# Free Convolution with A Free Multiplicative Analogue of The Normal Distribution

Ping Zhong

Indiana University Bloomington

July 23, 2013

Fields Institute, Toronto

# Outline

1. Density of the measures $\mu \boxplus \gamma_t$, where $\gamma_t$ is the semicircular distribution with variance $t$.
    - Voiculescu transform
    - $\boxplus$-infinitely divisible distributions
    - Density formula
2. Density of the measures $\mu \boxtimes \lambda_t$ and $\nu \boxtimes \sigma_t$, where $\lambda_t$ and $\sigma_t$ are free multiplicative analogue of the normal distributions.
    - $\Sigma$-transform
    - $\boxtimes$-infinitely divisible distributions
    - Density formula
    - Free multiplicative analogue of the normal distributions

# Voiculescu transform

- $\mathcal{M}_{\mathbb{R}}$: the set of probability measures on $\mathbb{R}$
- For a measure $\mu \in \mathcal{M}_{\mathbb{R}}$, we define the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ by

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, d\mu(t), \quad z \in \mathbb{C}^+.$$

We set $F_\mu(z) = 1/G_\mu(z)$, $z \in \mathbb{C}^+$, so that $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic.

- Voiculescu transform of $\mu$:

$$\varphi_\mu(z) = F_\mu^{-1}(z) - z.$$

- (Voiculescu '86, Maassen '92, Bercovici-Voiculescu '93)

$$\varphi_{\mu \boxplus \nu}(z) = \varphi_\mu(z) + \varphi_\nu(z).$$

# Infinite divisibility w.r.t. $\boxplus$

- Given $\nu \in \mathcal{M}_{\mathbb{R}}$, we say that $\nu$ is $\boxplus$-infinitely divisible if for every positive integer $n$, there exists a $\nu_{1/n} \in \mathcal{M}_{\mathbb{R}}$ such that

$$\nu = \underbrace{\nu_{1/n} \boxplus \nu_{1/n} \boxplus \cdots \boxplus \nu_{1/n}}_{n \text{ times}}.$$

- (Voiculescu '86, Bercovici-Voiculescu '93)

$\nu$ is $\boxplus$-infinitely divisible if and only if

$$\varphi_\nu(z) = \alpha + \int_{-\infty}^{+\infty} \frac{1+tz}{z-t} d\sigma(t).$$

where $\alpha \in \mathbb{R}$ and $\sigma$ is a finite positive measure on $\mathbb{R}$.

# Boundary set

- $F_\nu$ is a conformal map and its inverse map is $H : z \to z + \varphi_\nu(z)$.

- (Biane '97, Chistyakov-Götze '11) The boundary of the set $U = \{F_\nu(z) : z \in \mathbb{C}^+\}$ is the graph of a function defined on $\mathbb{R}$.

- If $x + iy \in \partial U$ and $y > 0$, then

$$1 = \int_{\mathbb{R}} \frac{1+t^2}{(x-t)^2+y^2} d\sigma(t). \tag{1}$$

# Biane's formula

- $\mu \in \mathcal{M}_{\mathbb{R}}$, $\gamma_t$: semicircular distribution.
- Subordination function: $F_t : F_{\mu \boxplus \gamma_t}(z) = F_\mu(F_t(z))$.
- The function $F_t$ can be regarded as the $F$-transform of a probability measure on $\mathbb{R}$, denoted by $\gamma_t \boxminus \mu$.

# Biane's formula

- $\mu \in \mathcal{M}_{\mathbb{R}}$, $\gamma_t$: semicircular distribution.
- Subordination function: $F_t : F_{\mu \boxplus \gamma_t}(z) = F_\mu(F_t(z))$.
- The function $F_t$ can be regarded as the $F$-transform of a probability measure on $\mathbb{R}$, denoted by $\gamma_t \boxright \mu$.
- $\gamma_t \boxright \nu$ is $\boxplus$-infinitely divisible and $\varphi_{\gamma_t \boxright \nu}(z) = z + tG_\mu(z)$.
- Set $\Omega_t := F_t(\mathbb{C}^+)$, then $\partial\Omega_t$ is the graph of a function.

# Biane's formula

- $\mu \in \mathcal{M}_{\mathbb{R}}$, $\gamma_t$: semicircular distribution.
- Subordination function: $F_t : F_{\mu \boxplus \gamma_t}(z) = F_\mu(F_t(z))$.
- The function $F_t$ can be regarded as the $F$-transform of a probability measure on $\mathbb{R}$, denoted by $\gamma_t \boxminus \mu$.
- $\gamma_t \boxminus \nu$ is $\boxplus$-infinitely divisible and $\varphi_{\gamma_t \boxminus \nu}(z) = z + tG_\mu(z)$.
- Set $\Omega_t := F_t(\mathbb{C}^+)$, then $\partial\Omega_t$ is the graph of a function.

- Given $x \in \mathbb{R}$, let $y_t(x) \in \mathbb{R}$ such that $x + iy_t(x) \in \partial\Omega_t$, and $f_t(x) = H_t(x + iy_t(x))$.

**Theorem (Biane '97)**

*The density of $\mu \boxplus \gamma_t$ is given by $p_t(f_t(x)) = \frac{y_t(x)}{\pi t}$.*

# Biane's formula

• $\mu \in \mathcal{M}_{\mathbb{R}}$, $\gamma_t$: semicircular distribution.

• Subordination function: $F_t : F_{\mu \boxplus \gamma_t}(z) = F_\mu(F_t(z))$.

• The function $F_t$ can be regarded as the $F$-transform of a probability measure on $\mathbb{R}$, denoted by $\gamma_t \boxminus \mu$.

• $\gamma_t \boxminus \nu$ is $\boxplus$-infinitely divisible and $\varphi_{\gamma_t \boxminus \nu}(z) = z + tG_\mu(z)$.

• Set $\Omega_t := F_t(\mathbb{C}^+)$, then $\partial\Omega_t$ is the graph of a function.

• Given $x \in \mathbb{R}$, let $y_t(x) \in \mathbb{R}$ such that $x + iy_t(x) \in \partial\Omega_t$, and $f_t(x) = H_t(x + iy_t(x))$.

**Theorem (Biane '97)**

*The density of $\mu \boxplus \gamma_t$ is given by* $p_t(f_t(x)) = \frac{y_t(x)}{\pi t}$.

• The density $p_t$ is analytic whenever it is positive.

• Biane also proved that the number of connected components of the interior of the support is a non-increasing function of $t$.

# Multiplicative free convolution on $\mathcal{M}_{\mathbb{T}}$

- $\mathcal{M}_{\mathbb{T}}$: the set of probability measures on $\mathbb{T}$; $\mathcal{M}_*$: the set of probability measures on $\mathbb{C}$ with nonzero mean.
- Given $\mu \in \mathcal{M}_{\mathbb{T}}$, we define

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{tz}{1 - tz} d\,\mu(t)$$

and set $\eta_\mu(z) = \psi_\mu(z)/(1 + \psi_\mu(z))$.
- $\eta_\mu : \mathbb{D} \to \mathbb{D}$ is analytic.

# Multiplicative free convolution on $\mathcal{M}_{\mathbb{T}}$

- $\mathcal{M}_{\mathbb{T}}$: the set of probability measures on $\mathbb{T}$; $\mathcal{M}_*$: the set of probability measures on $\mathbb{C}$ with nonzero mean.
- Given $\mu \in \mathcal{M}_{\mathbb{T}}$, we define

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{tz}{1-tz} d\,\mu(t)$$

and set $\eta_\mu(z) = \psi_\mu(z)/(1+\psi_\mu(z))$.

- $\eta_\mu : \mathbb{D} \to \mathbb{D}$ is analytic.
- For $\mu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$.

**Theorem (Voiculescu '87)**

*Given $\mu, \nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, then we have*

$$\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z).$$

# Subordination distributions

- $\lambda_t$: the free multiplicative analogue of the normal distribution on $\mathbb{T}$ such that $\Sigma_{\lambda_t}(z) = \exp\left(\frac{t}{2}\frac{1+z}{1-z}\right)$.
- Subordination function $\eta_t : \mathbb{D} \to \mathbb{D}$: $\eta_{\mu\boxtimes\lambda_t}(z) = \eta_\mu(\eta_t(z))$.
- Subordination distribution $\rho_t \in \mathcal{M}_\mathbb{T}$: $\eta_{\rho_t}(z) = \eta_t(z)$, for $t > 0$.

**Lemma**

*The measure $\rho_t$ is $\boxtimes$-infinitely divisible and its $\Sigma$-transform is*

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2}\int_\mathbb{T}\frac{1+\xi z}{1-\xi z}d\mu(\xi)\right). \tag{2}$$

- Let $\Phi_{t,\mu}(z) := z\Sigma_{\rho_t}(z)$, then $\Phi_{t,\mu}(\eta_{\rho_t}(z)) = z$ for all $z \in \mathbb{T}$.

- $\lambda_t$: the free multiplicative analogue of the normal distribution on $\mathbb{T}$ such that $\Sigma_{\lambda_t}(z) = \exp\left(\frac{t}{2}\frac{1+z}{1-z}\right)$.
- Subordination function $\eta_t : \mathbb{D} \to \mathbb{D}$: $\eta_{\mu \boxtimes \lambda_t}(z) = \eta_\mu(\eta_t(z))$.
- Subordination distribution $\rho_t \in \mathcal{M}_\mathbb{T}$: $\eta_{\rho_t}(z) = \eta_t(z)$, for $t > 0$.

**Lemma**

*The measure $\rho_t$ is $\boxtimes$-infinitely divisible and its $\Sigma$-transform is*

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2}\int_\mathbb{T} \frac{1+\xi z}{1-\xi z} d\mu(\xi)\right). \tag{2}$$

- Let $\Phi_{t,\mu}(z) := z\Sigma_{\rho_t}(z)$, then $\Phi_{t,\mu}(\eta_{\rho_t}(z)) = z$ for all $z \in \mathbb{T}$.
- The function $\eta_{\rho_t}$ is a conformal map.
- (Belinschi-Bercovici '05) $\eta_{\rho_t}$ extends to be a one-to-one and continuous function on $\overline{\mathbb{D}}$.

# A formula for the density of $\mu \boxtimes \lambda_t$

- $\mu \in \mathcal{M}_{\mathbb{T}}$; $\mu_t := \mu \boxtimes \lambda_t$.
- Subordination distribution: $\eta_{\mu_t} = \eta_\mu(\eta_{\rho_t}(z))$, $\rho_t \in \mathcal{ID}(\boxtimes, \mathbb{T})$.
- $\Phi_{t,\mu}(z) := z\Sigma_{\rho_t}(z)$.
- $\Omega_{t,\mu} := \{z \in \mathbb{D} : |\Phi_{t,\mu}(z)| < 1\} = \eta_{\rho_t}(\mathbb{D})$.

**Lemma**

*$\partial\Omega_{t,\mu}$ is the graph of a function of $\theta$.*

# A formula for the density of $\mu \boxtimes \lambda_t$

- $\mu \in \mathcal{M}_{\mathbb{T}}$; $\mu_t := \mu \boxtimes \lambda_t$.
- Subordination distribution: $\eta_{\mu_t} = \eta_\mu(\eta_{\rho_t}(z))$, $\rho_t \in \mathcal{ID}(\boxtimes, \mathbb{T})$.
- $\Phi_{t,\mu}(z) := z\Sigma_{\rho_t}(z)$.
- $\Omega_{t,\mu} := \{z \in \mathbb{D} : |\Phi_{t,\mu}(z)| < 1\} = \eta_{\rho_t}(\mathbb{D})$.

**Lemma**

*$\partial\Omega_{t,\mu}$ is the graph of a function of $\theta$.*

Given $\theta \in [-\pi, \pi]$, let $v_t(\theta) \in (0, 1]$ such that $v_t(\theta)e^{i\theta} \in \partial\Omega_{t,\mu}$.

**Theorem (Z. '12)**

*The density of $\mu_t$ is given by $p_t(\overline{\Psi_{t,\mu}(e^{i\theta})}) = -\ln(v_t(\theta))/\pi t$ , where $\Psi_{t,\mu} = \Phi_{t,\mu}(v_t(\theta)e^{i\theta})$.*

- The density $p_t$ is analytic whenever it is positive. The number of connected components of the interior of the support of $\mu_t$ is a non-increasing function of $t$.

# An example

- (Biane '97)

$$\int_{\mathbb{T}} \xi^n d\lambda_t(\xi) = Q_n(t)e^{-\frac{n}{2}t}, \tag{3}$$

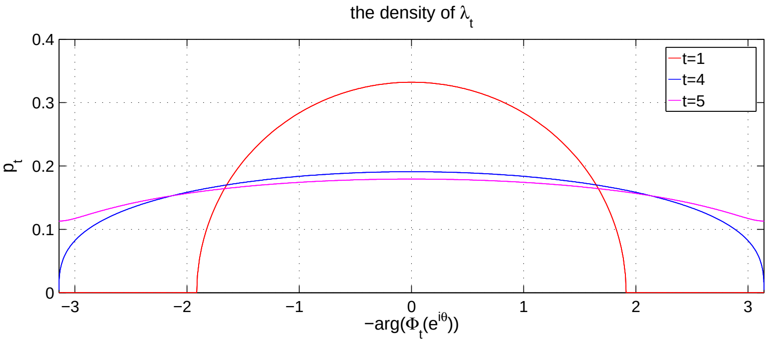
where $Q_n(t) = \sum_{k=0}^{n-1}(-1)^k \frac{t^k}{k!} n^{k-1}\binom{n}{k+1}$.

- Let $\mu = \delta_1$, $\Omega_t = \Omega_{t,\delta_1}$, $\Phi_t = \Phi_{t,\delta_1}$ and $U_t = U_{t,\delta_1}$, we recover Biane's result.

1. The support of $\lambda_t$ is $\mathbb{T}$ if $t \geq 4$.
2. Let $\theta_t = -\arccos\left(1 - \frac{t}{2}\right) - \frac{1}{2}\sqrt{(4-t)t}$, then the support of $\lambda_t$ is $\left\{e^{i\theta} : -\theta_t \leq \theta \leq \theta_t\right\} \subsetneq \mathbb{T}$ for $t < 4$.

**Proposition**

*The density of $\lambda_t$ is symmetric with respect to x-axis; and the density of $\lambda_t$ has a unique maximum at 1.*


the density of λ_t
0.4
0.3
0.2
0.1
0
p_t
t=1
t=4
t=5
−3
−2
−1
0
1
2
3
−arg(Φ_t(e^{iθ}))

# A formula for the density of $\nu \boxtimes \sigma_t$

- $\sigma_t$: $\Sigma_{\sigma_t}(z) = e^{\frac{t}{2}\frac{1+z}{z-1}}$; $\nu \in \mathcal{M}_{\mathbb{R}^+}$: $\nu \neq \delta_0$; $\nu_t := \nu \boxtimes \sigma_t$.

- $\tau_t$: $\eta_{\nu_t} = \eta_\nu(\eta_{\tau_t}(z))$, then $\tau_t \in \mathcal{ID}(\boxtimes, \mathbb{R}^+)$, and

$$\Sigma_{\tau_t}(z) = \Sigma_{\lambda_t}(\eta_\nu(z)) = \exp\left(\frac{t}{2}\int_0^{+\infty} \frac{1+\xi z}{\xi z - 1} d\nu(\xi)\right).$$

- Let $H_{t,\nu}(z) = z\Sigma_{\tau_t}(z)$, and let $\Gamma_{t,\nu}$ be the connected component of the set $\{z \in \mathbb{C}^+ : \Im(H_{t,\nu}(z)) > 0\}$ whose boundary contains the left half line $(-\infty, 0)$.

**Proposition (Belinschi-Bercovici '05)**

*The function $\eta_{\tau_t}$ extends to be a continuous function on $\mathbb{C}^+ \cup \mathbb{R}$, and $\eta_{\tau_t}$ is one to one on this set. If $\xi \in \mathbb{R}^+$ satisfies $\Im(\eta_{\tau_t}(\xi)) > 0$, then $\eta_{\tau_t}$ can be extended analytically to a neighborhood of $\xi$.*

**Lemma**

*Given $r \in (0, +\infty)$, then there exists only one $\theta \in [0, \pi)$, denoted by $u_t(r)$, such that $re^{i\theta} \in \partial(\Gamma_{t,\nu})$.*

Let $\Lambda_{r,\nu}(r) = H_{t,\nu}(re^{iu_t(r)})$.

**Theorem (Z. '12)**

*The measure $\nu_t$ has a density given by*

$$q_t\left(\frac{1}{\Lambda_{t,\nu}(r)}\right) = \frac{1}{\pi}\Lambda_{t,\nu}(r)\frac{u_t(r)}{t}$$

*for $r \in (0, +\infty)$.*

# The density of $\nu_t$

## Corollary

1. *The number of connected components of the interior of the support of* $\nu_t$ *is a non-increasing function of* $t$.
2. $\nu_t$ *is absolutely continuous and* $q_t$ *is analytic whenever it is positive.*
3. *If* $\nu$ *is compactly supported on* $(0, +\infty)$, *then the support of* $\nu_t$ *has only one connected component for large* $t$.

We then give a description of $\sigma_t$, part of results were known by Biane.

- Let $\nu = \delta_1$, $H_t(z) = H_{t,\delta_1} = z\exp(\frac{t}{2}\frac{z+1}{z-1})$, and $V_t = \{0 < r < +\infty : u_t(r) > 0\}$.

- Then $\nu_t = \sigma_t$ and $V_t = (x_1(t), x_2(t))$, where $x_1(t) = (2 + t - \sqrt{t(t+4)})/2$ and $x_2(t) = (2 + t + \sqrt{t(t+4)})/2 = 1/x_1(t)$.

$$\text{Let } x_3(t) = \frac{1}{\Lambda_t(x_2(t))} = \frac{2 + t - \sqrt{t(t+4)}}{2} \exp\left(-\frac{\sqrt{t(t+4)}}{2}\right)$$

and $x_4(t) = 1/x_3(t)$.

**Proposition**

*For $t > 0$, let $s_t(x)$ be the density of the measure $\sigma_t$ at $x$.*

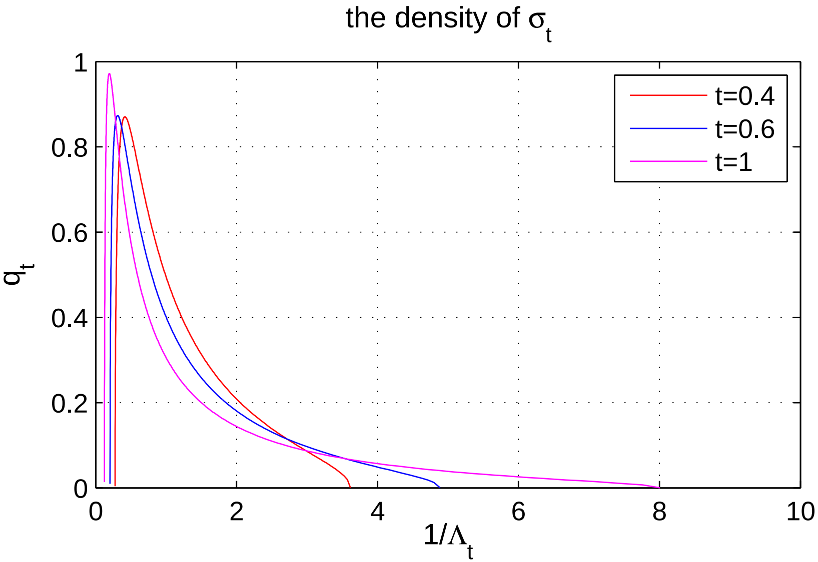
(1) *The support of the function $s_t$ is the interval $[x_3(t), x_4(t)]$.*

(2) *For $[\alpha, \beta] \subset (0, 1]$, we have $\sigma_t([\alpha, \beta]) = \sigma_t\left(\left[\frac{1}{\beta}, \frac{1}{\alpha}\right]\right)$.*

(3) *$s_t(x)x$ and $s_t$ are strictly decreasing functions of $x$ on the interval $[1, x_4(t))$.*

**Remark**

*For any integer $n$, let $a_n = 1 + \sqrt{2t/n}$, $b_n = 1/a_n$ and $\mu_n = (\delta_{a_n} + \delta_{b_n})/2$, then $\mu_n^{\boxtimes n} \to \sigma_t$ weakly as $n \to +\infty$.*


the density of σ_t
t=0.4
t=0.6
t=1
q_t
1/Λ_t
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10