Smooth solutions to portfolio liquidation problems under price-sensitive market impact

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[1]Based on Joint work with P. Graewe and E. Séré

Outline

- Portfolio liquidation/accquisition under market impact
  - liquidation with active orders
  - liquidation with active and passive orders
- Markovian Control Problem (with P. Graewe and E. Séré)
  - An HJB equation with singular terminal value
  - Existence of short-time solutions
  - Verification argument
- Non-Markovian Control Problem (with P. Graewe and J. Qiu)
  - A BSPDE with singular terminal value
  - Existence of solutions
  - Verification argument
- Conclusion

Portfolio Liquidation

Portfolio Liquidation

- Traditional financial market models assume that investors can buy sell arbitrary amounts at given prices
- This neglects *market impact*: large transactions (1%-3% of ADV, or more) move prices in an unfavorable direction

Portfolio Liquidation

- Economists have long studied models of optimal block trading
  - Their focus is often on informational asymmetries
  - *Stealth trading:* split large blocks into a series of smaller ones
- Mathematicians identified this topic only more recently
  - Their focus is often on 'structural models' (algorithmic trading)
  - Models of optimal portfolio liquidation give rise to novel stochastic control problems:
    - ('Liquidation') constraint on the terminal state
    - Value functions with *singular terminal value*
    - PDEs, BSDEs, BSPDEs, ... with singular terminal values

Portfolio Liquidation

- Almost all trading nowadays takes place in limit order markets.
  - Limit order book: list of prices and available liquidity
  - Limited liquidity available at each price level
- There are (essentially) two types of orders one can submit:
  - *active* orders submitted for immediate execution
  - *passive* orders submitted for future execution
- We allow active <u>and</u> passive orders; *price sensitive impact*
  - Markovian model: PDE with singular terminal condition
  - non-Markovian model: BSPDE with singular terminal condition

Liquidation with active orders

Consider an order to sell $X > 0$ shares by time $T > 0$:

- ξ_t *rate* of trading (control)
- $X_t = X - \int_0^t \xi_s \, ds$ remaining position (controlled state)
- S_t market/benchmark price (uncontrolled state)

The optimal liquidation problem is of the form

$$\min_{(\xi_t)} \mathbb{E}\left[\int_0^T f(\xi_t, S_t, X_t)\, dt\right] \quad \text{s.t. } X_{T-} = 0$$

The liquidation constraint results in a singularity of the value function:

$$\lim_{t \to T-} V(t, S, X) = \begin{cases} +\infty & \text{for } X \neq 0 \\ 0 & \text{for } X = 0 \end{cases}$$

Benchmark: linear temporary impact

For some martingale (S_t), the *transaction price* is given by

$$\widetilde{S}_t = S_t - \eta\xi_t \quad (\eta = \text{market impact factor}).$$

The *liquidity costs* are then defined as

$$\begin{aligned}\mathscr{C} &= \text{book value} - \text{revenue}\\ &= S_0 X - \int_0^T \widetilde{S}_t \xi_t \, dt = -\int_0^T X_t \, dS_t + \int_0^T \eta\xi_t^2 \, dt\end{aligned}$$

and the *expected liquidity costs* are

$$\mathbb{E}[\mathscr{C}] = \int_0^T \eta\xi_t^2 \, dt.$$

Usually, one minimizes expected liquidation + risk costs.

Literature review

- Almgren & Chriss (2000): mean-variance, S_t BM

$$\int_0^T \eta\xi_t^2 + \lambda\sigma^2 X_t^2 \, dt \;\longrightarrow \text{min}$$

- Gatheral & Schied (2011): time-averaged VaR, S_t GBM

$$\mathbb{E}\left[\int_0^T \eta\xi_t^2 + \lambda S_t X_t \, dt\right] \longrightarrow \text{min}$$

- Ankirchner & Kruse (2012): similar but $dS_t = \sigma(S_t)dW_t$

$$\mathbb{E}\left[\int_0^T \eta\xi_t^2 + \lambda(S_t) X_t^2 \, dt\right] \longrightarrow \; \text{min}$$

- and many others ....

Markovian Models

Liquidation with active and passive orders

Modeling the impact of active orders is comparably simple; the impact of passive orders is harder to model:

- how does the market react to passive order placement?
- using active and passive orders simultaneously may lead to market manipulation
- ....

To overcome this problem, we assume that passive orders are placed in a *dark pool*:

- passive orders are not openly displayed
- executed only when matching liquidity becomes available
- if executed, then at prices coming from some primary venue

Dark trading: *reduced trading costs vs. execution uncertainty*.

Liquidation with active and passive orders

We allow for active and passive orders:

- active order placements: $(\xi_t)_{t\in[0,T)}$
- passive order placements: $(\nu_t)_{t\in[0,T)}$

For $X_0 = X$ the portfolio dynamics is given by

$$dX_t = -\xi_t \, dt - \nu_t \, d\pi_t \quad \text{with} \quad X_{T-} = 0 \quad a.s.$$

Our value function is given by

$$\begin{aligned} & V(T,S,X) \\ = \quad & \inf_{(\xi,\nu)\in\mathscr{A}(T,X)} \mathbb{E}\left[\int_0^T \eta(S_t)|\xi_t|^p + \gamma(S_t)|\nu_t|^p + \lambda(S_t)|X_t|^p \, dt\right] \end{aligned}$$

where the coefficients $\eta, \sigma, \gamma, \lambda$ are nice enough and $p > 1$.

Remark (Power-structure of cost function)

Kratz (2012) and H & Naujokat (2013) consider the cost function

$$\mathbb{E}\left[\int_0^T \eta|\xi_t|^2 + \gamma|\nu_t|^{1} + \lambda|X_t|^2\,dt\right].$$

In this case, no passive orders are used after first execution. This property does not carry over to price-sensitive impact factors. We thus consider

$$\mathbb{E}\left[\int_0^T \eta(S_t)|\xi_t|^p + \gamma(S_t)|\nu_t|^{p} + \lambda(S_t)|X_t|^p\,dt\right].$$

Theorem (Structure of the Value Function)

The value function is of the form ('power-utility')

$$V(T,S,X) = v(T,S)|X|^p$$

and the optimal controls are:

$$\xi_t^* = \frac{v(T-t,S_t)^\beta}{\eta(S_t)^\beta} X_t, \quad \nu_t^* = \frac{v(T-t,S_t)^\beta}{\gamma(S_t)^\beta + v(T-t,S_t)^\beta} X_t,$$

where $\beta := \frac{1}{p-1} > 0$ *and the "inflator" v solves the PDE*

$$v_T = \frac{1}{2}\sigma^2(S) v_{SS} + \underbrace{\lambda(S) - \frac{1}{\beta\eta(S)^\beta} v^{\beta+1} - \theta\left(v - \frac{\gamma(S)v}{(\gamma(S)^\beta + v^\beta)^{1/\beta}}\right)}_{F(S,v)}.$$

Boundary condition for v

The final position when following ξ^* and ν^* is

$$X \exp\left(-\int_0^T \frac{v(T-t,S_t)^\beta}{\eta(S_t)^\beta}\,dt\right)\prod_{0\le t<T}^{\Delta\pi_t\neq 0}\left(1-\frac{v(T-t,S_t)^\beta}{\gamma(S_t)^\beta+v(T-t,S_t)^\beta}\right).$$

- To ensure $X^*_{T-}=0$ one needs

$$\frac{v(T-t,S)^\beta}{\eta(S)^\beta}\longrightarrow\infty \quad \text{as } t\to T \text{ (uniformly in } S\text{)}.$$

- Through a-priori estimates one shows that

$$v(T,S)\sim\frac{\eta(S)}{T^{\frac{1}{\beta}}} \quad \text{as } T\to 0 \text{ uniformly in } S.$$

If $\eta\equiv const$, no passive orders, then this holds automatically.

Theorem (PDE for v)

After a change of variables, the inflator v is the unique classical solution of

$$v_t = \frac{1}{2}\Delta v - \frac{1}{2}\sigma'(x)\nabla v + F(x, v)$$

such that

$$v(t, x) \to 0 \quad \text{as } t \to 0 \text{ uniformly in } x.$$

This solution satisfies:

$$v(t, x) \sim \frac{\eta(x)}{t^{\frac{1}{\beta}}} \quad \text{as } t \to 0 \text{ uniformly in } x.$$

Remark

- *The operator* $A = \frac{1}{2}\Delta - \frac{1}{2}\sigma'(x)\nabla$ *generates an analytic (yet not strongly continuous) semigroup* e^{tA} *in* $C(\mathbb{R})$ *and a priori bounds give that any short-time solution extends to a global solution.*
- *For the short-time solution, we express the asymptotics in terms of an equation:*

$$v(t,x) = \frac{\eta(x)}{t^{\frac{1}{\beta}}} + \text{'correction'}$$

Existence of a short-time solution

Our ansatz is to additively separate the "leading singular term":

$$v(t,x) = \frac{\eta(x)}{t^{\frac{1}{\beta}}} + \frac{u(t,x)}{t^{\frac{1}{\beta}+1}}, \quad u(t,x) \in \mathscr{O}(t^2) \text{ as } t \to 0 \text{ uniformly in } x$$

Results in an evolution equation in $C(\mathbb{R})$ for the correction term:

$$u'(t) = Au + f(t, u(t)), \quad u(0) \equiv 0,$$

with the singular nonlinearity of the form:

$$f(t, u(t)) = \dots \sum_{k=2}^{\infty} \dots \left(\frac{u(t)}{t\eta} \right)^k \dots .$$

Remark

We move the singularity from the terminal condition into the non-linearity in such a way that it causes no harm.

Existence of a short-time solution

The contraction argument giving a short-time solution by a fixed point of the operator

$$\Gamma(u)(t) = \int_0^t e^{(t-s)A} f(s, u(s))\, ds$$

is then carried out in the space

$$E = \{u \in C([0, \delta]; C(\mathbb{R})) : \|u\|_E < \infty\}$$

where

$$\|u\|_E = \sup_{t \in (0,\delta]} \|t^{-2} u(t)\}$$

Theorem (Existence of solutions)

The operator Γ has a fixed point for all sufficiently small $t \in [0, T]$.

Lemma

It is enough to consider only strategies that yield monotone portfolio processes. For such strategies

$$\mathbb{E}\left[v(T-t, S_t)|X_t^{\xi,\nu}|^p\right] \longrightarrow 0 \quad \text{as } t \to T.$$

Theorem (Value Function)

The value function for our control problem is

$$V(T, S, X) = v(T, X)|X|^p.$$

Non-Markovian Models

Probability space

Consider a probability space $(\Omega, \bar{\mathscr{F}}, \{\bar{\mathscr{F}}_t\}_{t\geq 0}, \mathbb{P})$ with $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ being generated by three mutually independent processes:

- *m*-dimensional Brownian motion W;
- *m*-dimensional Brownian motion B;
- stationary Poisson point process J on $\mathscr{Z} \subset \mathbb{R}^l$ with
  - *finite* characteristic measure : $\mu(dz)$;
  - counting measure $\pi(dt, dz)$ on $\mathbb{R}_+ \times \mathscr{Z}$; and
  - $\{\tilde{\pi}([0,t] \times A)\}_{t\geq 0}$ a martingale where

$$\tilde{\pi}([0,t] \times A := \pi([0,t] \times A) - t\mu(A).$$

- The filtration generated by W is denoted $\mathscr{F}$.

The control problem

- The controlled process is

$$x_t = x - \int_0^t \xi_s \, ds - \int_0^t \int_{\mathscr{Z}} \rho_s(z) \, \pi(dz, ds); \quad x_{T-} = 0$$

the set of admissible strategies is the set of all pairs

$$(\xi, \rho) \in \mathscr{L}^2_{\bar{\mathscr{F}}}(0, T) \times \mathscr{L}^4_{\bar{\mathscr{F}}}(0, T; L^2(\mathscr{Z})) \text{ with } x_{T-} = 0 \quad a.s.$$

- The uncontrolled factors follow the dynamics

$$y_t = y + \int_0^t b_s(y_s, \omega) \, ds + \int_0^t \bar{\sigma}_s(y_s, \omega) \, dB_s + \int_0^t \sigma_s(y_s, \omega) \, dW_s$$

where the processes $b(y, \cdot), \sigma(y; \cdot), \bar{\sigma}(y, \cdot)$ are $\mathscr{F}$-adapted.

The value function

Just as above, the objective is to minimize the cost functional

$$J_t(x_t, y_t; \xi, \rho) =: \mathbb{E}\left[\int_0^T \Big(\eta_s(y_s,\omega)|\xi_s|^2 + \lambda_s(y_s,\omega)|x_s|^2\Big)\, ds \right.$$
$$\left. + \int_{[0,T]\times\mathscr{Z}} \gamma_s(y_s, z, \omega)|\rho_s(z)|^2\, \mu(dz)ds\right]$$

The resulting value function is

$$V_t(x, y) =: \operatorname*{ess\,inf}_{\xi,\rho} J_t(x_t, y_t; \xi, \rho)\Big|_{x_t=x, y_t=y}$$

Hamilton-Jacobi-Bellman Equation

We expect the value function $V_t(x,y)$ to satisfy the BSPDE:

$$\begin{cases} -\,dV_t(x,y) \\ \quad = \Big[\mathrm{tr}\left(\frac{1}{2}\left(\sigma_t\sigma_t^{\mathscr{T}} + \bar\sigma_t\bar\sigma_t^{\mathscr{T}}\right)\partial^2_{yy}V_t(x,y) + \partial_y\Psi_t(x,y)\sigma_t^{\mathscr{T}}(y)\right) \\ \qquad + b_t^{\mathscr{T}}\partial_y V_t(x,y) + \operatorname*{ess\,inf}_{\xi,\rho}\Big\{\eta_t|\xi|^2 + \lambda_t|x|^2 - \xi\partial_x V_t(x,y) \\ \qquad + \int_{\mathscr{Z}}\left(V_t(x-\rho,y) - V_t(x,y) + \gamma_t(y,z)|\rho|^2\right)\mu(dz)\Big\}\Big]\,dt \\ \qquad - \Psi_t(x,y)\,dW_t, \quad (t,x,y)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d; \\ V_T(x,y) = (+\infty)\,1_{x\neq 0}, \quad (x,y)\in\mathbb{R}\times\mathbb{R}^d. \end{cases}$$

A solution is a pair of adapted processes (V,Ψ) s.t. (i) ... (ii) ....

Hamilton-Jacobi-Bellman Equation

Making the same ansatz as before:

$$V_t(x,y) = u_t(y)x^2 \quad \text{and} \quad \Psi_t(x,y) = \psi_t(y)x^2,$$

we now obtain a BSPDE for the inflator. It is of the form:

$$(\mathscr{E}) \begin{cases} -du_t(y) = \Big[\mathrm{tr}\left(a_t \partial^2_{yy} u_t(y) + \partial_y \psi_t(y) \sigma_t^{\mathscr{T}} \right) + b_t^{\mathscr{T}} \partial_y u_t(y) \\ \qquad\qquad - \int_{\mathscr{Z}} \frac{|u_t(y)|^2}{\gamma(t,y,z) + u_t(y)} \mu(dz) - \frac{|u_t(y)|^2}{\eta_t(y)} + \lambda_t(y) \Big] \, dt \\ \qquad\qquad - \psi_t(y) \, dW_t, \quad (t,y) \in [0,T] \times \mathbb{R}^d; \\ u_T(y) = +\infty, \quad y \in \mathbb{R}^d. \end{cases}$$

Theorem (Verification Theorem)

Suppose (u, ψ) *is a solution to BSPDE* $(\mathscr{E})$ *such that ... and a.s.*

$$\frac{c_0}{T-t} \leq u_t(y) \leq \frac{c_1}{T-t}.$$

Then

$$V(t,y,x) := u_t(y)x^2, \quad (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d,$$

coincides with the value function for almost every $y \in \mathbb{R}^d$*, and the optimal (feedback) control is given by*

$$(\xi_t^*,\ \rho_t^*(z)) = \left(\frac{u_t(y_t)x_t}{\eta_t(y_t)},\ \frac{u_t(y_t)x_{t-}}{\gamma_t(z,y_t) + u_t(y_t)} \right).$$

Theorem (Existence of solutions)

Our BSPDE ($\mathscr{E}$) *admits a unique solution* (u, ψ) *such that ... and*

$$\frac{c_0}{T-t} \leq u_t(y) \leq \frac{c_1}{T-t}, \quad \mathbb{P} \otimes dt \otimes dy - a.e. \tag{1}$$

Under suitable stronger conditions on σ *we have that*

$$V(t, y, x) := u_t(y)x^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \tag{2}$$

coincides with the value function for every $y \in \mathbb{R}^d$.

Remark

The proof is based on the penalization method; consider BSPDEs

$$\begin{cases} -du_t^N(y) = \Big[tr\left(a_t \partial_{yy}^2 u_t^N(y) + \partial_y \psi_t^N(y) \sigma_t^{\mathscr{T}}\right) + b_t^{\mathscr{T}} \partial_y u_t^N(y) \\ \qquad\qquad - \int_{\mathscr{Z}} \frac{|u_t^N(y)|^2}{\gamma(t,y,z) + u_t^N(y)} \mu(dz) - \frac{|u_t^N(y)|^2}{\eta_t^N(y)} + \lambda_t^N(y) \Big] dt \\ \qquad\qquad - \psi_t^N(y)\, dW_t, \quad (t,y) \in [0,T] \times \mathbb{R}^d; \\ u_T^N(y) = N, \quad y \in \mathbb{R}^d. \end{cases}$$

and establish their convergence. Converge has to be fast enough. This is the hard part which our method in the Markovian case bypassed.

Conclusion

- We studied control problems with singular terminal conditions arising in models of optimal portfolio liquidation
- In the Markovian framework we showed that the HJB PDE has a strong solution, and ...
- ... obtained detailed information about the degree of the singularity at the terminal time.
- In the non-Markovian framework we solved a BSPDE with singular terminal condition by means of penalization, and ...
- ... also obtained detailed information about the degree of the singularity at the terminal time.
- Open problem: permanent market impact
- Major open problem: different powers for active and passive orders (possible for non-price dependent impact functions).

Thanks