# Utility indifference valuation for non-smooth payoffs on a market with some non tradable assets

- Joint work with G. Benedetti (Paris-Dauphine, CREST) -

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# Contents

# Utility indifference pricing (UIP): Motivation

- Goal: pricing in incomplete markets introducing agent's risk aversion.
- Focus on non-smooth payoffs. The motivation comes from structural models for energy markets: e.g., in Aïd, Campi and Langrené (2012) the spot price essentially is

$$P_T = g(\overline{C}_T - D_T) \sum_{1}^{d} h_i S_T^i \mathbf{1}_{\{\sum_1^{i-1} C_T^j \leq D_T \leq \sum_1^i C_T^j\}}$$

where $g(x) = (1/\epsilon)\mathbf{1}_{x \leq \epsilon} + (1/x)\mathbf{1}_{x \geq \epsilon}$ (i.e. capped above for $x > 0$ small).

- Other important example: call options on spread $(P_T - h_i S_T^i - K)_+$, building blocks for power plant evaluation using real option approach.

# Motivation

- Incomplete market, thus need for pricing/hedging criterion.
- local risk minimization in Aïd, Campi, Langrené (MF, 2012)
- We focus on exponential UIP, i.e. $U(x) = -e^{-\gamma x}$, $\gamma > 0$.
- In stock markets (with non-traded assets): El Karoui-Rouge, Davis, Becherer, Henderson, Hobson, Monoyios, Imkeller, Ankirchner, Frei, Schweizer and many others (survey by Henderson & Hobson (2009) for more info on UIP).
- In energy market literature, see Benth et al. (2008) for certainty equivalent principle, without trading on fuel markets.
- In our case, the payoff may depend on both assets, quite unusual in the UIP literature for markets with traded and non-traded assets.
- Sircar and Zariphopoulou (2005) deal with $f(S_T, X_T)$, but with $f$ smooth and both $S$ and $X$ univariate

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# Our contributions

- In a multivariate Markovian model with B&S tradable and mean-reverting non-tradable assets,
- we give a characterization of UIP of some $f$ as the solution $Y$ to a BSDE beyond the usual assumptions of boundedness and $\exists$ of exp moments.
- It's nonetheless difficult to interpret the $Z$ of this BSDE as the optimal hedging strategy.
- To do that, we consider European claims $f(S_T, X_T)$, under some growth conditions on $f$ and its derivatives.
- We deduce from it some asymptotic expansions for prices and strategies.

# The Model: Dynamics of tradable assets

Let $(\Omega, \mathbb{F}, P)$ be a filtered prob space where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the natural filtration generated by a $(n+d)$-dim BM $W = (W^S, W^X)$.

## Tradable assets

The tradable assets $S^i$, $i = 1, \ldots, n$ have dynamics

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_{i\cdot} dW_t^S, \quad 1 \leq i \leq n \tag{1}$$

In a more compact way

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^S, \tag{2}$$

where $W^S$ is a $n$-dim BM, $\sigma$ is a $n \times n$ invertible vol matrix.

In this (sub-)market, $\exists$ a unique EMM $Q^0 \sim P$ for $S$.

# The Model: Dynamics of non tradable assets

## Nontradable assets

They follow (generalized) OU processes

$$dX_t^i = (b_t^i - \alpha_i(t) X_t^i)dt + \beta_i(t) dW_t^X.$$

for $i = 1, \ldots, d$. We denote $\beta_{\cdot i}$ the $i$-th column of the matrix $\beta$.

The agent wealth process is

$$V_t^v(\pi) = v + \int_0^t \pi_u'(\mu du + \sigma dW_u^S) = v + \int_0^t \pi_u' \sigma(\theta du + dW_u)$$

where $\theta = \sigma^{-1}\mu$. We define the sets

$$\mathcal{H} = \{\pi : V^0(\pi) \text{ is a } Q-\text{supermartingale } \forall Q \in \mathcal{M}_E^a\}$$

$$\mathcal{H}_b = \{\pi : V^0(\pi) \text{ is uniformly bdd below by a constant}\}$$

where $\mathcal{M}_E^a$ is the set of all abs cont MM with finite entropy for $S$.

# Definition of UIP

## Definition

Let $f \in L^0(\mathcal{F}_T)$. The buyer UIP $p$ of $f$ is the solution to

$$\sup_\pi E\left[-e^{-\gamma\left(V_T^{v-p}(\pi)+f\right)}\right] = \sup_\pi E\left[-e^{-\gamma V_T^v(\pi)}\right] \quad (3)$$

where the sup is over $\mathcal{H}$ or $\mathcal{H}_b$ (cf Owen & Zitkovic (09)).
*The optimal hedging strategy* $\Delta$ is the difference between the max $\hat{\pi}^f$ and $\hat{\pi}^0$ in resp. the LHS and RHS of (3), i.e. $\Delta = \hat{\pi}^f - \hat{\pi}^0$.

## Main example : Forward contracts on the spot

$$f = P_T = g(\overline{C}_T - D_T)\sum_{i=1}^n h_i S_T^i \mathbf{1}_{\{\sum_{l=1}^{i-1} C_T^l \le D_T \le \sum_{l=1}^{i} C_T^l\}}$$

which is not bounded nor smooth. Usually $f$ is bounded or has exponential moments (BSDE) or it is smooth (PDE).

# UIP & BSDE : bounded payoffs

Set $Z = (Z^S, Z^X)$ and consider the pricing BSDE

$$Y_t = f - \int_t^T \left( \frac{\gamma}{2} \|Z_s^X\|^2 + \mu' \sigma^{-1} Z_s^S \right) ds - \int_t^T Z_s dW_s \qquad (4)$$

## A starting point

Suppose $f$ is bounded. Then $p = Y_0$, where $(Y, Z)$ is the unique solution of BSDE (4) satisfying

$$E\left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T \|Z_t\|^2 dt \right] < \infty$$

Moreover, the optimal hedging strategy is given by $\Delta_t = -\sigma^{-1} Z_t^S$.

Ref. Rouge and El Karoui (2000), or adapting Hu et al. (2005).

# UIP & BSDE : unbounded payoffs

Assume that the claim $f$ satisfies

$$V_T^{v_1}(\pi^1) \leq f \leq V_T^{v_2}(\pi^2), \quad v_i \in \mathbb{R}, \pi^i \in \mathcal{H}. \tag{5}$$

with $V_T^{v_i}(\pi^i) \in L^1(Q^0)$.

**Proposition**

*Under Assumption (5) the pricing BSDE above admits a solution. Moreover, if*

$$\sup_{Q \in \mathcal{M}_E^a} E^Q[f_n - f] \to 0, \qquad \inf_{Q \in \mathcal{M}_E^a} E^Q[f_n - f] \to 0$$

*where* $f_n = (-n) \vee f \wedge n$, *then* $p = Y_0$.

The condition above is in our case easy to handle thanks to the product structure of $\mathcal{M}_E^a$ (recall independence of $S$ and $X$).

# UIP & BSDE II : unbounded payoffs

The proof is based on the following steps (based on Briand and Hu (2007)) :

- Consider the pricing BSDE under $Q^0$ with $f_n = f \wedge n \vee (-n)$ instead of $f$

$$Y_t = f_n + \int_t^T g(Z_s)ds - \int_t^T Z_s dW_s^0, \quad g(z) = -\gamma/2\|z^X\|^2,$$

  which admits a bounded solution $(Y^n, Z^n)$.
- Using our super/sub-hedging bounds on $f$, prove that $|Y^n| \leq L$ for some cont mart $L$.
- With this bound, define $\tau_k = \inf\{t : L_t > k\} \wedge T$ and proceed as in Briand and Hu (2005), i.e. paste the solutions on each $(\tau_k, \tau_{k+1}]$.
- Last part by using Owen/Zitkovic (2009).

# European payoff case: heuristics

To get more info on the process $Z$ (thus on the hedging strategy), we consider European payoffs.
Notation: $A = (S, X)$ for processes and $a = (s, x)$ for their values.

Since $f = f(S_T, X_T)$ we look for a solution to (4) of the form $Y_t = \varphi(t, A_t)$ where $\varphi$ solves

$$\begin{cases} \mathcal{L}\varphi - \frac{\gamma}{2}\sum_{i=1}^{d}(\beta'_{\cdot j}\varphi_x)^2 = 0 \\ \varphi(T, a) = f(a) \end{cases} \tag{6}$$

with

$$\mathcal{L}\varphi = \varphi_t + (b - \alpha x)\varphi_x + \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma'_j s^i s^j \varphi_{s^i s^j} + \frac{1}{2}\sum_{i,j=1}^{d}\beta_i\beta'_j\varphi_{x^i x^j}.$$

If $f$ is regular enough (not too much) we expect $\Delta \propto Z^S \propto \varphi_s$.

# Assumptions on $f$

Two types of assumptions for $f = f(S_T, X_T)$.

## Continuous non-smooth payoffs (CONT)

- $f$ is continuous and
- a.e. differentiable with left and right derivatives growing polynomially in $s$, uniformly in $x$.

## Discontinuous payoffs (DISC)

- $f$ is bdd below.
- Finitely many discontinuities (only wrt $x$).
- $f$ is a.e. differentiable such that:
- $f_{s^i}$ is bdd and $f_{s^i} = O(1/s^i)$ for $s^i$ large, uniformly in $x$.
- $|f_{x^j}(s, x)| \leq C(1 + \|s\|^q)$ for some $q \geq 0$, for all $j$, for some constant $C$ independent of $x$.

# The main result

## Theorem

1. *Under (CONT) or (DISC) the price* $\varphi$ *of the claim* $f$ *is viscosity solution of*

$$\mathcal{L}\varphi - \frac{\gamma}{2}\sum_{j=1}^{d}(\beta'_{\cdot j}\varphi_x)^2 = 0, \quad \varphi(T,a) = f(a)$$

*on* $[0,T) \times \mathbb{R}_+^n \times \mathbb{R}^d$, *which is also differentiable wrt* $(s,x)$.

2. *The optimal hedging strategy is given by*

$$\Delta_t = -\sigma^{-1}Z_t^S = -\sigma^{-1}\sigma(S_t)\varphi_s(t,A_t),$$

*where* $(Y,Z)$ *is solution to the pricing BSDE,* $\sigma(S)$ *is the matrix whose* $i$*-th row is* $\sigma_i S^i$.

# Step 1 : an auxiliary problem with compact controls

- Consider this problem first :

$$-\mathcal{L}\varphi + h^m(\beta'\varphi_x) = 0, \quad \varphi(T, a) = f(a)$$

with $h^m(q) = \sup_{\delta \in \mathcal{B}^m(\mathbb{R}^d)} \left\{ -q\delta - \frac{1}{2\gamma}\|\delta\|^2 \right\}$, $m > 0$.

When $m \to \infty$ this PDE becomes the one we are interested in.

- The associated BSDE under $Q^0$ is

$$Y_t^m = f - \int_t^T h^m(Z_r^{X,m})dr - \int_t^T Z_r^m dW_r^0 \tag{7}$$

# Step 1 : an auxiliary problem with compact controls

- When $f$ is smooth (or non-smooth with poly growth), we can prove $\exists$ of a classical (viscosity) solution to the PDE such that:
- Probabilistic representation of the spacial derivatives of $\varphi^m$ (as in Zhang (2005))

$$\varphi_a^m(t,a) = E_{t,a}^0\left[f(A_T)N_T - \int_t^T h^m(Z_r^{X,m})N_r dr\right] \quad (8)$$

where $N$ is a process depending only on the forward dynamics, it is very simple in our case.
In particular, $\varphi^m$ is differentiable wrt spacial variables.

# Step 2 : $m \to \infty$

- When $f$ smooth, one can prove (as in Pham (2002)) that our PDE admits a classical solution, which is the UIP.
- When $f$ is non-smooth satisfying e.g. (CONT), take $f^l \to f$ ($l \to \infty$) with $f^l$ smooth. Taking $f^l$ as terminal cond in our PDE, we get a classical sol $\varphi^l = \lim_m \varphi^{m,l}$ (as before).
- We want to pass to the limit in Zhang's representation as $m, l \to \infty$ to get the differentiability of $\varphi$ viscosity sol of our PDE.
- To do so, we use the (uniform) estimates inherited from (CONT):

$$|\varphi^{m,l}_{s^i}(t,a)| + |\varphi^{m,l}_{x^j}(t,a)| \leq C\|s\|^q,$$

allowing dom convergence to get the differentiability of $\varphi$.

# Discontinuous payoffs

Idea: approximate $f$ with a smooth sequence $f^l$, and prove that the derivatives of the price $\varphi^l$ will not explode for $t < T$.
Example: digital payoff $f(x) = \mathbf{1}_{[0,\infty)}(x)$ no traded assets. Setting $\alpha = 0$ we have $\varphi^l_x(T-t,x) \to g(t,x)$, where $g$ solves the Burgers' equation

$$g_t + \gamma g_x g = \frac{1}{2}\beta^2 g_{xx}$$

which has the solution

$$g(t,x) = \frac{\beta e^{-\frac{x^2}{2\beta^2 t}}(1 - e^{-\frac{\gamma}{\beta^2}})}{\gamma\sqrt{2\pi t}\left[(e^{-\frac{\gamma}{\beta^2}} - 1)\Phi\left(\frac{x}{\beta\sqrt{t}}\right) + 1\right]}$$

We deduce $\varphi^l_x(T-t,x) \le \frac{C}{\sqrt{T-t}}$, uniformly in $l$. BUT not applicable with traded assets!

# Step 3 : the optimal strategy

- Approximate $f$ again with a sequence $f^l$, bdd for each $l$.
- The corresponding optimal strategies with the claims $f^l$ are given by $\hat{\pi}^l_t = -\sigma^{-1}\sigma(S_t)\varphi^l_s(t, A_t) + \frac{1}{\gamma}\sigma^{-2}\mu$ and the value functions are

$$u^l(t, v, a) = E_{t,a}\left[-e^{-\gamma\left(V^v_T(\hat{\pi}^l)+f^l\right)}\right].$$

- By the growth assumptions in $s$ (uniform in $x$) we deduce from previous results that $u^l \to u$ where

$$u(t, v, a) = E_{t,a}\left[-e^{-\gamma\left(V^v_T(\hat{\pi})+f\right)}\right]$$

for some optimal $\hat{\pi}$. We would like to identify $\hat{\pi}$ with $\tilde{\pi}_t := -\sigma^{-1}\sigma(S_t)\varphi_s(t, A_t) + \frac{1}{\gamma}\sigma^{-2}\mu$.

# Step 3 : the optimal strategy

- By the reverse Fatou's Lemma

$$\limsup_{l} E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi}^l)+f^l\right)}\right] \leq E_{t,a}\left[\lim_{l} -e^{-\gamma\left(V_T^v(\hat{\pi}^l)+f^l\right)}\right]$$

  where the limit on the LHS is in probability.

- $V_T^v(\hat{\pi}^l) \to V_T^v(\tilde{\pi})$ in $L^2(\Omega, P)$, hence in probability. In the same way, $f^l \to f$ in probability.

- Therefore

$$E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi})+f\right)}\right] \leq E_{t,a}\left[-e^{-\gamma\left(V_T^v(\tilde{\pi})+f\right)}\right]$$

  implying that $\tilde{\pi}$ is indeed optimal (remark that it is in $\mathbb{H}^2(\mathbb{R}^n, Q)$ for any $Q \in \mathcal{M}_V$, therefore it lies in $\mathcal{H}_M$).

# Asymptotic expansion: The price

Reformulating a result in Monoyios (2012), we get under (CONT) or (DISC)

$$\varphi(t,a) = p^0(t,a) - \frac{\gamma}{2} E^0_{t,a}\left[\int_t^T \|\beta p^0_x\|^2(s,A_s)ds\right] + O(\gamma^2)$$

where $p^0(t,a) = E^0_{t,a}[f(A_T)]$ is the price under the MMM $Q^0$.

**Remark**

The zero-th order term is the price we obtained via the local risk min approach. It has been computed for many power derivatives in Aïd et al. (2012).
We computed explicitly the first order term in the expansions above for forward contracts.

# Asymptotic expansions: The opt hedging strategy

Under (CONT) and assuming $f_x$ bounded, we have the following expansions for the derivatives of $\varphi$:

$$\varphi_{x^i}(t,a) = E^0_{t,a}\left[f_{x^i}(A_T)\right] - \gamma E^0_{t,a}\left[f_{x^i}(A_T)\int_t^T \beta\varphi^0_x dW^X_u\right] + O(\gamma^2)$$

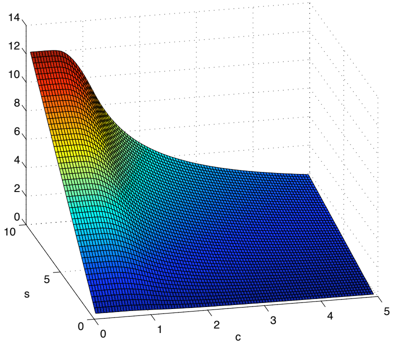
$$\varphi_{s^i}(t,a) = E^0_{t,a}\left[f_{s^i}(A_T)\right] - \gamma E^0_{t,a}\left[f_{s^i}(A_T)\int_t^T \beta\varphi^0_x dW^X_u\right] + O(\gamma^2)$$

where $\varphi^0_{x^i}(t,a) = E^0_{t,a}\left[f_{x^i}(A_T)\right]$.

Expansions for the optimal hedging strategy can be derived from these results.

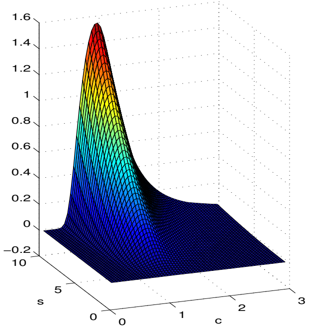
# Example

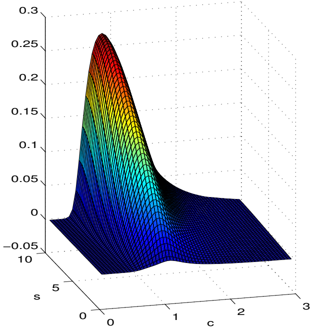
Forward contract with one fuel $f(s,c) = sg(c)$, where $c$: OU process for difference between demand and capacity, and $g(c) = \min\left(M, \frac{1}{c}\right) \mathbf{1}_{\{c>0\}} + M\mathbf{1}_{\{c\le 0\}}$. No-arbitrage price of a forward contract at a given time to maturity $T-t = 0.5$. Parameter values: $\sigma = \beta = 0.3$, $\alpha = 0.2$, $\frac{1}{M} = 0.8$.



# Example

Absolute difference in the price (left) and hedging strategy (right), under no-arbitrage and utility indifference evaluation (with $\gamma = 5$) of a forward contract.



# Thanks for your attention!