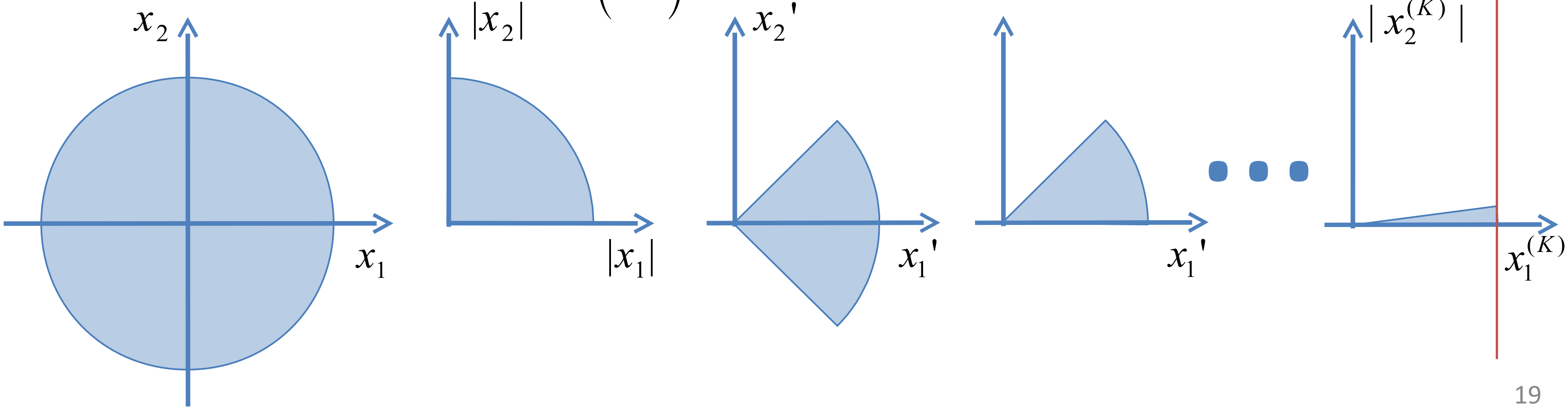
$$\sim n \cdot \frac{1}{\sqrt{\varepsilon}}$$



# Facts

- Solving *p*-cone programs
  - "efficient" approximation of SOC$\left(C_2\right)$
    - efficient (Ben-Tal, Nemirovski)
      - using tower of variables, suffices to describe 3D cone,
      - rely on rotational invariance to describe unit ball,
      - number of inequalities $\sim n \cdot \ln\left(\frac{1}{\varepsilon}\right)$



# Facts

- Solving *p*-cone programs
  - "efficient" approximation of $C_p$ ?
    - cannot be extended in straightforward manner
      - using tower of variables, suffices to describe 3D cone,
      - rotational invariance is lost !
    - for *p* = powers of 2, can build "cascading" construction
      - use SOC to approximate epigraph of $y = x^2$ ,
      - number of inequalities
    - for *p* = rational, becomes prohibitively expensive

# Facts

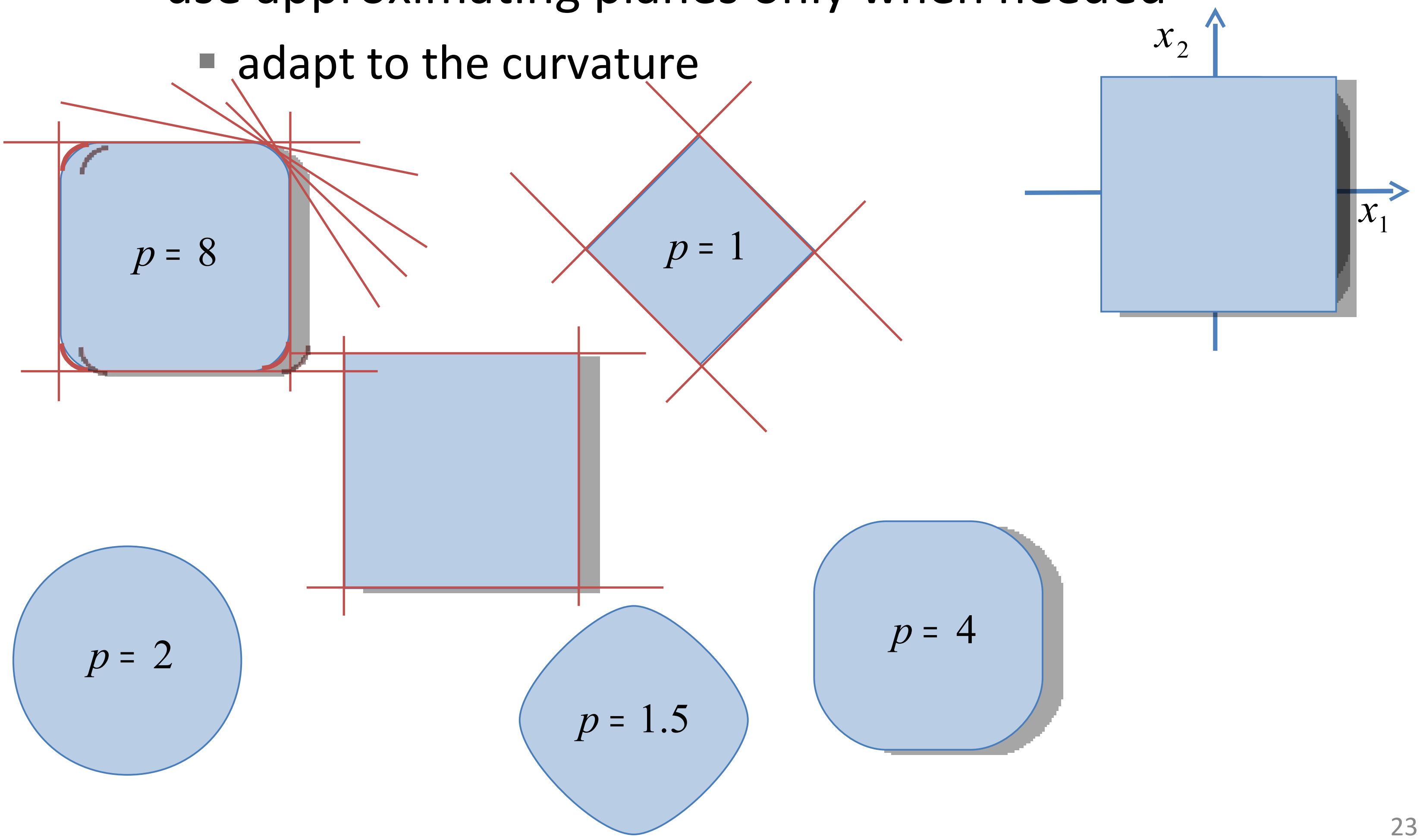
- Solving *p*-cone programs
  - interior-point methods and barriers
  - "efficient" approximation

| *p* | Barrier $\theta_f$ native | Barrier $\theta_f$ SOC | LP approximation |
|---|---|---|---|
| 1 | $2n+1$ | $2n+1$ | $2n+1$ |
| $\infty$ | $2n+1$ | $2n+1$ | $2n+1$ |
| 2 | 2 | 2 | $n \ln(1/\varepsilon)$ |
| 2k | 4 (2k ) | $> 2n\,k$ | $n\,k \ln(1/\varepsilon)$ |
| $m / q$ | $4n$ | $> 2n\,(m+q)$ | (too large ☺) |

**p** **Barrier** **LP approximation**

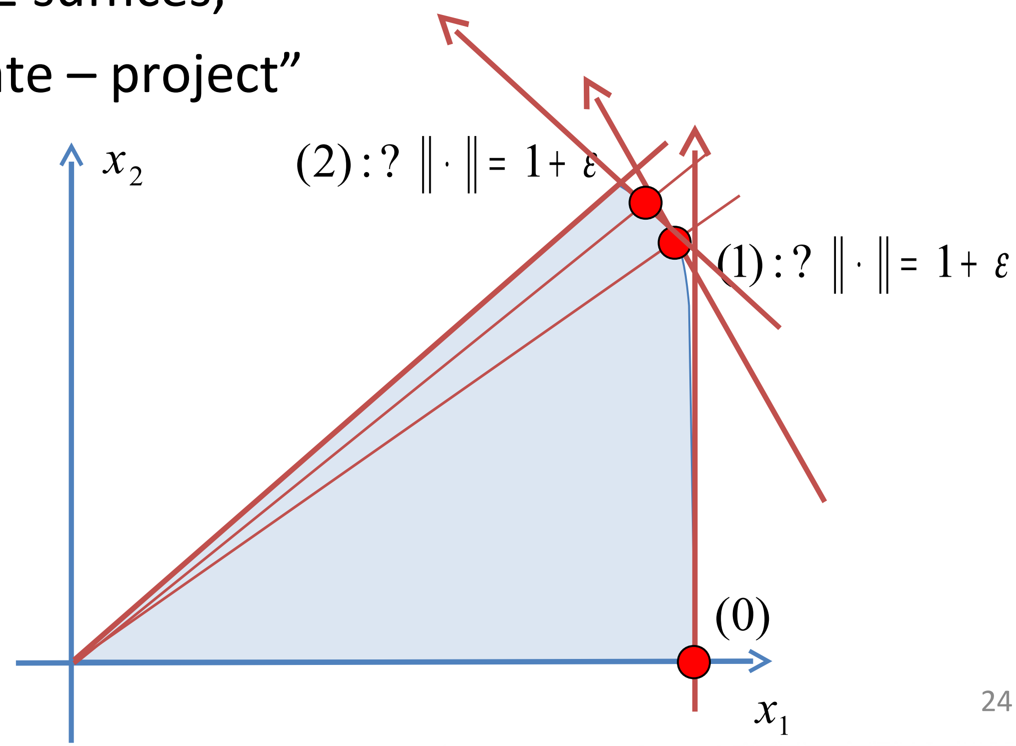
# “Greedy” approximation

- The idea
  - use approximating planes only when needed
    - adapt to the curvature



# “Greedy” approximation

- The idea
  - use approximating planes only when needed
    - adapt to the curvature ,
    - by coordinate symmetry suffices to consider first octant,
    - by duality, $p > 2$ suffices,
    - “tangent – locate – project”



# "Greedy" approximation

- Lazy bound
  - let the two points be $(x_0, y_0)$, $(x_1, y_1)$
  - intersecting $(1+\varepsilon)\cdot(x_1, y_1)$: $y_1 = \frac{1}{y_0^{p-1}} \cdot \frac{1}{1+\varepsilon} - \left(\frac{x_0}{y_0}\right)^{p-1} x_1$
  - boundary

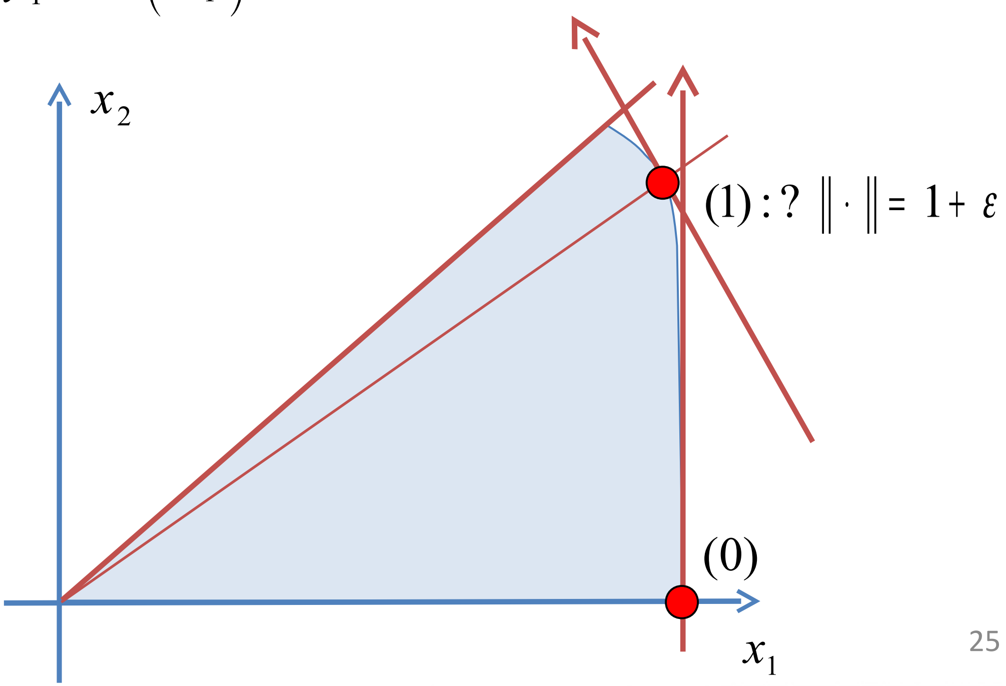
    $(x_1, y_1) \in \partial C_p$: $y_1 = \frac{1}{y_1^{p-1}} - \left(\frac{x_1}{y_1}\right)^{p-1} x_1$
  - combining

$$\frac{x_1}{y_1} < \frac{x_0}{y_0},$$

$$\frac{1}{y_1^{p-1}} < \frac{1}{1+\varepsilon} \cdot \frac{1}{y_0^{p-1}},$$

$$y_1 > \sqrt[p-1]{1+\varepsilon} \cdot y_0,$$

$$\# \sim \frac{1}{\ln(1+\varepsilon)} \sim \frac{1}{\varepsilon}$$



# "Greedy" approximation

- Tight bounds and complexity
  - number of inequalities

$$\sim n \cdot \frac{1}{\sqrt{\varepsilon}}$$

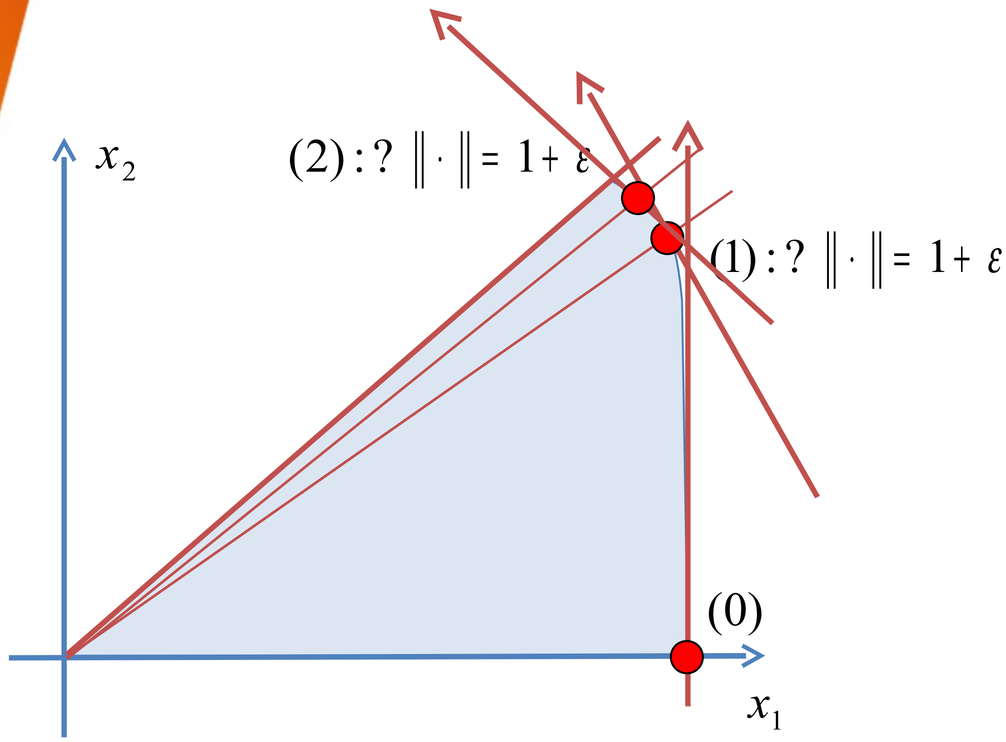
    - can establish upper and lower bound of the same order
  - comparing to naïve equi-spaced scheme get

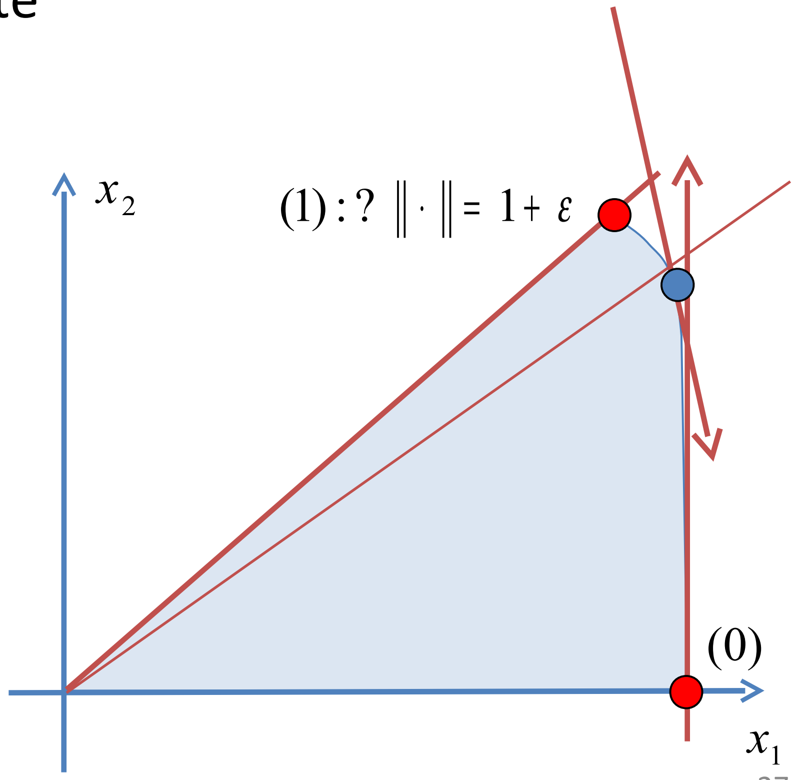
$$\lim_{\varepsilon \to 0} \frac{\text{"greedy"\# inequalites}}{\text{"naive"\# inequalites}} = p - 1$$

    - for large *p* the difference will be large

# "Greedy" approximation

- One extension
  - "greedy" error is strictly below ε due to projections,
  - improve by targeting exact error
    - "tangent/tangent – locate"



# "Greedy" approximation

- One extension
  - "greedy" error is strictly below ε due to projections,
  - improve by targeting exact error
    - "tangent/tangent – locate"
  - number of inequalities

$$\sim n \cdot \frac{1}{\sqrt{\varepsilon}}$$

    - roughly 2 times less than "greedy"

# Facts

- Solving *p*-cone programs
  - interior-point methods and barriers
  - “efficient” approximation

| $p$ | Barrier $\theta_f$ native | Barrier $\theta_f$ SOC | LP approximation |
|---|---|---|---|
| $1$ | $2n + 1$ | $2n + 1$ | $2n + 1$ |
| $\infty$ | $2n + 1$ | $2n + 1$ | $2n + 1$ |
| $2$ | $2$ | $2$ | $n \ln(1/\varepsilon)$ |
| $2k$ | $4\,(2k)$ | $> 2n\,k$ | $n\,k \ln(1/\varepsilon)$ |
| $m / q$ | $4n$ | $> 2n\,(m + q)$ | (too large ☺) |

p | Barrier | LP approximation

$$O\left(n \cdot \frac{1}{\sqrt{\varepsilon}}\right)$$

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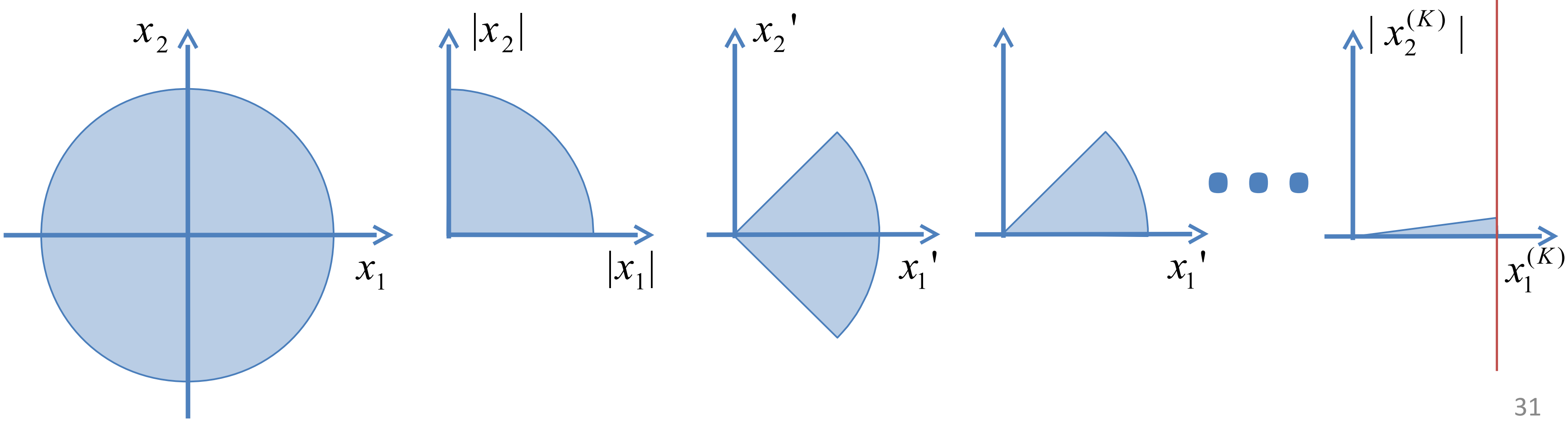
# Moving further

# Moving further

- Lessons from BTN ☺
  - other geometric primitives
    - reflect,
    - rotate,
    - fold onto
  - only the boundary really matters!



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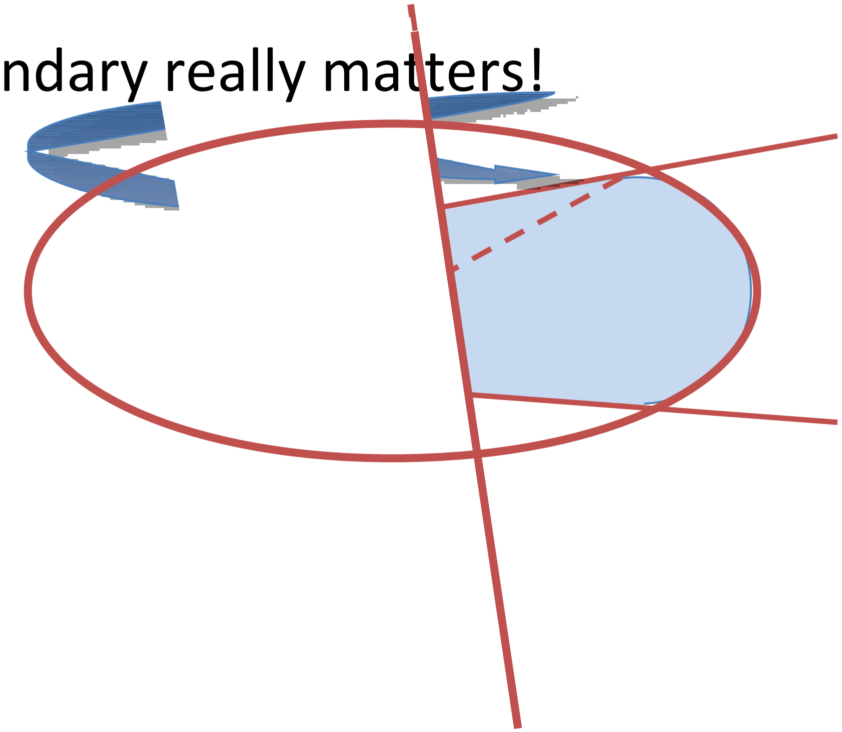
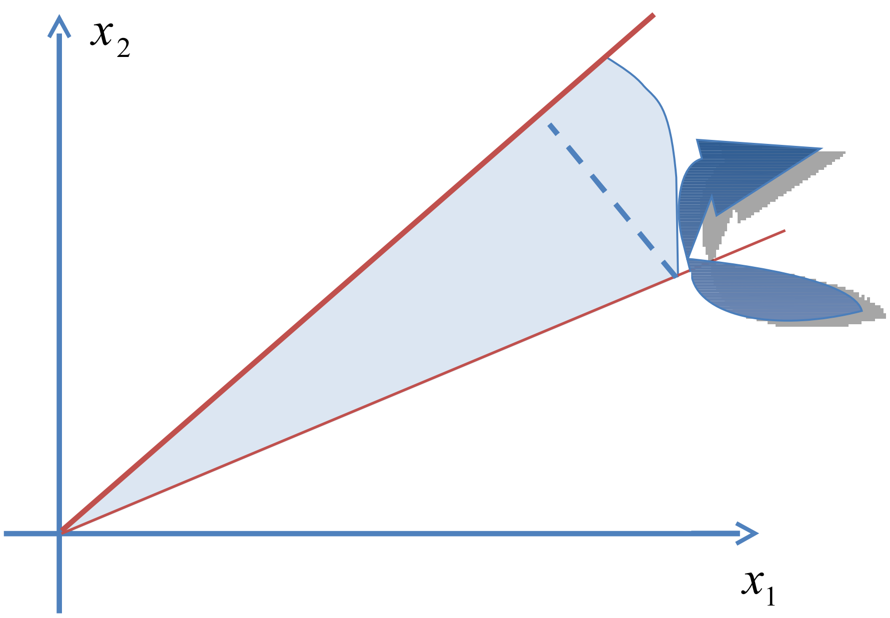


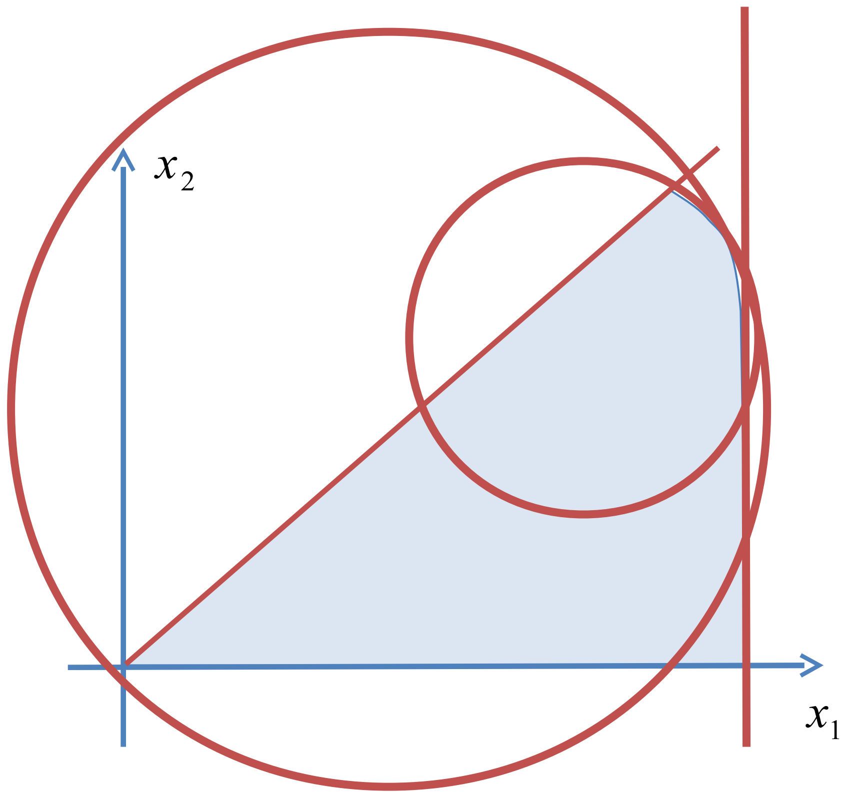
# Moving further

- Boundary curvature and more insights
  - curvature = radius of inscribed circle,
  - curvature = "steering" when driving at a const speed,
  - increasing with arc-length for $p > 2$ (recall duality),
  - octant may be folded "onto itself" , etc.



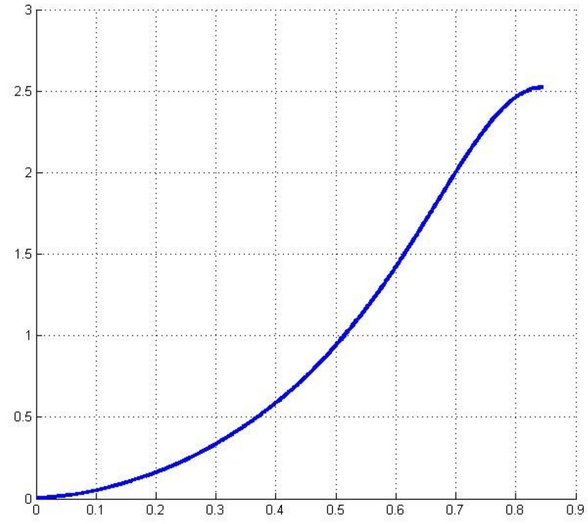
# Moving further

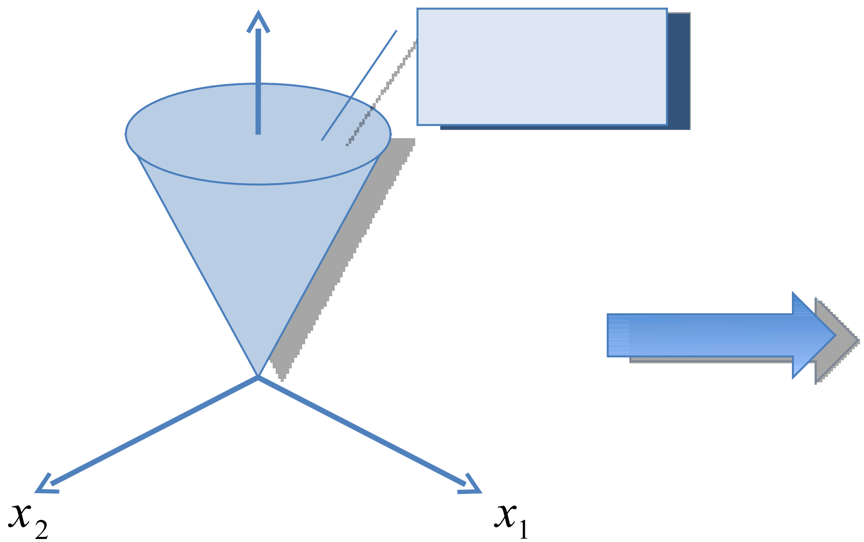
- Fitting with constant curvature
  - constant curvature = Euclidean ball
    - recall SOC has efficient polyhedral approximation,
    - “jerk-and-lock” the steering wheel ☺

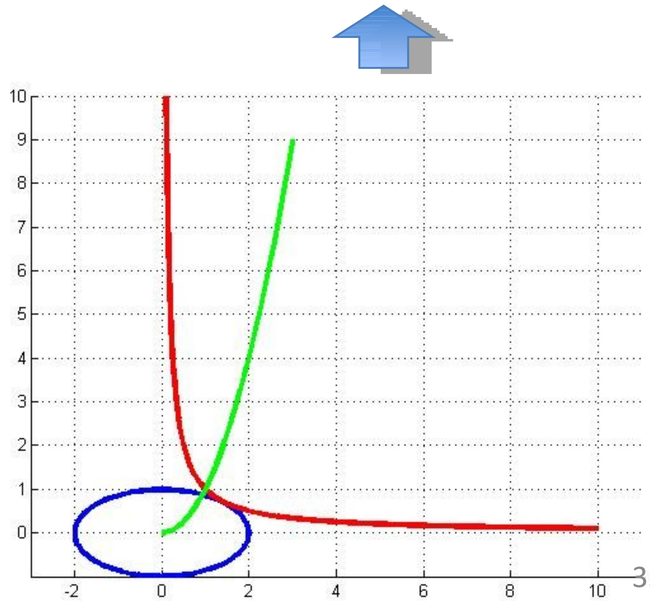


# Moving further

- Fitting with variable curvature
  - SOC conic sections
    - parabola,
    - ellipse,
    - hyperbola





# Moving further

- Fitting with variable curvature"
  - SOC conic sections
    - parabola, ellipse, hyperbola
  - fit general quadratics
    - instead of "jerk-and-lock" use smooth steering pattern …

| ***p* = 4** | | | | |
|---|---|---|---|---|
| K: ε = 10-K | # LP | # ball | # parabola | # general quadratic |
| 1 | 2 | 2 | 2 | 2 |
| 2 | 6 | 3 | 3 | 3 |
| 3 | 16 | 7 | 7 | 4 |
| 4 | 49 | 14 | 18 | 7 |
| 5 | 153 | 29 | 41 | 10 |
| ***p* = 4** | | | | |

# Hopes and conclusions

- Despite *p*-norm not being rotationally-invariant, believe that
  true polyhedral approximation complexity
  is not far from that of SOC…

**THANK YOU!**

**p.s.: looking for a PDF**