Mean Field Consumption-Accumulation Games with Congestion

Minyi Huang

School of Mathematics and Statistics
Carleton University, Ottawa

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Some background

- Mean field game theory: Competitive decision with a large number of agents
- Stochastic growth theory: Consumption and investment optim.
  - Optimal control of a whole sector of an economy
  - More generally: Nash games of N agents (producers)
  - The pioneering work (Brock and Mirman, J. Econ. Theory, 1972); a nice survey (Olson and Roy, 2006)

A historical note: how numbers matter

- The vision of von Neumann and Morgenstern at the very early time ...

von Neumann and Morgenstern (1944, pp. 12)

"... When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory becomes possible."

"... In all fairness to the traditional point of view this much ought to be said: It is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. An almost exact theory of a gas, containing about 10^{25} freely moving particles, is incomparably easier than that of the solar system, made up of 9 major bodies ... This is, of course, due to the excellent possibility of applying the laws of statistics and probability in the first case."

- The development of mean field game theory very much confirms this vision

Related literature: mean field game models (this is only a partial list)

- J.M. Lasry and P.L. Lions (2006a,b, JJM'07): Mean field equilibrium; O. Gueant (JMPA'09); GLL'11 (Springer): Human capital optimization
- G.Y. Weintraub et. el. (NIPS'05, Econometrica'08): Oblivious equilibria for Markov perfect industry dynamics; S. Adlakha, R. Johari, G. Weibtraub, A. Goldsmith (CDC'08): further generalizations with OEs
- M. Huang, P.E. Caines and R.P. Malhame (CDC'03, 04, CIS'06, TAC'07): Decentralized ε-Nash equilibrium in mean field dynamic games; M. Nourian, Caines, et. al. (TAC'12): collective motion and adaptation; A. Kizilkale and P. E. Caines (Preprint'12): adaptive mean field LQG games
- T. Li and J.-F. Zhang (IEEE TAC'08): Mean field LQG games with long run average cost; M. Bardi (Net. Heter. Media'12) LQG
- H. Tembine et. al. (GameNets'09): Mean field MDP and team; H. Tembine, Q. Zhu, T. Basar (IFAC'11): Risk sensitive mean field games

Related literature (ctn)

- A. Bensoussan et. al. (2011, 2012, Preprints) Mean field LQG games (and nonlinear diffusion models).
- H. Yin, P.G. Mehta, S.P. Meyn, U.V. Shanbhag (IEEE TAC'12): Nonlinear oscillator games and phase transition; Yang et. al. (ACC'11); Pequito, Aguiar, Sinopoli, Gomes (NetGCOOP'11): application to filtering/estimation
- D. Gomes, J. Mohr, Q. Souza (JMPA'10): Finite state space models
- V. Kolokoltsov, W. Yang, J. Li (preprint'11): Nonlinear markov processes and mean field games

Related literature (ctn)

- Z. Ma, D. Callaway, I. Hiskens (IEEE CST'13): recharging control of large populations of electric vehicles
- Y. Achdou and I. Capuzzo-Dolcetta (SIAM Numer.'11): Numerical solutions to mean field game equations (coupled PDEs)
- R. Buckdahn, P. Cardaliaguet, M. Quincampoix (DGA'11): Survey
- R. Carmona and F. Delarue (Preprint'12): McKean-Vlasov dynamics for players, and probabilistic approach
- R. E. Lucas Jr and B. Moll (Preprint'11): Economic growth (a trade-off for individuals to allocate time for producing and acquiring new knowledge)
- Huang (2010); Nguyen and Huang (2012); Nourian and Caines (2012); Bensoussan et al (2013): Major player models.

Related literature (ctn):

Mean field type optimal control:

- D. Andersson and B. Djehiche (AMO'11): Stochastic maximum principle
- J. Yong (Preprint'11): control of mean field Volterra integral equations
- T. Meyer-Brandis, B. Oksendal and X. Y. Zhou (2012): SMP.

There is a single decision maker who has significant influence on the mean of the underlying state process.

A player in a mean field game (except major player models) has little impact on the mean field.

Our plan

- Introduce mean field dynamics into production
- Mean field game and decentralized optimization
- Illustrate long time nonlinear behaviour (via a simple model)
  - The success of mean field games relies on the fact:
    - Agents *can rationally anticipate the mean field behaviour*
  - The model here shows new challenges.
    - Very different from long run ergodic stochastic control

Classical stochastic one-sector growth model: Review

The one-sector economy at stage t involves two basic quantities:

- κ_t: the capital stock (used for investment)
- c_t: consumption

The next stage output y_{t+1}:

$$y_{t+1} = f(\kappa_t, r_t), \quad t = 0, 1, \dots,$$

- $f(\cdot, \cdot)$: called the production function
- r_t: random disturbance
- y_0: the initial output, given
- If the output remaining after investment is all consumed, one has the constraint $\kappa_t + c_t = y_t$

The objective is to maximize the expected discounted sum utility

$$E \sum_{t=0}^{\infty} \rho^t \nu(c_t),$$

- $\nu(c_t)$: utility from consumption, usually concave on $[0, \infty)$

Issues of interest: sustainability, extinction (like whale hunting), business cycles and other oscillatory behaviors, etc.

The model is closely related to the theory of optimal savings and portfolio optimization which usually consider linear production functions.

Stochastic growth theory was pioneered by Brock and Mirman (1972), J. Econ. Theory.

Notation in the mean field model

Keep track of the notation (for the main part):

u_t^i:	control (investment)
X_t^i:	state (production output)
N:	number of players in the game
$c_t, c_t^i, c(\cdot)$:	consumption
$V_i(x,t)$:	value function
$G(p,W), g$:	growth coefficient in production
W:	white noise
p:	aggregate investment
γ:	HARA utility exponent

Mean field production dynamics

- N agents involved in a certain type of production activity
- X_t^i: output (or wealth) of agent i, $1 \le i \le N$
- $u_t^i \in [0, X_t^i]$: investment (so no borrowing)
- $c_t^i = X_t^i - u_t^i$: amount for consumption
- $u_t^{(N)} = (1/N)\sum_{j=1}^N u_t^j$: aggregate investment level

The next stage output of agent i, measured by the unit of capital, is

$$X_{t+1}^i = G(u_t^{(N)}, W_t^i)u_t^i, \qquad t \ge 0, \tag{3.1}$$

We may think of $u_t^{(N)}$ as a quantity measured according to a macroscopic unit.

Motivation for the mean field production dynamics

- Some literature addresses negative externalities.
- For example, Barro and Sala-I-Martin (Rev. Econ. Stud., 1992) considered the so-called congestion effect

$$y = Ak(Gv/K)^{\alpha}, \quad \alpha \in (0,1)$$

  - y: output of a private producer; k: its input;
  - K: aggregate input of n producers;
  - Gv: total resource provided by the government.

For similar models, see (Liu and Turnovsky, J. Pub. Econ., 2005)

The utility functional

The expected discounted sum utility is

$$J_i(u^i, u^{-i}) = E\sum_{t=0}^{T} \rho^t v(X_t^i - u_t^i),$$

$\rho \in (0, 1]$: the discount factor; $c_t^i = X_t^i - u_t^i$: consumption

We take

$$v(z) = \frac{1}{\gamma} z^\gamma, \quad z \geq 0, \qquad \gamma \in (0, 1).$$

Assumptions

(A1) (i) Each sequence $\{W_t^i, t \in \mathbb{Z}_+\}$ consists of i.i.d. random variables with support D_W and distribution function F_W. The N noise sequences $\{W_t^i, t \in \mathbb{Z}_+\}$, $i = 1, \ldots, N$ are i.i.d. (ii) The initial states $\{X_0^i, 1 \le i \le N\}$ are i.i.d. positive random variables with distribution F_{X_0} and mean m_0, which are also independent of the N noise sequences.

(A2) (i) The function G: $[0, \infty) \times D_W \to [0, \infty)$ is continuous; (ii) for a fixed $w \in D_W$, $G(z, w)$ is a decreasing function of $z \in [0, \infty)$.

(A3) (iii) $EG(0, W) < \infty$ and $EG(p, W) > 0$ for each $p \in [0, \infty)$.

(A2) implies congestion effect: when the aggregate investment level increases, the production becomes less efficient.

Further interpretation of (A2):

- Diminishing return of the sector (although no notable d.r. to individual scale)
- Illustration: Suppose the production structure

$$G(z, W) = g(z)W, \qquad g \text{ decreasing.}$$

Let $X_t^{(N)} = \frac{1}{N}\sum_{i=1}^N X_t^i$. Then

$$\begin{aligned} X_t^{(N)} &= g(u_t^{(N)})u_t^{(N)} EW_0^1 + \frac{g(u_t^{(N)})}{N}\sum_{i=1}^N u_t^i(W_t^i - EW_t^i) \\ &= g(u_t^{(N)})u_t^{(N)} EW_0^1 + \text{“small fluctuation”} \end{aligned}$$

To do (in the Nash certainty equivalence approach)

- Step 1 – Approximate the aggregate investment $(u_t^{(N)})_{t=0}^{T-1}$ by a sequence $(p_0, \ldots, p_{T-1})$.
- Step 2 – Solve an optimal control problem with a representative agent
  - Obtain control $\hat{u}_t^i = (\mathrm{coeff}_t)\hat{X}_t^i$ (linear feedback).
  - Determine closed loop state $\hat{X}_t^i$.
- Step 3 – Consistency requirement

$$\lim_{N\to\infty}(1/N)\sum_{i=1}^{N}\hat{u}_t^i = \lim_{N\to\infty}(1/N)\sum_{i=1}^{N}(\mathrm{coeff}_t)\hat{X}_t^i = (\mathrm{coeff}_t)E\hat{X}_t^i = p_t.$$

- Step 4 – Show ε_N-Nash property for $(\hat{u}_t^1, \ldots, \hat{u}_t^N)$.

Step 1: mean field limit

Now agent i considers the optimal control problem with dynamics

$$X_{t+1}^i = G(p_t, W_t^i)u_t^i, \qquad t \geq 0, \tag{3.2}$$

where $u_t^i \in [0, X_t^i]$.

The utility functional is now written as

$$\bar{J}_i(u^i, (p_t)_0^{T-1}, 0) = E \sum_{t=0}^{T} \rho^t v(X_t^i - u_t^i), \tag{3.3}$$

Step 2: optimal control

Further denote the utility functional with initial time t

$$\bar{J}_i(u^i, (p_l)_t^{T-1}, t) = E\sum_{s=t}^{T} \rho^{s-t} v(X_s^i - u_s^i),$$

which is affected only by $(p_l)_t^{T-1}$. The value function is defined as

$$V_i(x,t) = \sup_{\{u_s^i\}_{s=t}^{T}} E[\bar{J}_i(u^i, (p_l)_t^{T-1}, t) | X_t^i = x].$$

We have the dynamic programming equation

$$V_i(x,t) = \max_{0\le u_i \le x} \left[v(x-u_i) + \rho E V_i(G(p_t, W_t^i)u_i,\ t+1) \right],$$

where $t = 0, 1, \dots, T-1$.

Step 2

We look for a solution

$$V_i(x,t) = \frac{1}{\gamma} D_t^{\gamma-1} x^\gamma. \tag{3.4}$$

Denote $\Phi(z) = \rho E G^\gamma(z, W)$ and $\phi(z) = \Phi^{\frac{1}{\gamma-1}}(z)$.

Theorem *(i)* The value function $V_i(x,t)$ takes the form (3.4), where

$$D_t = \frac{\phi(p_t) D_{t+1}}{1+\phi(p_t) D_{t+1}}, \quad t \le T-1, \quad D_T = 1. \tag{3.5}$$

(ii) The optimal control has the feedback form

$$u_t^i = \frac{X_t^i}{1+\phi(p_t) D_{t+1}}, \quad t \le T-1, \quad u_T^i = 0. \tag{3.6}$$

Step 3: consistency

We have the closed-loop

$$X_t^i = G(p_t, W_t^i)\frac{X_t^i}{1+\phi(p_t)D_{t+1}}.$$

By symmetry, $\lim_{N\to\infty} Eu_t^{(N)} = Eu_t^i =: \Lambda_t(p_0, \ldots, p_{T-1})$.

Define the operator Λ (an example to follow)

$$\begin{array}{ccc} (p_0, \ldots, p_{T-1}) & \longmapsto & \Lambda(p_0, \ldots, p_{T-1}) := (\Lambda_0, \ldots, \Lambda_{T-1})(p_0, \ldots, p_{T-1}). \\ \text{assumed mean field} & \longmapsto & \text{actual mean field of closed-loop} \end{array}$$

Theorem Λ has a fixed point in a rectangle region.

By this fixed point, the initially assumed mean field is regenerated by the agents taking individual optimal responses.

Step 3: consistency

Example:

We take $T = 2$. Let (p_0, p_1) be given. Then

$$\Lambda_0(p_0, p_1) = \frac{(1+\phi_1)EX_0^i}{1+\phi_1+\phi_1\phi_0}, \quad \Lambda_1(p_0, p_1) = \frac{EG(p_0, W_0^i)EX_0^i}{1+\phi_1+\phi_1\phi_0}.$$

where $\phi_t = \phi(p_t)$ and we recall $\phi(z) = \Phi^{\frac{1}{\gamma-1}}(z)$.

Step 4: ε-Nash

Theorem The set of NCE based strategies $\{\check{u}^i_t, 0 \le t \le T, 1 \le i \le N\}$ is an ε_N-Nash equilibrium, i.e., for any $i \in \{1, \dots, N\}$,

$$\sup_{u^i} J_i(u^i, \check{u}^{-i}) - \varepsilon_N \le J_i(\check{u}^i, \check{u}^{-i}) \le \sup_{u^i} J_i(u^i, \check{u}^{-i}), \qquad (3.7)$$

where u^i is a centralized strategy and $0 \le \varepsilon_N \to 0$ as $N \to \infty$.

- I did not achieve $\varepsilon_N = O(1/\sqrt{N})$ since no Lipschitz condition is assumed on G (to use a linear inequality for error estimate and then Gronwall's lemma)
- The performance estimate only uses certain dominated convergence argument

Discussion

- One might think the generalization to infinite horizon is straightforward.
  - Unfortunately this is not the case.
  - We can see rich group behavior by developing out-of-equilibrium analysis.

The model

Consider the model

$$X_{t+1}^i = G(p, W_t^i)u_t^i, \quad t \geq 0. \tag{4.1}$$

Let the utility functional be

$$J_{[0,\infty)} = \frac{1}{\gamma} E \sum_{i=0}^{\infty} \rho^t (X_t^i - u_t^i)^\gamma. \tag{4.2}$$

Proposition Assume $X_0^i > 0$. Then $\sup J_{[0,\infty)} < \infty$ if and only if (H) $\rho E G^\gamma(p, W) < 1$.

- We suppose "by luck" p is already an equilibrium state (i.e., optimal responses of many agents exactly replicate it).
- We can use this model to diagnose dynamic properties of the mean field system under self-optimizing behavior.

Assume (H). The stationary dynamic programming equation

$$V_i(x) = \max_{u^i} \left[v(x - u^i) + \rho E V_i(G(p, W) u^i) \right].$$

We try a solution of the form

$$V_i(x) = \frac{1}{\gamma} D^{\gamma - 1} x^{\gamma}, \quad D > 0.$$

The optimal control is

$$u_t^i = \frac{X_t^i}{1 + \phi(p) D}, \tag{4.3}$$

where $D > 0$ satisfies

$$D = \frac{\phi(p) D}{1 + \phi(p) D}, \tag{4.4}$$

$\phi(p) = [\rho E G^{\gamma}(p, W)]^{\frac{1}{\gamma - 1}} > 1.$

The NCE approach gives

$$\begin{cases} 1+\phi(p)D=\phi(p), \\ EG(p,W)=1+\phi(p)D \qquad (\text{so} = \phi(p)) \\ EX_0^i = p(1+\phi(p)D) \end{cases} \tag{4.5}$$

where $\phi(p)=[\rho EG^\gamma(p,W)]^{\frac{1}{\gamma-1}}$.

- The first equation is due to optimal reponse/control
- The second and third equations are the consistency condition on p
  - The second equation means the mean of the aggregate investment is preserved
  - The third equation means the initial state mean is "right"

question

We rewrite

$$\begin{cases} 1+\phi(p)D = \phi(p), \\ EG(p,W) = 1+\phi(p)D \qquad (\text{so} = \phi(p)) \\ EX_0^i = p(1+\phi(p)D) \end{cases} \tag{4.6}$$

Suppose the above NCE based u_i is applied by an infinite population which has its initial mean EX_0^i just slightly different from the "right one" as above.

Can the "right mean" be restored when time goes to infinity? Or at least, this system can tolerate the small error?

Definition The pair (p, D) is called a relaxed stationary mean field (RSMF) solution of (4.1)-(4.2) if

$$\begin{cases} 1 + \phi(p)D = \phi(p) \\ EG(p, W) = 1 + \phi(p)D (= \phi(p)) \end{cases} \tag{4.7}$$

holds and satisfies $D > 0$, $p \geq 0$ and $\rho EG^{\gamma}(p, W) < 1$.

Here the pair (p, D) is so called since the restriction on the initial mean is removed.

We give an existence result for a class of models with multiplicative noise, i.e.,

$$G(p, W) = g(p)W \tag{4.8}$$

for some function $g > 0$ on $[0, \infty)$, $W \geq 0$ and $EW = 1$.

Theorem Assume (i) $0 < \rho < 1$, (ii) g is continuous and strictly decreases on $[0, \infty)$, (iii) $\rho E(W^\gamma)g(0) \geq 1$, and $\rho E(W^\gamma)g(\infty) < 1$ where $g(\infty) = \lim_{p \to \infty} g(p)$.

Then there exists a unique RSMF solution.

The mean field dynamics with RMFS

The closed-loop state equation is given by

$$X_{t+1}^i = \frac{G(u_t^{(N)}, W_t^i) X_t^i}{\phi(\hat{p})}, \quad 1 \le i \le N.$$

We obtain the relation

$$u_{t+1}^i = \frac{G(u_t^{(N)}, W_t^i)}{\phi(\hat{p})} u_t^i, \quad t \ge 0. \tag{4.9}$$

The mean of $\hat{u}_t^i$ should coincide with p_t, and hence we derive the mean field dynamics

$$p_{t+1} = \frac{p_t EG(p_t, W)}{\phi(\hat{p})}, \quad t \ge 0, \tag{4.10}$$

where $p_0 = Eu_0^i$.

Stable equilibrium

For cases 1)-3), we take $g(p) = \frac{1}{\rho E(W^\gamma)} \cdot \frac{C}{1+(C-1)p^3}$.

Case 1). $C = 1.2$, so $p_{t+1} = \frac{1.2p_t}{1+0.2p_t^3} =: Q_C(p_t)$, $Q'_C(\hat{p}) = 0.5$ (slope).

Case 2). If $C = 2$, so $p_{t+1} = \frac{2p_t}{1+p_t^3}$, $Q'_C(\hat{p}) = -0.5$.

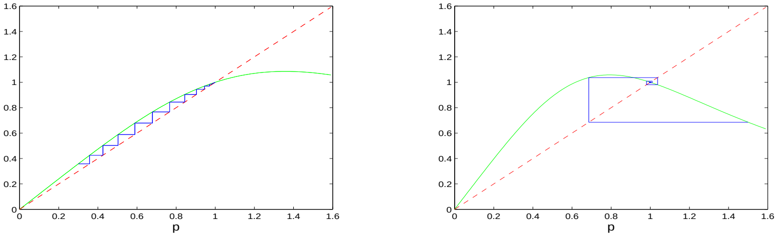


Figure : The iteration of $Q_C(p)$ and stable equilibria. Left: $C = 1.2$; if $0 < p_0 < 1$, the iteration converges to 1 monotonically. Right: $C = 2$.

Limit cycle

Case 3). Take $C = 4$ (i.e., now more sensitive to the MF) to obtain

$$p_{t+1} = \frac{4p_t}{1+3p_t^3} =: Q_C(p_t), \quad Q'_C(\hat{p}) = -1.25,$$

which has a limit cycle determined by the two points on the graph.

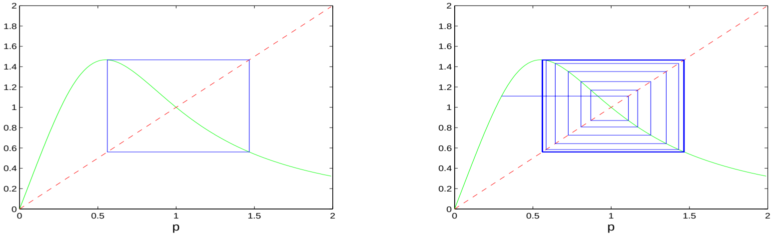
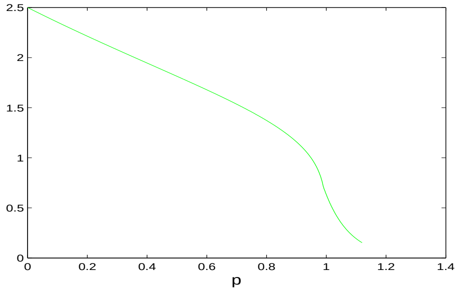


Figure : The iteration of $Q_C(p)$, $C = 4$. Left: the limit cycle. Right: the convergence to the limit cycle with an initial value $p_0 = 0.3$.

Chaotic mean field dynamics

Define $g(p) = \frac{h(p)}{\rho EW^\gamma}$ for the production function (see (4.8)), where $EW = 1$. (See graph of h below)

The sharp decrease near $p = 1$ models a "pollution effect": when the aggregate investment level approaches a certain reference level (here equal to 1), the production becomes very inefficient.



Chaotic mean field dynamics

$$p_{t+1} = p_t h(p_t) =: Q(p_t).$$

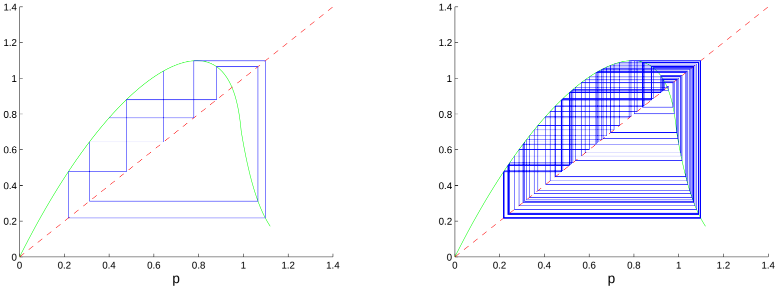


Figure : Initial condition $p_0 = 0.4$. Left: 8 iterates; right: 100 iterates.

See (Li and Yorke, 1975) for definition of chaos. Also see (J. Benhabib ed., Cycles and Chaos in Economic Equil., Princeton Univ. Press, 1992)

Discussion

- The RSMF solution can be used as a classifier of the model.
- When the RSMF solution detects a stable equilibrium, one may propose an asymptotically constant mean field and do consistent MF approximations even if the initial state mean is not the "right one" (Huang, CDC'13).

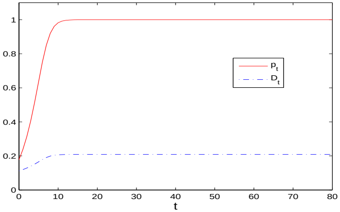


Figure : Infinite horizon mean field approx. with transient behavior.

Discussion

- When oscillatory behavior is diagnosed, is there any hope to develop consistent MF approximations from a certain class of functions? No answer known.

Concluding remarks

Potential extensions:

- Continuous time mean field production modeling. Example: a mean field version of the AK model

$$dk_t^i = (A(k_t^{(N)})k_t^i - \mu k_t^i - c_t^i)dt - \sigma k_t^i dw_t^i$$

  where $-(\mu k_t^i dt + \sigma k_t^i dw_t^i)$ models stochastic depreciation. Note: (R. Merton, Rev. Econ. Stud., 1975) was the first to introduce a continuous time stochastic growth model.
    - No chaos in this case
    - Limit cycles are still possible
- Mean field effect depending on distribution instead of just mean (empirical first order moment)