Option Hedging with Market Impact

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Outline

1. Background and previous work
2. The problem
3. Formulation
4. Solution
5. Examples and applications

Work with Tianhui Michael Li

Princeton: Bendheim Center and ORFE

Equity price swings on July 19 2012

(one day prior to options expiration)

http://www.bloomberg.com/news/2012-07-20/hourly-price-swings-whipsaw-investors-in-ibm-coke-mcdonald-s.html

Investors Whipsawed by Price Swings in IBM, Coca-Cola

By Inyoung Hwang, Whitney Kisling & Nina Mehta - Jul 20, 2012 5:22 PM ET

Investors in three of the biggest Dow Jones Industrial Average (INDU) stocks were whipsawed by price swings that repeated every hour yesterday, fueling speculation the moves were a consequence of computerized trading.

Shares of International Business Machines Corp. (IBM), McDonald's Corp. (MCD) and Coca-Cola Co. (KO) swung between successive lows and highs in intervals that began near the top and bottom of each hour, data compiled by Bloomberg show. While only IBM finished more than 1 percent higher, the intraday patterns weren't accompanied by any breaking news in the three companies where $3.42 billion worth of shares changed hands.

Marko Kolanovic, global head of derivatives and quantitative strategy at JPMorgan & Chase Co

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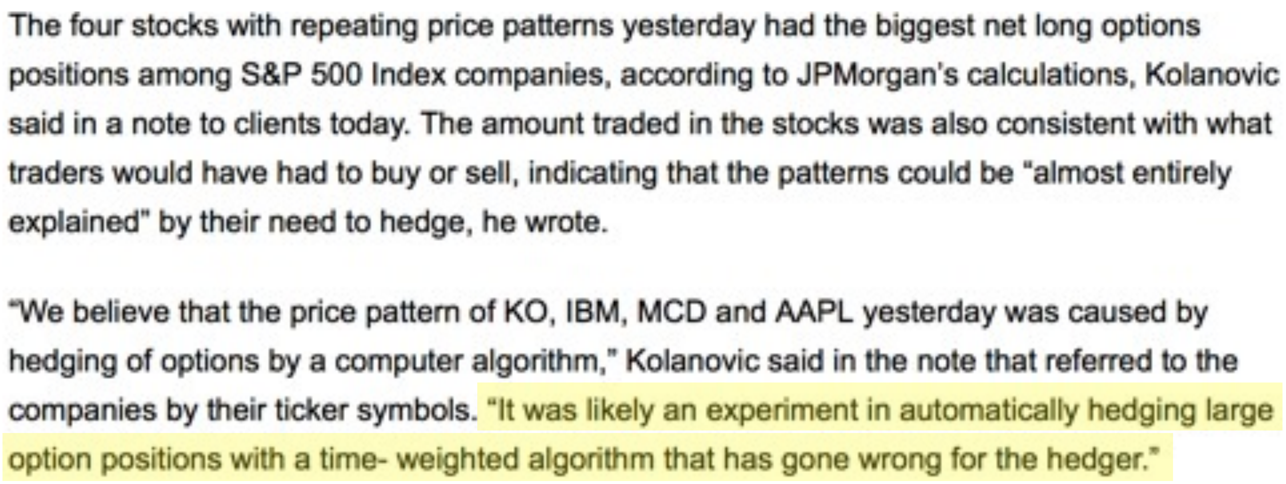

The four stocks with repeating price patterns yesterday had the biggest net long options positions among S&P 500 Index companies, according to JPMorgan's calculations, Kolanovic said in a note to clients today. The amount traded in the stocks was also consistent with what traders would have had to buy or sell, indicating that the patterns could be "almost entirely explained" by their need to hedge, he wrote.

"We believe that the price pattern of KO, IBM, MCD and AAPL yesterday was caused by hedging of options by a computer algorithm," Kolanovic said in the note that referred to the companies by their ticker symbols. "It was likely an experiment in automatically hedging large option positions with a time- weighted algorithm that has gone wrong for the hedger."

CA CHEUVREUX QUANTITATIVE RESEARCH

QUANT **NOTE**

28th of August, 2012

What does the saw-tooth pattern on US markets on 19 July 2012 tell us about the price formation process

The saw-tooth patterns observed on four US securities on 19 July provide us with an opportunity to comment on common beliefs regarding the market impact of large trades; its usual smoothness and amplitude, the subsequent "reversal" phase, and the generic nature of market impact models.
This underscores the importance of taking into account the motivation behind a large trade in order to optimise it properly, as we already emphasised in *Navigating Liquidity 6*.

We used different intraday analytics to work out what happened: pattern-matching techniques, market impact models, order flow imbalances and PnL computations of potential stat. arb intraday strategies. After looking at open interests of derivatives on these stocks, we conclude that repetitive automated hedging of large-exposure derivatives lay behind this behaviour. This is an opportunity to understand how a very crude trading algorithm can impact the price formation process ten times more than is usually the case.

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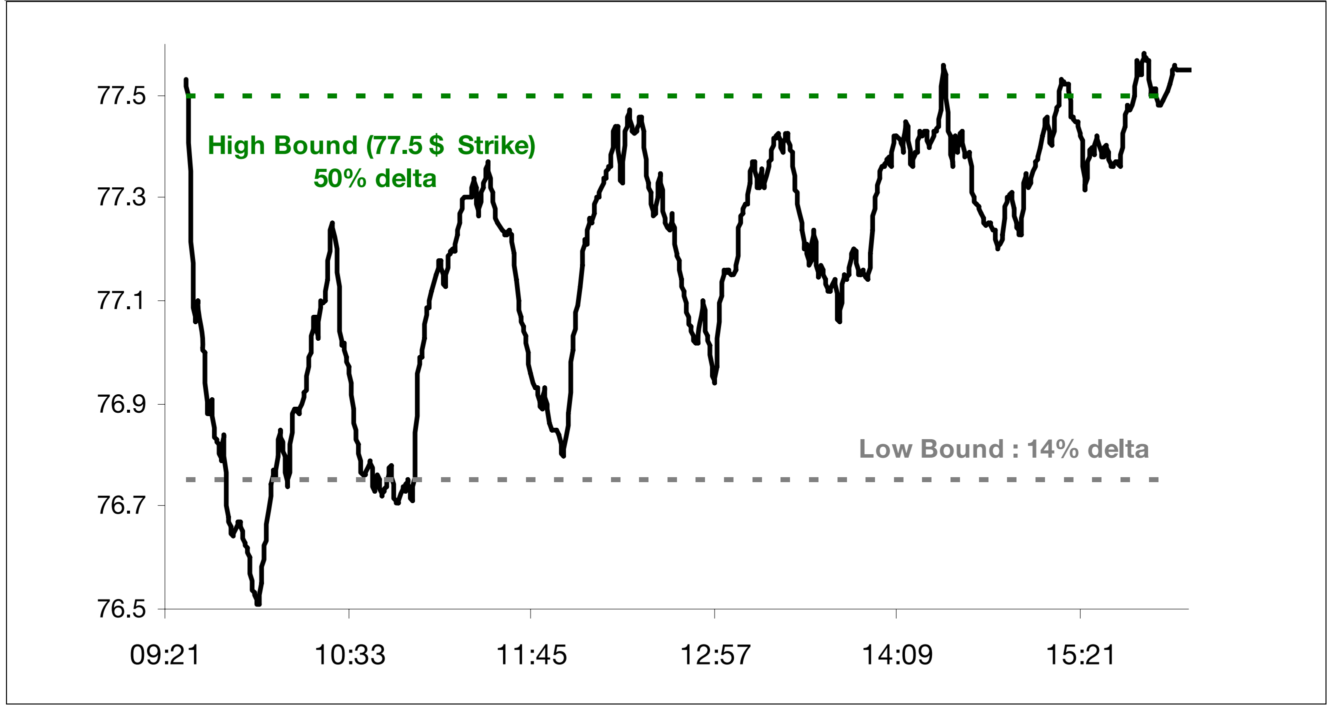
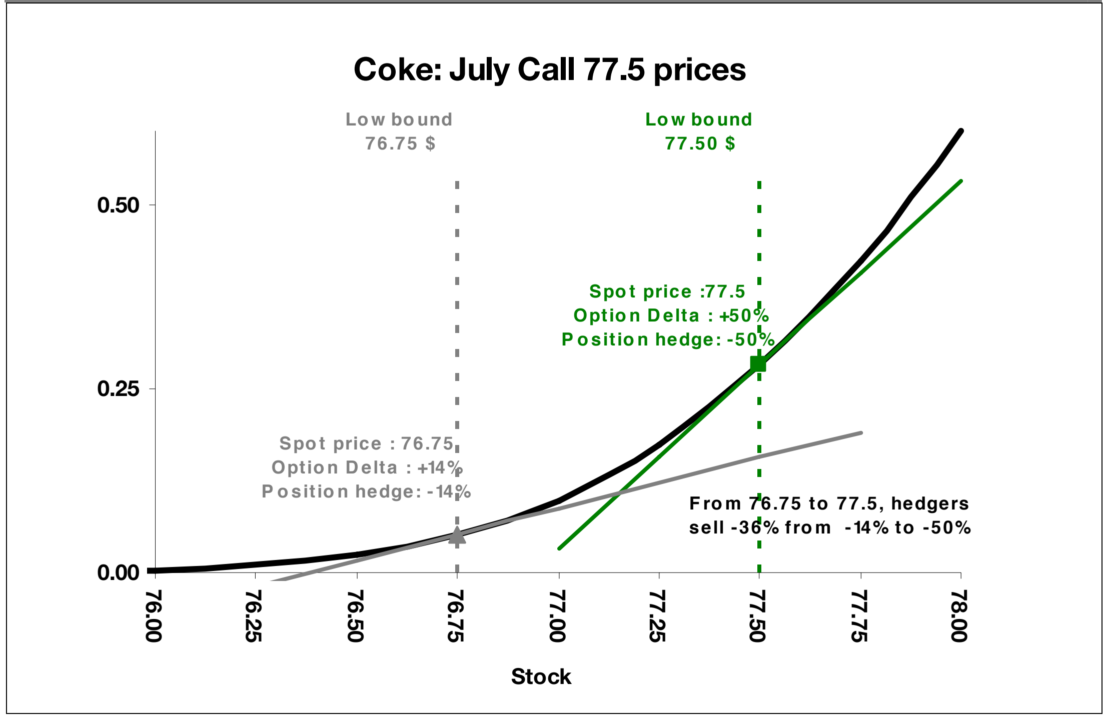
Chevreux: sawtooths caused by options hedging

FIGURE 18: SAW-TOOTH ON COCA-COLA AND 77.50 CALL



Source: Crédit Agricole Cheuvreux Quantitative Research

FIGURE 19: LONG VOLATILITY DELTA HEDGING



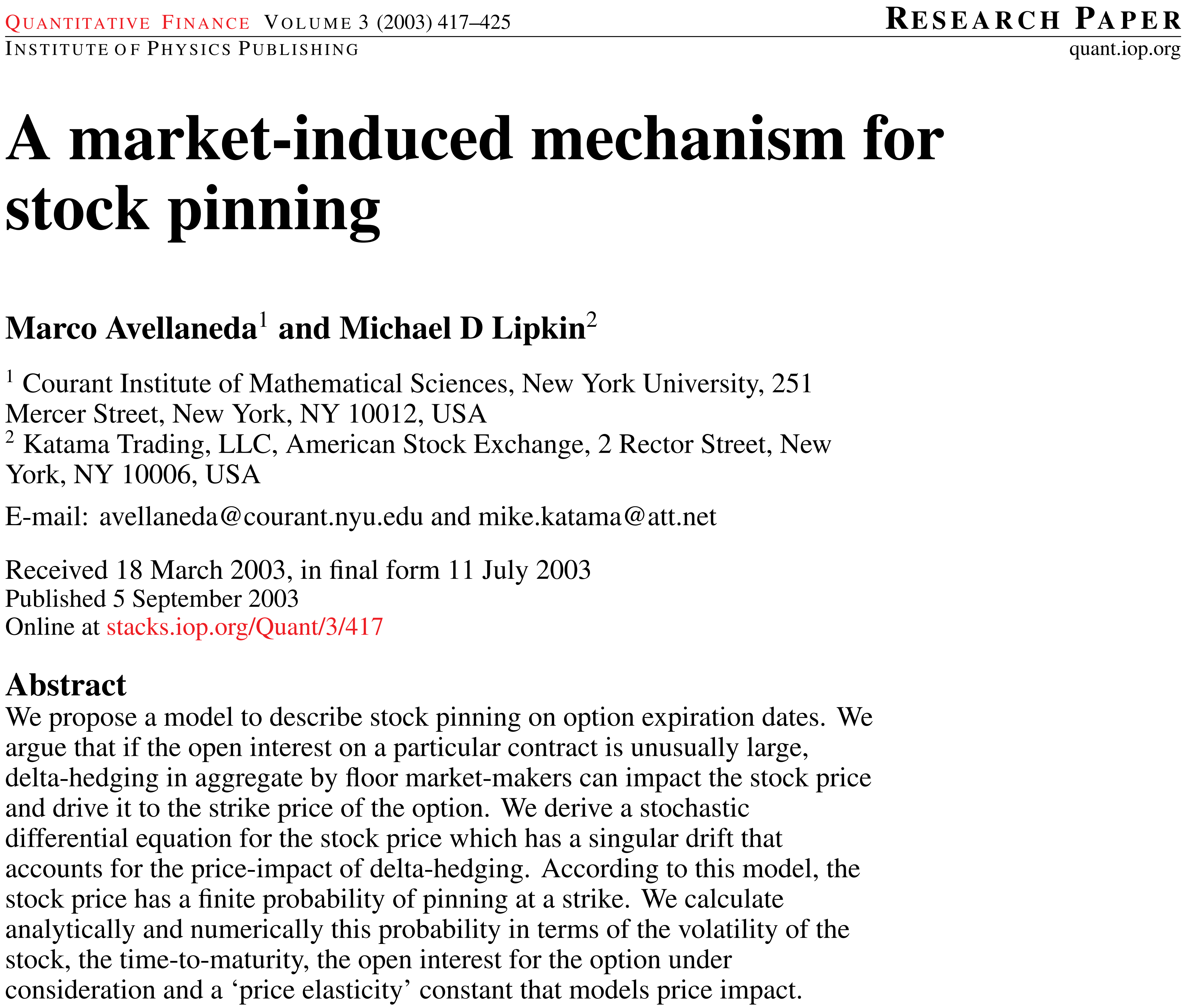
Source: Crédit Agricole Cheuvreux Quantitative Research

For a very large open interest position encompassing a large gamma, a significant move in the stock price will have disastrous effects for a basic rudimental hedger such as the one described above.

- Repetitive delta hedging seems to be the most plausible explanation

Simplistic hedging of large gamma options is a plausible explanation for the "saw-tooth" trading pattern. This explanation is consistent with the main features of this phenomenon: timing, aggressiveness, impact, predictability and information leakage which is what characterises those "saw-tooth" patterns. Fortunately, large option positions are most often managed dynamically in a continuous way, and discrete archaic hedging processes have almost disappeared in modern-day markets.

(we show how to do this better)



QUANTITATIVE FINANCE VOLUME 3 (2003) 417–425 RESEARCH PAPER
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A market-induced mechanism for stock pinning

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Abstract

We propose a model to describe stock pinning on option expiration dates. We argue that if the open interest on a particular contract is unusually large, delta-hedging in aggregate by floor market-makers can impact the stock price and drive it to the strike price of the option. We derive a stochastic differential equation for the stock price which has a singular drift that accounts for the price-impact of delta-hedging. According to this model, the stock price has a finite probability of pinning at a strike. We calculate analytically and numerically this probability in terms of the volatility of the stock, the time-to-maturity, the open interest for the option under consideration and a ‘price elasticity’ constant that models price impact.

Avellaneda & Lipkin 2003

Linear permanent market impact

$$\frac{\Delta S}{S} = EQ, \qquad (1)$$

where Q represents the number of shares traded, S is the stock price, ΔS is the change in stock price associated with a trade of size Q and E is a stock-specific proportionality constant

Solution near expiration ⇒ pinning at strike

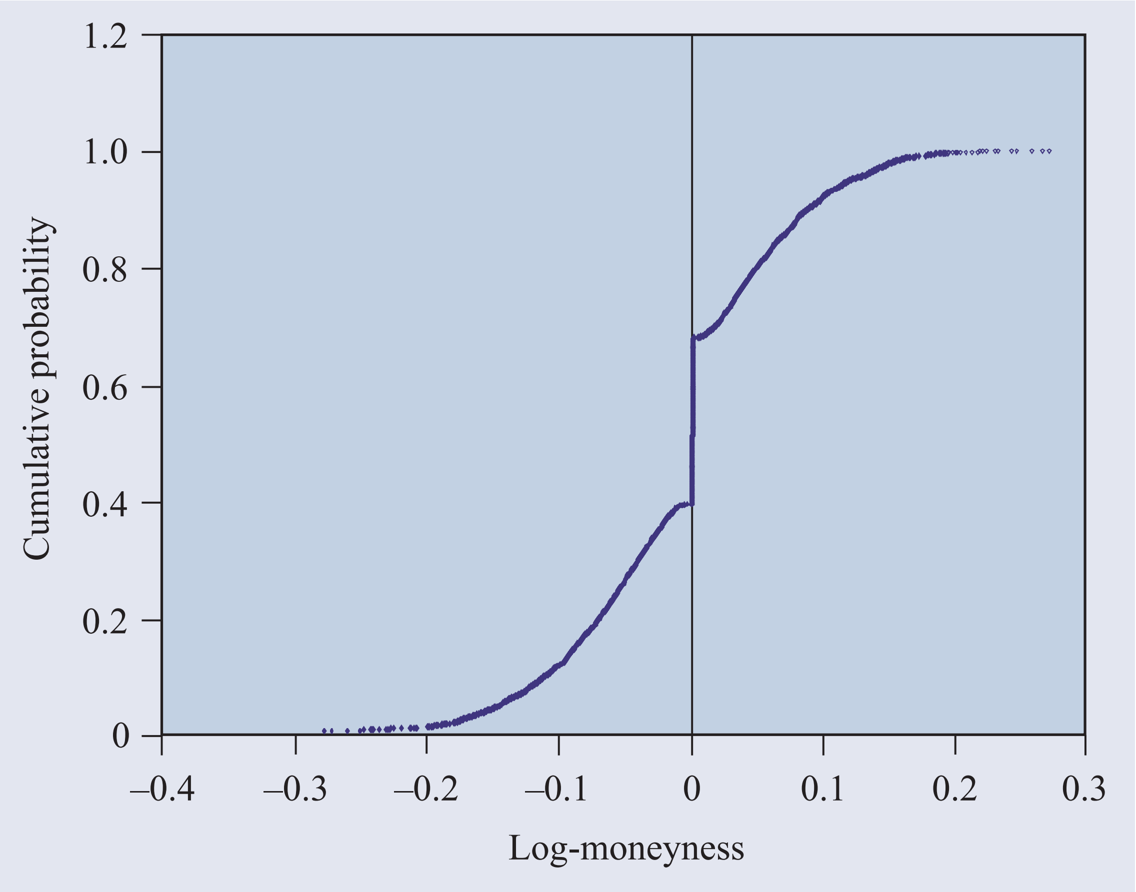

Figure 6. Cumulative probability distribution function computed by Monte Carlo simulation. The step corresponds to the fact that a finite fraction of the paths is pinned at the strike.

Quantitative Finance, Vol. 8, No. 8, December 2008, 823–831

Modeling stock pinning

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This paper investigates the effect of hedging strategies on the so-called pinning effect, i.e. the tendency of stock's prices to close near the strike price of heavily traded options as the expiration date nears. In the paper we extend the analysis of Avellaneda and Lipkin, who propose an explanation of stock pinning in terms of delta hedging strategies for long option positions. We adopt a model introduced by Frey and Stremme and show that, under the original assumptions of the model, pinning is driven by two effects: a hedging-dependent drift term that pushes the stock price toward the strike price and a hedging-dependent volatility term that constrains the stock price near the strike as it approaches it. Finally, we show that pinning can be generated by simulating trading in a double auction market. Pinning in the microstructure model is consistent with the Frey and Stremme model when both discrete hedging and stochastic impact are taken into account.

Jeannin, Iori, & Samuel 2008

$$\mathrm{d}S(t) = \underline{n\hat{L}S(t)\frac{\partial\Delta}{\partial S}\,\mathrm{d}S(t)} + n\hat{L}S(t)\left(\frac{\partial\Delta}{\partial t}\,\mathrm{d}t + \underline{\frac{1}{2}\frac{\partial^2\Delta}{\partial S^2}\,\mathrm{d}\langle S(t)\rangle}\right)$$
$$+\,\sigma S(t)\,\mathrm{d}W(t),$$

added relative to Avellaneda & Lipkin

The model further assumes that traders do not take into account feedback effects when rebalancing their portfolio.

Thus the stock price still follows a diffusion process,

$$\mathrm{d}S(t) = b(t, S(t))S(t)\,\mathrm{d}t + v(t, S(t))S(t)\,\mathrm{d}W(t), \qquad (6)$$

but with a new drift and volatility given by

$$b(t, S(t)) = \frac{n\hat{L}}{1 - n\hat{L}S(t)(\partial\Delta/\partial S)} \times \left\{\frac{\partial\Delta}{\partial t} + \frac{1}{2}\frac{\partial^2\Delta}{\partial S^2}\frac{\sigma^2 S^2(t)}{\left[1 - n\hat{L}S(t)(\partial\Delta/\partial S)\right]^2}\right\},$$

and

$$\boxed{v(t, S(t)) = \frac{\sigma}{1 - n\hat{L}S(t)(\partial\Delta/\partial S)}.}$$

modified volatility

Other academic work

Frey & Stremme 1997
Sircar & Papanicolaou 1998
Schönbucher & Wilmott 2000

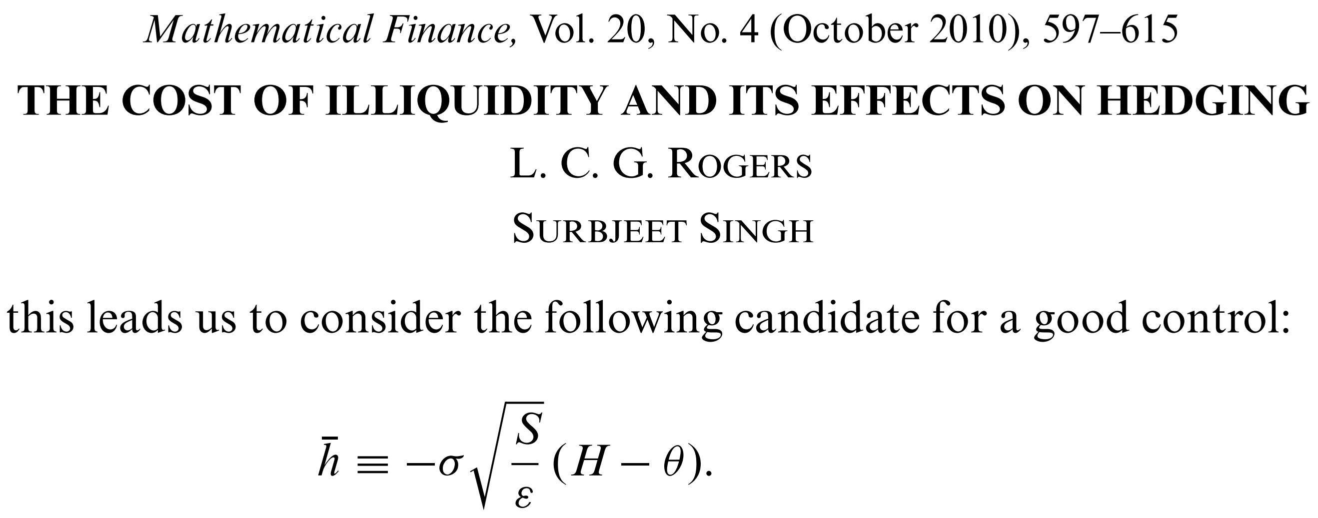
- ad hoc impact model (permanent)
- modified volatility

Closest related work

Rogers & Singh 2010

temporary impact only

no global effects



Mathematical Finance, Vol. 20, No. 4 (October 2010), 597–615

THE COST OF ILLIQUIDITY AND ITS EFFECTS ON HEDGING

L. C. G. ROGERS

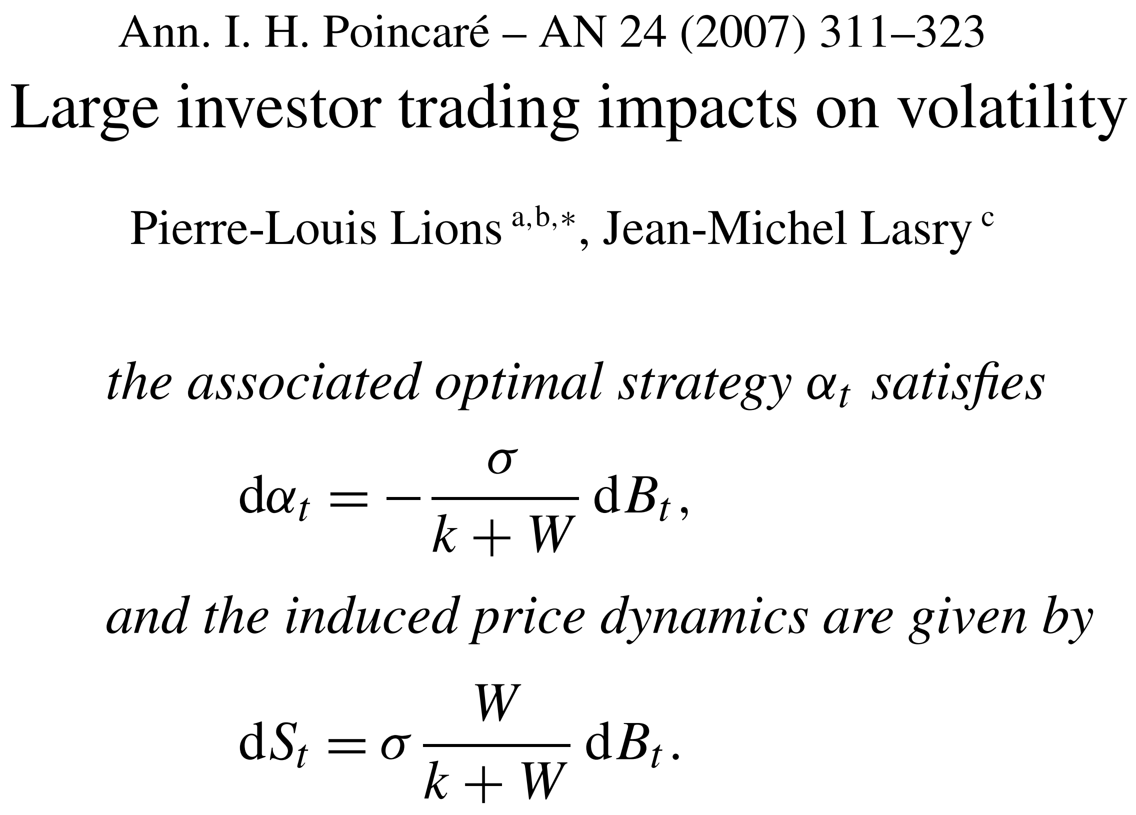
SURBJEET SINGH

this leads us to consider the following candidate for a good control:

$$\bar{h} \equiv -\sigma\sqrt{\frac{S}{\varepsilon}}\,(H-\theta).$$

Lions & Lasry 2006,2007

permanent impact only



Ann. I. H. Poincaré – AN 24 (2007) 311–323

Large investor trading impacts on volatility

Pierre-Louis Lions [a,b,*], Jean-Michel Lasry [c]

the associated optimal strategy α_t *satisfies*

$$\mathrm{d}\alpha_t = -\frac{\sigma}{k+W}\,\mathrm{d}B_t,$$

and the induced price dynamics are given by

$$\mathrm{d}S_t = \sigma\frac{W}{k+W}\,\mathrm{d}B_t.$$

FRBNY ECONOMIC POLICY REVIEW / JUNE 1998 35

Dealers' Hedging of Interest Rate Options in the U.S. Dollar Fixed-Income Market

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John E. Kambhu

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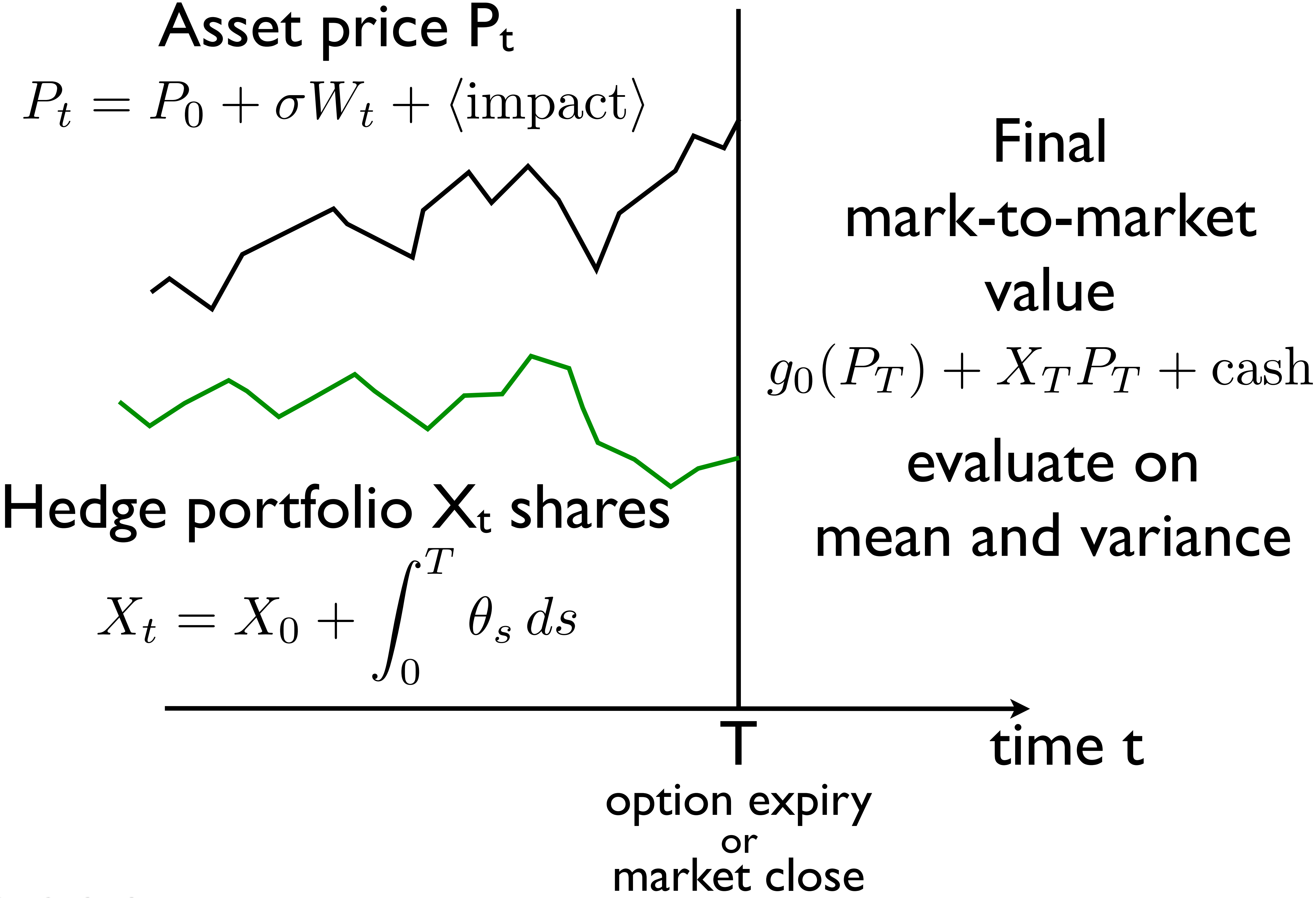
we address two questions: First, are dealers' hedge adjustments large enough to affect trading volume and liquidity in the most common hedging instruments? Second, what effects might potential hedging difficulties have on risk premia in options prices and the structure of the market for over-the-counter interest rate options?

CONCLUSION

Our analysis suggests that transaction volume in underlying markets is large enough for dealers to manage the price and liquidity risks they incur through the intermediation of price risk in selling interest rate options.

2. Option hedging (version 1)



Asset price P_t

$$P_t = P_0 + \sigma W_t + \langle \text{impact} \rangle$$

Hedge portfolio X_t shares

$$X_t = X_0 + \int_0^T \theta_s \, ds$$

Final mark-to-market value

$$g_0(P_T) + X_T P_T + \text{cash}$$

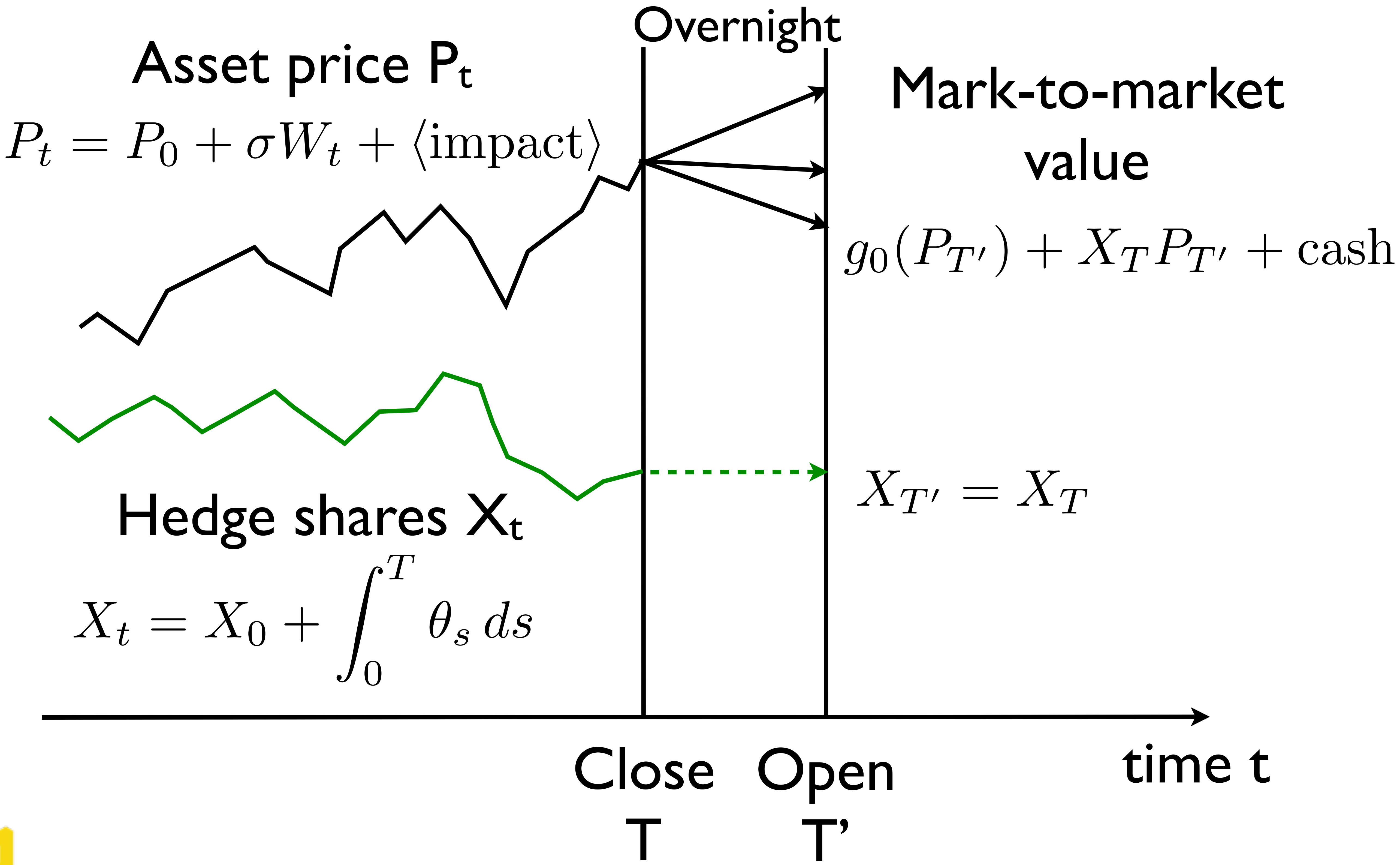
evaluate on mean and variance

T

time t

option expiry
or
market close

Option hedging (version 2)



Overnight

Asset price P_t

$$P_t = P_0 + \sigma W_t + \langle \text{impact} \rangle$$

Mark-to-market value

$$g_0(P_{T'}) + X_T P_{T'} + \text{cash}$$

$$X_{T'} = X_T$$

Hedge shares X_t

$$X_t = X_0 + \int_0^T \theta_s \, ds$$

Close T

Open T'

time t

Questions of this paper

1. What is a reasonable market model?
2. What are optimal hedge solutions?
3. How do they compare to Black-Scholes?

Applications

1. Broker execution algorithm:

   Client specifies Δ and Γ (possibly varying)
   Execute to achieve optimal hedge at close
   one direction trading (buy or sell)

2. What does option hedging do to market?

   Seller must hedge
   What does his hedging do to price process?

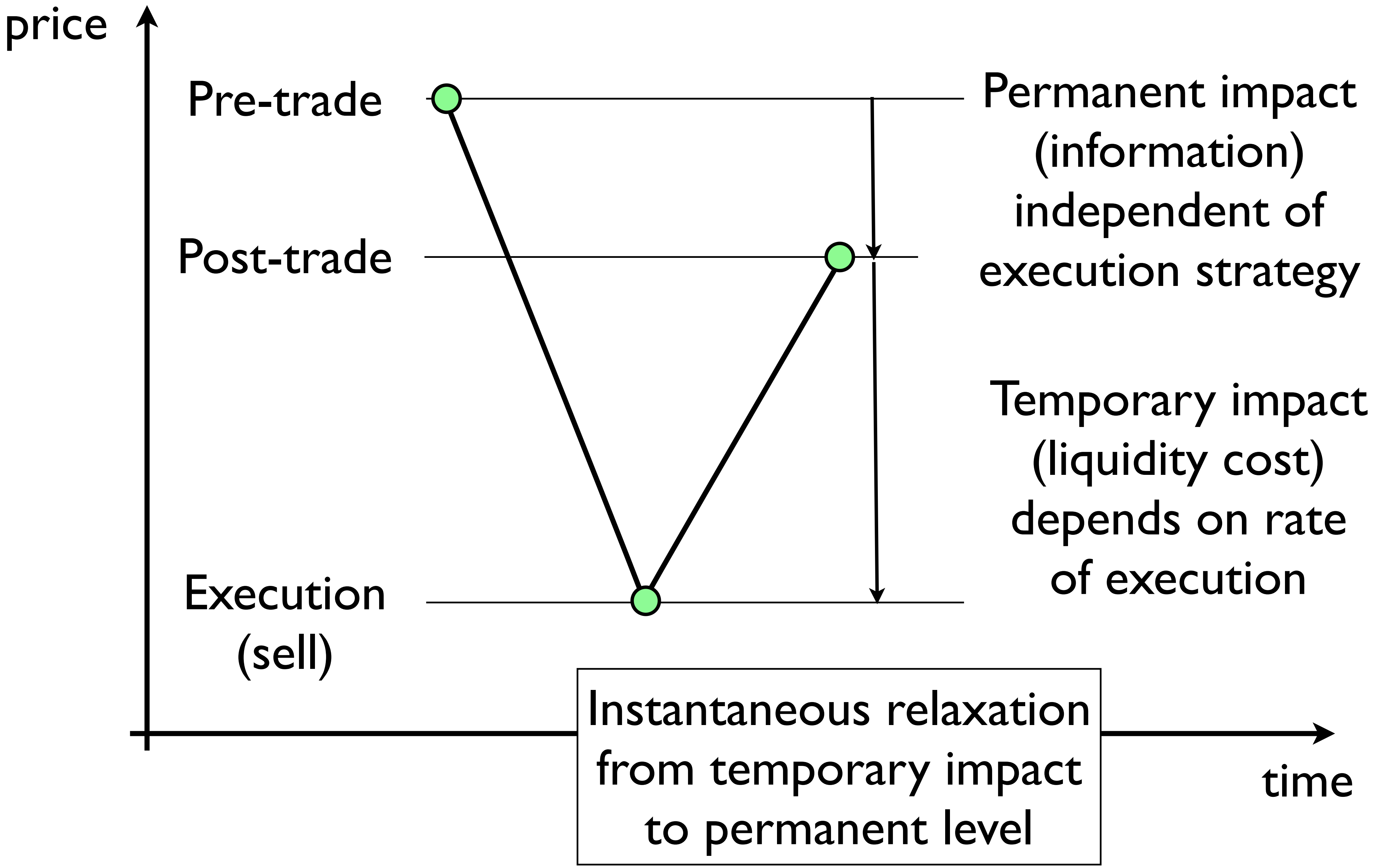
Market impact models

Two types of market impact
(both active, both important):

- Permanent
  due to information transmission
  affects public market price
- Temporary
  due to finite instantaneous liquidity
  "private" execution price not reflected in market

Many richer structures are possible

Temporary vs. permanent market impact



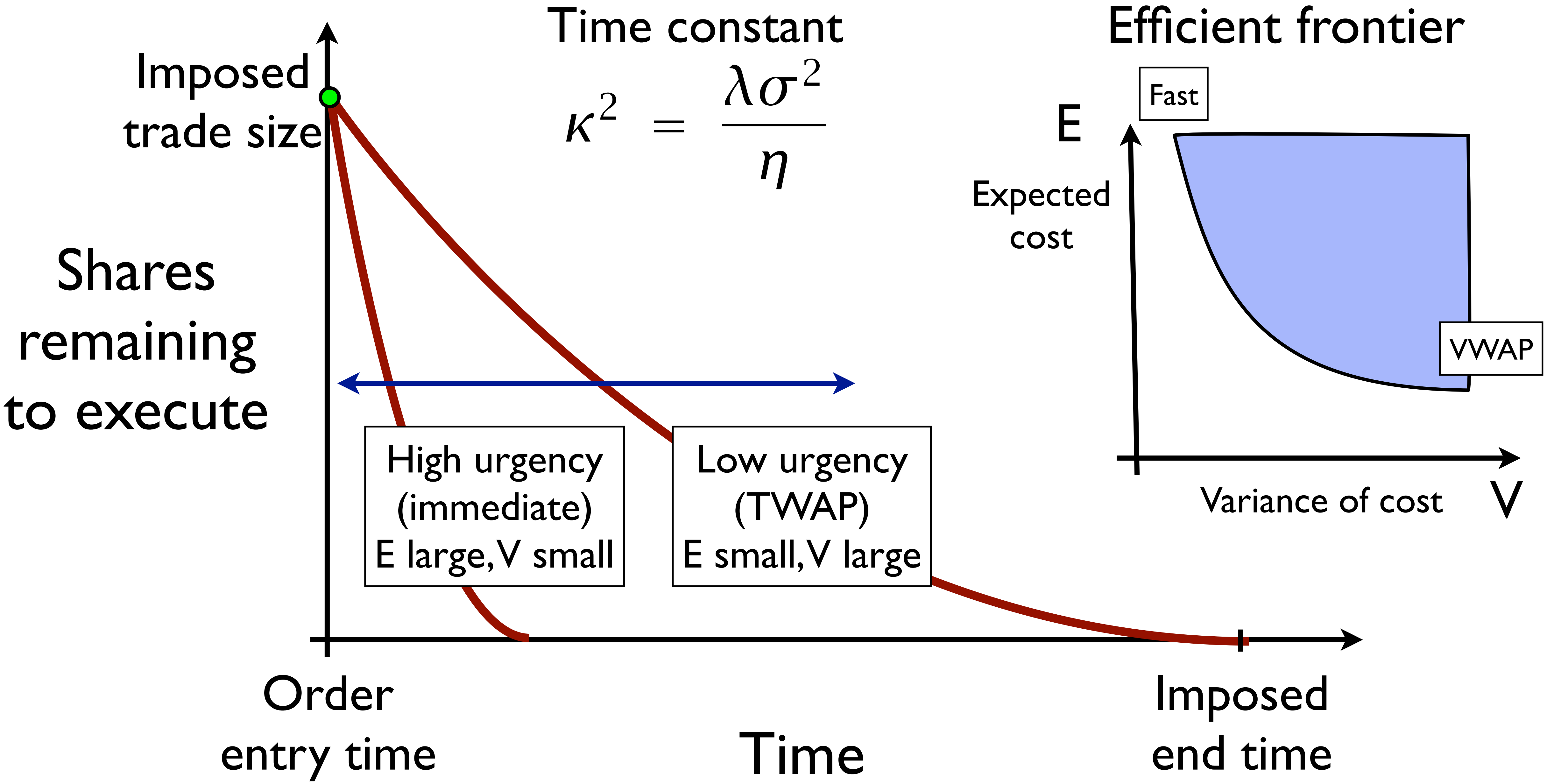
Permanent impact (information) independent of execution strategy

Temporary impact (liquidity cost) depends on rate of execution

Instantaneous relaxation from temporary impact to permanent level

Jim Gatheral: richer time structures for decay

Large literature on market impact models: optimal execution of given trade program



Time constant

$$\kappa^2 = \frac{\lambda\sigma^2}{\eta}$$

Permanent impact

$$X_t = X_0 + \int_0^T \theta_s \, ds$$

$$\theta_t \; = \; \text{instantaneous } \textit{rate} \text{ of trading}$$

$$dP_t \; = \; \sigma \, dW_t \; + \; G(\theta_t) \, dt$$

Linear to avoid round-trip arbitrage (Huberman & Stanzl, Gatheral)

(Schönbucher & Wilmott 2000: knock-out option--also need temporary impact)

$$G(\theta) \; = \; \nu \, \theta$$

$$\boxed{P_t \; = \; P_0 \; + \; \sigma \, W_t \; + \; \nu (X_t - X_0)}$$

(independent of path)

Cost to execute net X shares = $\frac{1}{2} \nu X^2$

Temporary impact

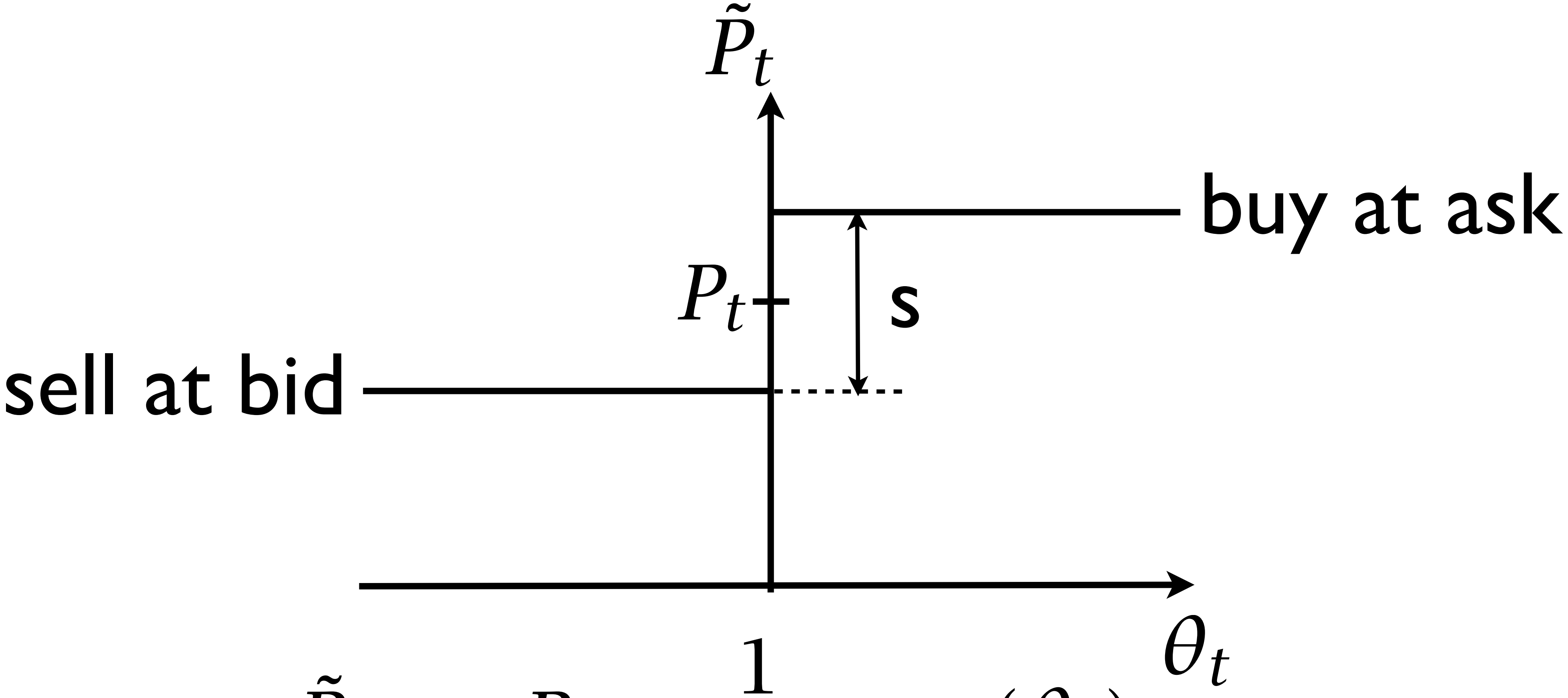
We trade at $\tilde{P}_t \neq P_t$

$\tilde{P}_t$ depends on instantaneous trade rate θ_t

$$\tilde{P}_t = P_t + H(\theta_t)$$

Require finite instantaneous trade rate
⇒ imperfect hedging

Example: bid-ask spread



$$\tilde{P}_t = P_t + \frac{1}{2} s \operatorname{sgn}(\theta_t)$$

"Linear" model: cost to trade $\theta_t \Delta t$ shares

$$\frac{1}{2} s \operatorname{sgn}(\theta_t) \cdot \theta_t \Delta t = \frac{1}{2} s |\theta_t| \Delta t$$

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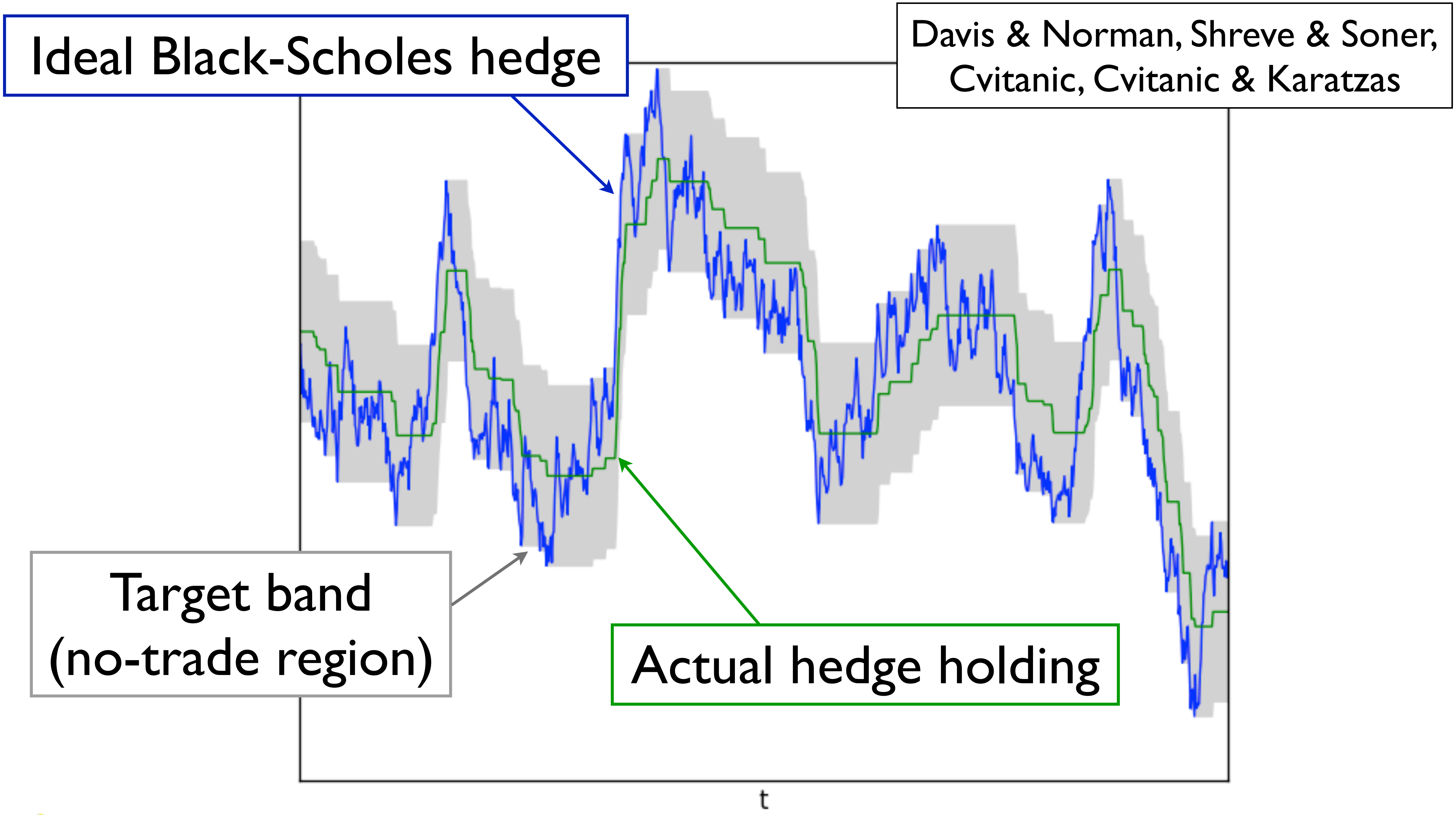
Solutions with bid-ask spread cost



Ideal Black-Scholes hedge

Davis & Norman, Shreve & Soner, Cvitanic, Cvitanic & Karatzas

Target band (no-trade region)

Actual hedge holding

Critique of linear cost model

independent of trade size

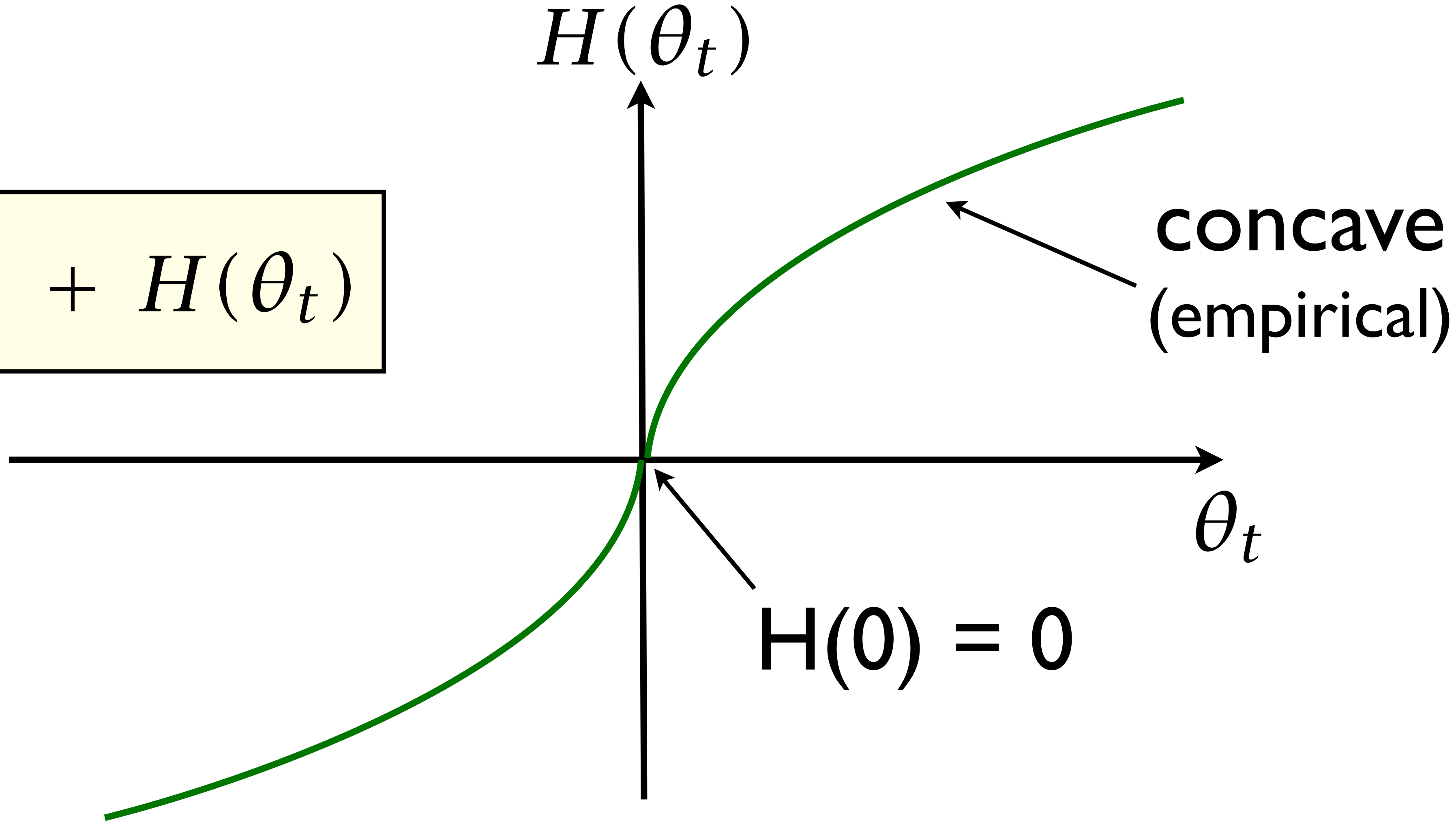
- not suitable for large traders

in practice, effective execution near midpoint

- spread cost not consistent with modern cost models
- liquidity takers act as liquidity providers

Proportional temporary cost model

$$\boxed{\tilde{P}_t \;=\; P_t \;+\; H(\theta_t)}$$



Linear for simplicity $H(\theta) \;=\; \frac{1}{2}\lambda\,\theta$

$\Longrightarrow$ Quadratic cost: $H(\theta)\cdot\theta\,\Delta t \;=\; \frac{1}{2}\lambda\,\theta^2\,\Delta t$

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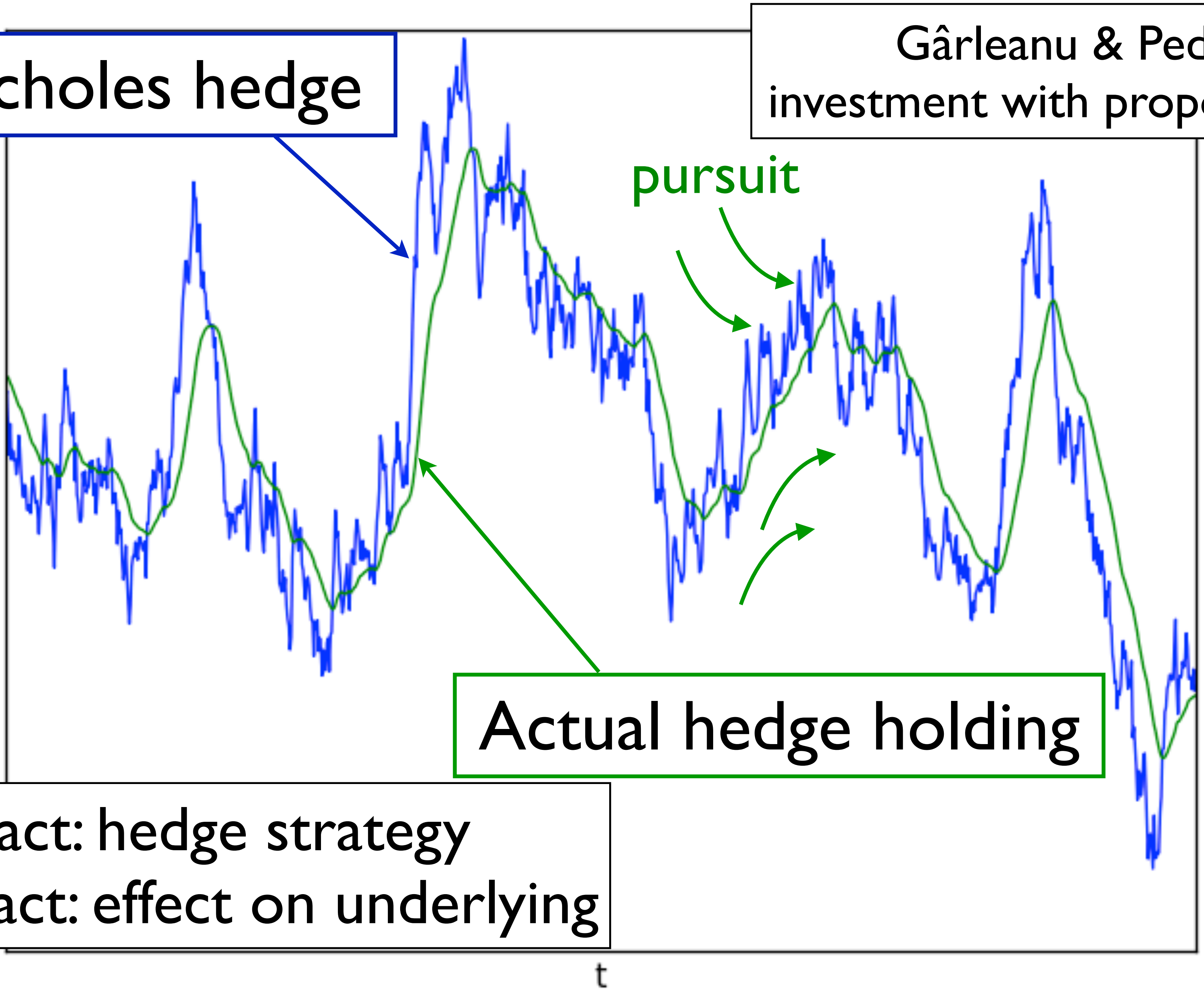
Our solutions with proportional cost



Ideal Black-Scholes hedge

Gârleanu & Pedersen:
investment with proportional cost

pursuit

Actual hedge holding

Temporary impact: hedge strategy
Permanent impact: effect on underlying

$$\theta_t = -\kappa\, h(\kappa(T-t)) \cdot (X_t - \text{target})$$

3. Formulation

Market model

Hedge holding: $X_t = X_0 + \int_0^T \theta_s \, ds$

Public market price: $P_t = P_0 + \sigma W_t + \nu (X_t - X_0)$

Private trade price: $\tilde{P}_t = P_t + \frac{1}{2} \lambda \theta_t$

$$\mathcal{F}_t = \text{filtration of } W_t$$

strategies measurable in $\mathcal{F}_t$

Black-Scholes option value

$g(T, p) = g_0(p)$ Final value specified

Intermediate values defined by Black-Scholes PDE

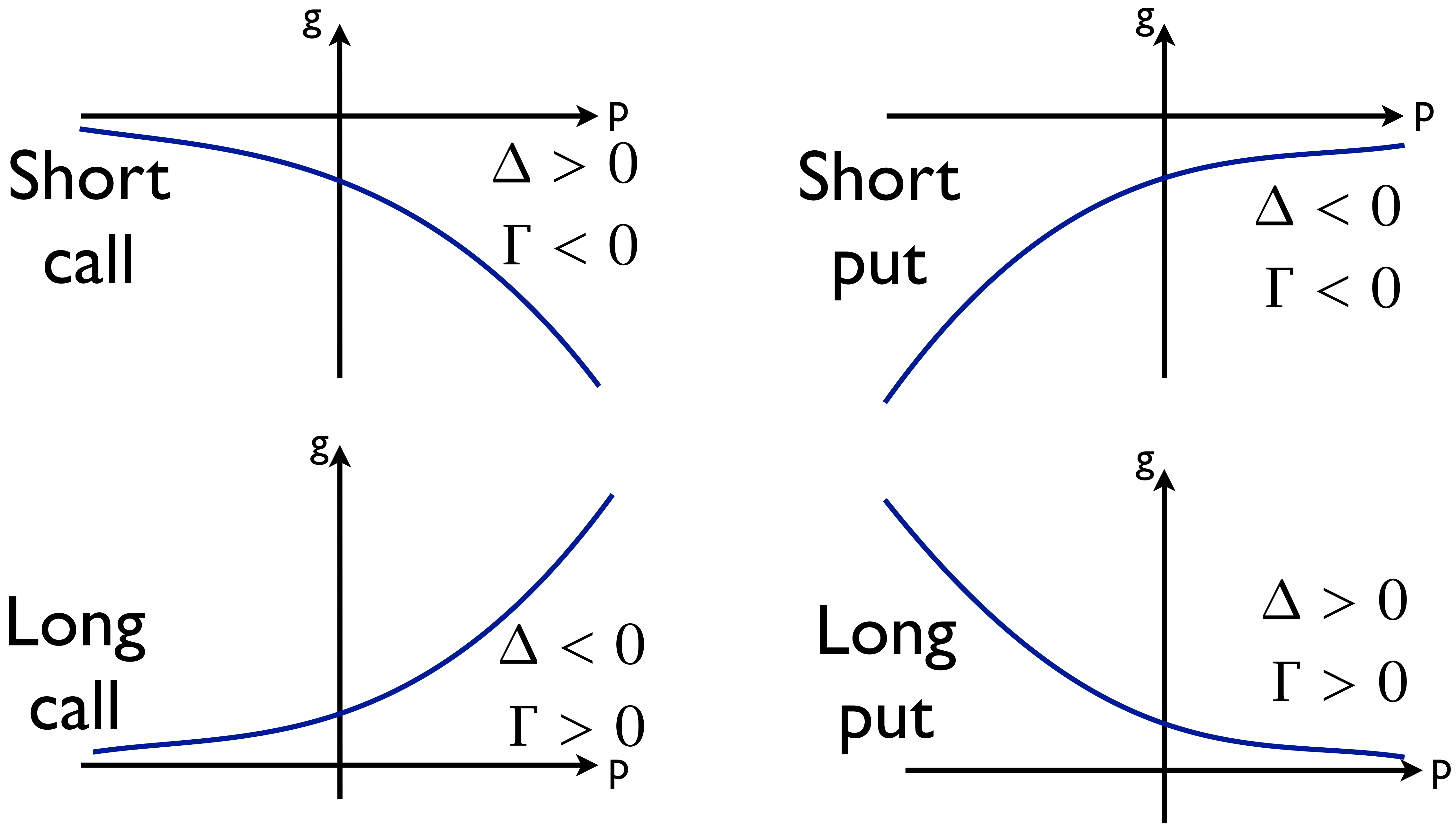
Def: $g(t, p), \qquad t < T, p \in \mathbb{R}$

$$\dot{g} + \frac{1}{2}\sigma^2 g'' = 0$$

Def:
$$\begin{aligned} \Delta(t, p) &= -g'(t, p) \quad \text{(what you want to hold to hedge)} \\ \Gamma(t, p) &= -\Delta'(t, p) = g''(t, p) \end{aligned}$$

g(t,p) = option payout to position holder

$$\Delta(t,p) = -g'(t,p) \qquad \Gamma(t,p) = -\Delta'(t,p) = g''(t,p)$$



Short call

$\Delta > 0$

$\Gamma < 0$

Short put

$\Delta < 0$

$\Gamma < 0$

Long call

$\Delta < 0$

$\Gamma > 0$

Long put

$\Delta > 0$

$\Gamma > 0$

Γ = sign and size of hedger’s option position

$\Gamma>0$ (long the option)

- g’ increasing in P
- Δ decreasing in P
- Permanent price impact pushes P *toward* you
- Hedging is *easy* (unless you over-control)

$\Gamma<0$ (short the option)

- g’ decreasing in P
- Δ increasing in P
- Permanent price impact pushes P *away* from you
- Hedging is *hard*

Final portfolio value

$$R_T \;=\; g(T, P_T) \;+\; X_T P_T \;-\; \int_0^T \tilde{P}_t\, \theta_t\, dt$$

Option value Portfolio value Cash spent

Mark to market
without transaction costs

Should include permanent
impact in liquidation cost: $-\frac{1}{2}\nu X^2$

We neglect: gives manipulation opportunities
dominated by risk aversion

Integrate by parts

$$\begin{aligned} R_T &= R_0 + \int_0^T (X_t + g'(t,P_t))\,dP_t - \frac{1}{2}\lambda\int_0^T \theta_t^2\,dt \\ &= R_0 + \int_0^T Y_t\,dP_t - \frac{1}{2}\lambda\int_0^T \theta_t^2\,dt \\ &= R_0 + \int_0^T Y_t\,\sigma\,dW_t + \underbrace{\int_0^T Y_t\,\nu\theta_t\,dt}_{\text{positive when } Y_t,\,\theta_t \text{ same sign}} - \frac{1}{2}\lambda\int_0^T \theta_t^2\,dt \end{aligned}$$

Initial value: $R_0 = g(0,P_0) + X_0 P_0$ (constant)

Mis-hedge: $Y_t = X_t - \Delta(t,P_t) = X_t + g'(t,P_t)$

Mean-variance evaluation

$$\boxed{R_T \;=\; R_0 \;+\; \int_0^T Y_t\,\sigma\,dW_t \;+\; \nu \int_0^T Y_t\,\theta_t\,dt \;-\; \frac{1}{2}\lambda \int_0^T \theta_t^2\,dt}$$

mis-hedge $$Y_t \;=\; X_t \;+\; g'(t,P_t) \;\neq\; 0$$

smooth

infinite variation

R_T is random: optimize expectation and variance

$$\mathbb{E}R_T \;=\; R_0 \;+\; \nu\,\mathbb{E}\int_0^T Y_t\,\theta_t\,dt \;-\; \frac{1}{2}\lambda\,\mathbb{E}\int_0^T \theta_t^2\,dt$$

$$\mathrm{Var}\,R_T \;=\; \text{complicated}$$

Variance of R_T

Neglect uncertainty of market impact term in comparison with price uncertainty

$$\operatorname{Var} R_T \;\approx\; V \;\equiv\; \operatorname{Var} \int_0^T \sigma Y_t \, dW_t \;=\; \sigma^2 \mathbb{E} \int_0^T Y_t^2 \, dt$$

Small portfolio size, or "market power"
(Almgren/Lorenz 2007, Almgren 2012)

$$\mu \;=\; \frac{\lambda X / T}{\sigma \sqrt{T}}$$

price impact of trading whole position

price change from volatility

("Mean-quadratic-variation" Forsyth et al 2012)

Mean-"variance" objective

Risk aversion $\frac{1}{2}\gamma$

$$\inf_{\theta\in\Theta}\mathbb{E}\left[\frac{1}{2}\gamma\sigma^2\int_0^T Y_t^2\,dt-\nu\int_0^T Y_t\,\theta_t\,dt+\frac{1}{2}\lambda\int_0^T\theta_t^2dt\right]$$

Running mishedge

Permanent impact

Temporary impact

(T = option expiration)

Version 2: Overnight risk

T = market close today
T’ = market open tomorrow

$$
\begin{aligned}
R_{T'} &= g(T', P_{T'}) \; + \; X_T P_{T'} \; - \int_0^T \tilde{P}_t \, \theta_t \, dt \\
&= R_T + Y_T (P_{T'} - P_T) + \int_T^{T'} \Big[g'(t, P_t) - g'(T, P_T) \Big] \, dP_t \\
&= R_T \; + \; Y_T \, \Delta P_T \; - \; \xi
\end{aligned}
$$

$\Delta P_T, \xi$ have mean zero

Version 2 objective function

Random variables

ξ distribution depends only on P_T

ΔP_T mean 0, independent of $\mathcal{F}_T$

$$\inf_{\theta\in\Theta}\mathbb{E}\Big[\frac{1}{2}\gamma\Big(Y_T\,\Delta P_T-\xi\Big)^2 \xleftarrow{} $$

Terminal mishedge

$$+\frac{1}{2}\gamma\sigma^2\int_0^T Y_t^2\,dt-\nu\int_0^T Y_t\,\theta_t\,dt+\frac{1}{2}\lambda\int_0^T\theta_t^2 dt\Big]$$

4. Solution

Value function: time t, price p, mis-hedge y

$$
\begin{aligned}
J(t,p,y) \;=\; \inf_{\theta_s : t \le s \le T} \mathbb{E}\Bigg[& \frac{1}{2}\gamma\Big(Y_T\,\Delta P_T - \xi\Big)^2 \\
& + \frac{1}{2}\gamma\sigma^2\int_t^T Y_s^2\,ds - \nu\int_t^T Y_s\,\theta_s\,ds + \frac{1}{2}\lambda\int_t^T \theta_s^2 ds \\
& \Bigg|\; P_t = p,\; Y_t = y \;\Bigg]
\end{aligned}
$$

Dynamic programming

HJB PDE:

$$
\begin{aligned}
0 \;&=\; \inf_{\theta} \left\{ \frac{1}{2}\gamma\sigma^2 y^2 \;-\; y\nu\theta \;+\; \frac{1}{2}\lambda\theta^2 \;+\; J_t \;+\; (1+\nu\Gamma)\theta J_y \;+\; \nu\theta J_p \right\} \\
&\qquad +\; \frac{1}{2}\sigma^2 J_{pp} \;+\; \sigma^2\Gamma J_{py} \;+\; \frac{1}{2}\sigma^2\Gamma^2 J_{yy} \\
&=\; \frac{1}{2}\gamma\sigma^2 y^2 \;-\; \frac{1}{2\lambda}\Big[\, \nu(y - J_p) - (1+\nu\Gamma)J_y \,\Big]^2 \\
&\qquad +\; J_t \;+\; \frac{1}{2}\sigma^2 J_{pp} \;+\; \sigma^2\Gamma J_{py} \;+\; \frac{1}{2}\sigma^2\Gamma^2 J_{yy}
\end{aligned}
$$

optimal control

$$
\theta \;=\; \frac{1}{\lambda}\Big(\nu y \;-\; (1+\nu\Gamma)J_y \;-\; \nu J_p\Big)
$$

Ansatz: quadratic in y

$$J(t,p,y) \;=\; \frac{1}{2} A_2(T-t)\, y^2 \;+\; A_1(T-t,p)\, y \;+\; A_0(T-t,p)$$

A₂ independent of price

not consistent unless A₁ constant in p

$$\dot{A}_2 = \gamma\sigma^2 \;-\; \frac{1}{\lambda}\Big[\nu\big(1-A_1'\big) - (1+\nu\Gamma)A_2\Big]^2$$

$$\dot{A}_1 = \frac{\sigma^2}{2} A_1'' + \frac{1}{\lambda}\Big[\nu\big(1-A_1'\big) - (1+\nu\Gamma)A_2\Big] \cdot \Big[\nu A_0' + (1+\nu\Gamma)A_1\Big]$$

$$\dot{A}_0 = -\frac{1}{2\lambda}\Big[\nu A_0' + (1+\nu\Gamma)A_1\Big]^2 + \sigma^2\Gamma A_1' + \frac{\sigma^2}{2}\Big[\Gamma^2 A_2 + A_0''\Big]$$

Solvable in 2 special cases

(A) Constant gamma

$$g'(t, P_t) = g'(t, P_0) + \Gamma \left(P_t - P_0 \right)$$

Γ measures position size and size

(B) No permanent impact

$$\nu = 0$$

(no feedback)

(A) Constant Γ

$$A_1 = 0, \qquad A_0(T-t,p) = A_0(T_t)$$

$$\boxed{\theta_t = -\underbrace{\kappa}_{\text{rate coefficient}}\, \underbrace{h\Big(\kappa(1+\nu\Gamma)(T-t)\Big)}_{\text{function of time remaining}}\, \underbrace{Y_t}_{\text{Instantaneous mishedge}}}$$

time constant $\kappa = \sqrt{\dfrac{\gamma\sigma^2}{\lambda}}$ risk / temporary impact

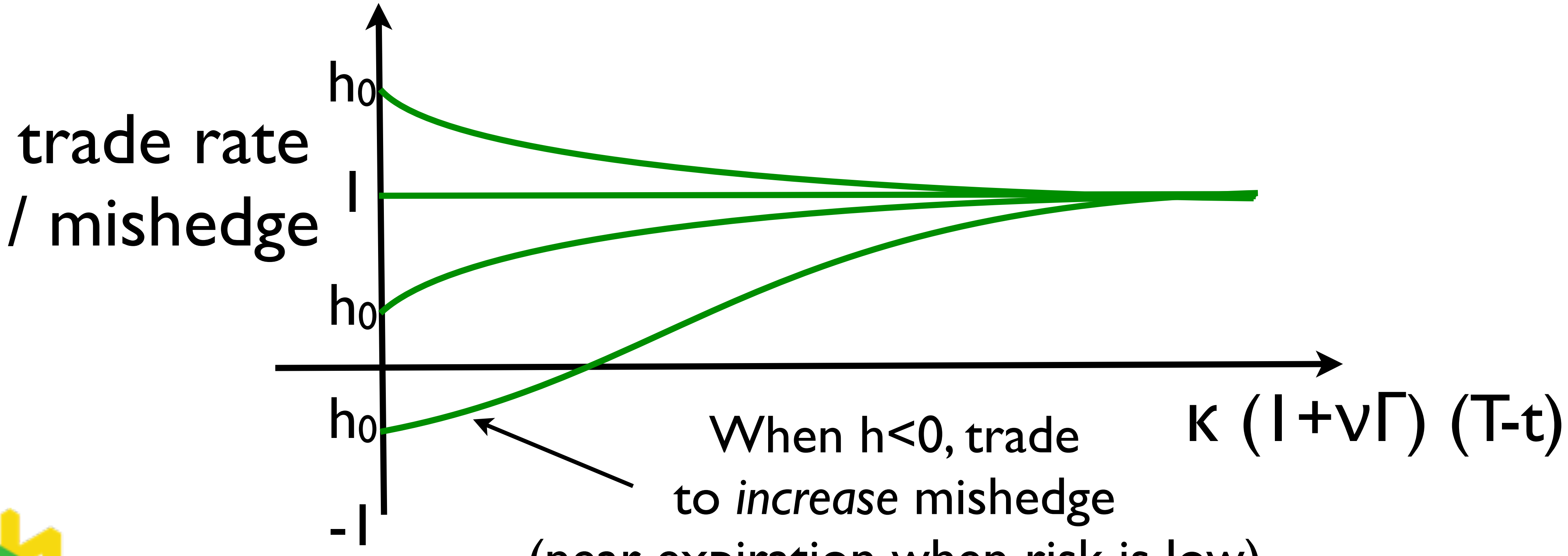
same as for optimal execution

$$G = 1 + \nu\Gamma \qquad h_0 = \frac{G\gamma\sigma_T^2 - \nu}{\lambda\kappa}$$

Need G ≥ 0 and h_0 ≥ -1

Permanent impact ν not too big

$$h(x) = \begin{cases} \tanh(x + \tanh^{-1}(h_0)), & -1 < h_0 < 1 \\ 1, & h_0 = 1 \\ \coth(x + \coth^{-1}(h_0)) & h_0 > 1 \end{cases}$$



When h<0, trade to *increase* mishedge (near expiration when risk is low)

Summary of hedge strategy

Far from expiration, h=1 $\theta_t = -\kappa Y_t$

Near expiration $\theta_t = -\kappa h Y_t$

h increases if overnight risk large
h decreases if overnight risk small
h becomes negative (!) if no overnight risk

What happens to price process?

constant Γ

$$\Delta(t, P_t) \;=\; \Delta_0 - \Gamma(P_t - P_0)$$

$$\begin{aligned} Y_t &= X_t - \Delta \\ &= X_t - X_0 + \Gamma(P_t - P_0) \end{aligned}$$

dynamic hedge $\theta_t = -\kappa\, Y_t \quad (h = 1)$

$$\begin{aligned} dX_t &= \theta_t\, dt \\ dP_t &= \sigma\, dW_t \;+\; \nu\, \theta_t\, dt \end{aligned}$$

Mis-hedge
$$\begin{aligned} dY_t &= dX_t + \Gamma\, dP_t \\ &= -\kappa(1+\nu\Gamma)\, Y_t\, dt + \sigma\,\Gamma\, dW_t \end{aligned}$$

$$Y_t = \sigma\Gamma \int_{-\infty}^{t} e^{-\kappa(1+\nu\Gamma)(t-s)}\, dW_s$$

$$\langle {Y_t}^2 \rangle = \frac{\sigma^2\Gamma^2}{2\kappa(1+\nu\Gamma)} = \frac{\sigma^2\Gamma^2}{2(1+\nu\Gamma)}\sqrt{\frac{\lambda}{\gamma\sigma^2}}$$

$$\text{Mean mis-hedge} \;\propto\; (\text{temporary impact } \lambda)^{1/4}$$

$$\text{Total liquidity cost} = \int_0^T \lambda\, \theta_t^2\, dt = \int_0^T \lambda\, \kappa^2\, Y_t^2\, dt$$

$$\sim \lambda\, \kappa^2\, T\, \frac{\sigma^2\Gamma^2}{\kappa} \sim T\, \sigma^3\, \Gamma^2\, \gamma^{1/2}\, \lambda^{1/2}$$

As λ→0, perfect hedge, cost→0

What happens to price process?

def:
$$Z_t = -\nu(X_t - X_0) + (P_t - P_0)$$

$$dZ_t = \sigma\, dW_t \longleftarrow \text{same as in } P_t$$

$$\begin{aligned} P_t &= P_0 + \frac{1}{1+\nu\Gamma}(\nu Y_t + Z_t) \\ &= P_0 + \frac{\sigma}{1+\nu\Gamma}\left(W_t + \nu\Gamma \int_0^t e^{-\kappa(1+\nu\Gamma)(t-s)} dW_s \right) \end{aligned}$$

modified volatility

stationary $\propto 1/\sqrt{\kappa}$

Γ<0: hedger is short the option

1+νΓ<1: overreaction, increased volatility

Γ>0: hedger is long the option

1+νΓ>1: underreaction, reduced volatility

“Signature plot”

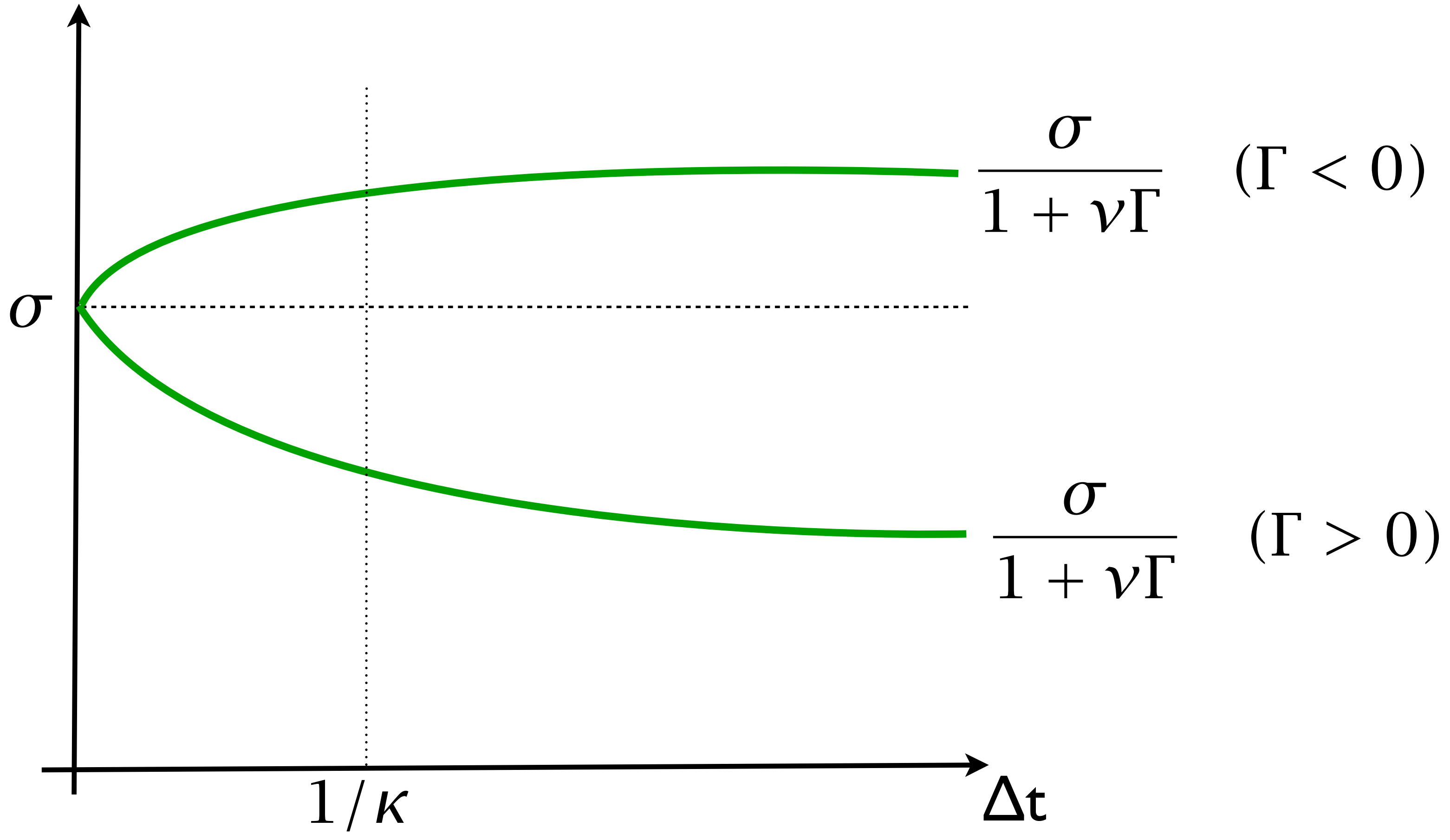
$$\tilde{\kappa} = (1 + \nu\Gamma)\kappa$$

unconditional ($\mathcal{F}_0$)

$$\frac{1}{s-t}\mathbb{E}(P_s - P_t)^2 =$$

$$= \left(\frac{\sigma}{1+\nu\Gamma}\right)^2 \left(1 + (2+\nu\Gamma)\nu\Gamma\,\frac{1-e^{-\tilde{\kappa}(s-t)}}{\tilde{\kappa}(s-t)}\right)$$

Volatility measured on time interval Δt



Nondimensional parameter

$$1 + \nu\Gamma$$

$$\nu = \frac{\text{price change due to market impact}}{\text{shares executed}}$$

$$\Gamma = \frac{\text{shares needed to adjust hedge}}{\text{market price change}}$$

Problems unless $|\nu\Gamma| \lesssim 1$

Extensions

(B) General Γ, no permanent impact

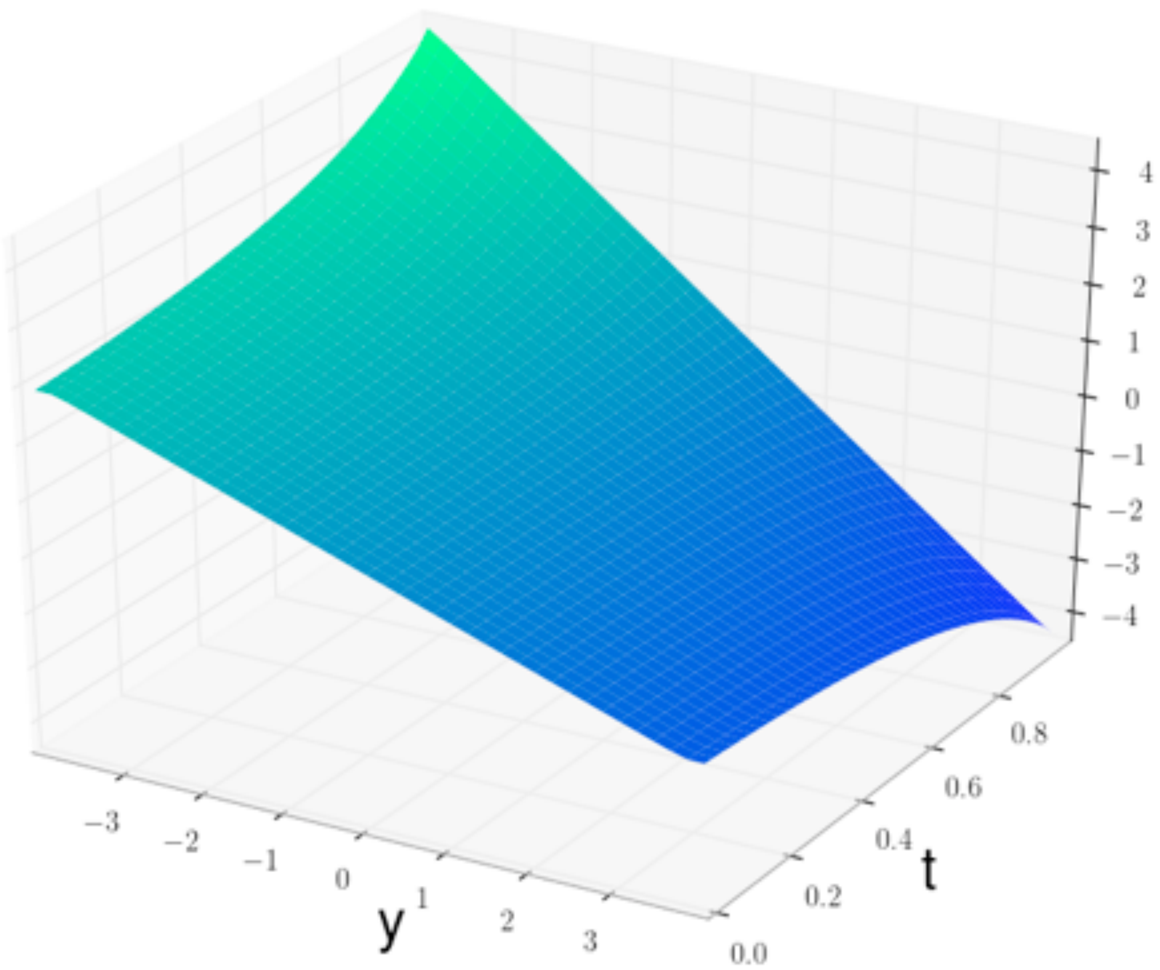
$$\theta_t \;=\; -\kappa\, h(T-t)\, Y_t \;+\; \text{bias term}$$

Restriction on sign of trading

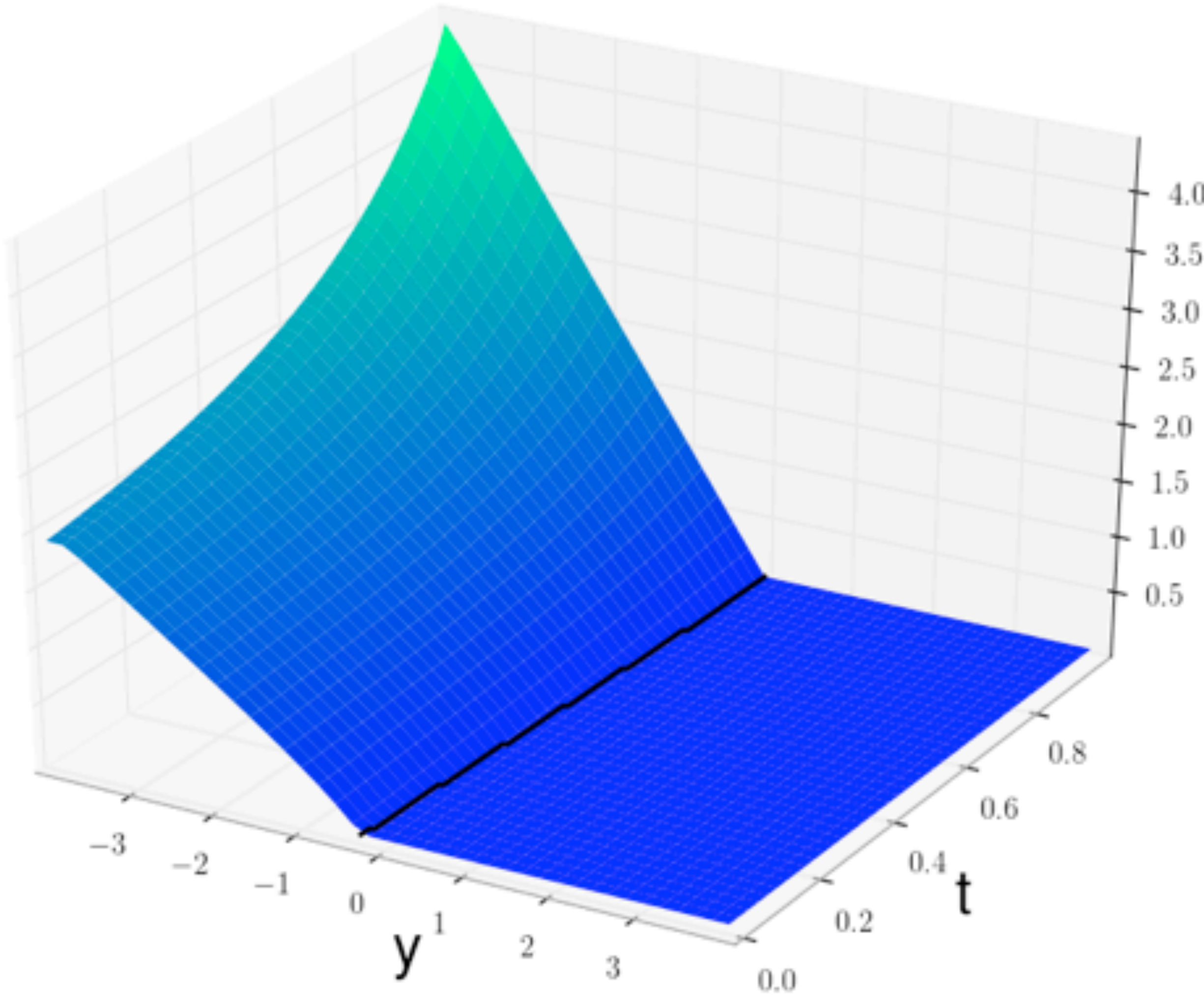
$$\inf_{\theta\in\Theta^+}\left\{\cdots\right\}$$

solve numerically

Restricted sign



Unrestricted strategy



Restricted strategy

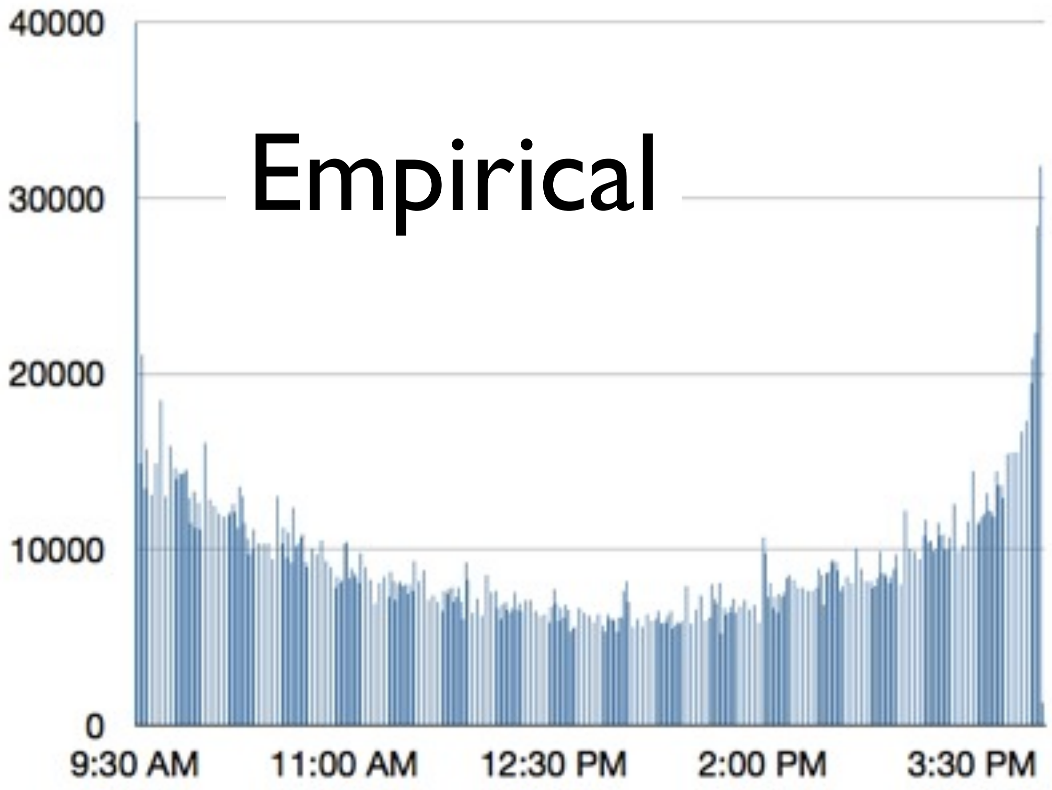
5. Applications

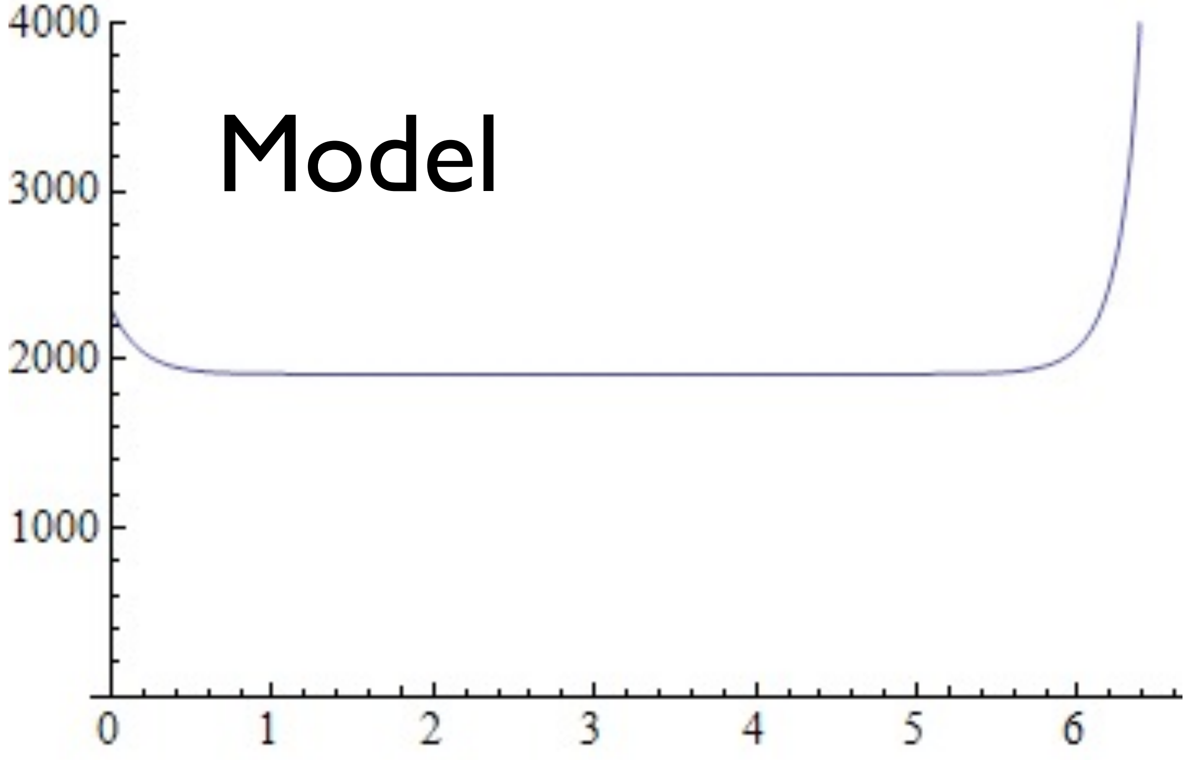
Price pinning to strike near expiration

- if hedgers are net long Γ
- local asymptotics near strike and expiry

Intraday volume patterns

- hedge near open and close





Discrete-time hedging

- If evaluate at start of interval Δt

trade rate $$\theta = \frac{\nu - (1 + \nu\Gamma)G\,A_2}{\lambda}\,y$$

leads to overshoot and sawtooth pattern

- Full optimization gives "implicit" scheme

$$\theta = \frac{\nu - (1 + \nu\Gamma)G\,A_2}{\lambda + (1 + \nu\Gamma)^2 A_2 \Delta t}\,y$$

Conclusions

Simple market impact model
  - temporary/permanent
  - linear model

Explicit solution (at least for constant Γ)
  - hedge position tracks toward Black-Scholes

Large hedger can change volatility
  - market impact on implied and realized volatility