Twisted parking symmetric functions and free multiplicative convolution

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Free Multiplicative Convolution

- We will use algebraic framework
- $\mathcal{D}_{alg} = \{\mu \colon \mathbb{C}[X] \to \mathbb{C} : \mu \text{ linear}, \mu(1) = 1\}$
- Operation $\boxtimes$ on $\mathcal{D}_{alg}$ corresponds to multiplication of free elements in a noncommutative probability space $(\mathcal{A}, \varphi)$:
  if μ is the distribution of $a \in \mathcal{A}$ and ν is the distribution of $b \in \mathcal{A}$ and a is free from b, then $\mu \boxtimes \nu$ is the distribution of ab.
- There are explicit formulas for writing moments of $\mu \boxtimes \nu$ in terms of moments of μ and ν
- $\mathcal{G} = \{\mu \in \mathcal{D}_{alg} : \mu(X) = 1\}$ is a commutative group

What is the structure of $\mathcal{G}$?

- **Theorem.** (Voiculescu 1987)

$$S\colon \mathcal{G} \xrightarrow{\sim} \mathbb{C}[[z]]^{\times}$$

(multiplicative group of power series with constant term $= 1$)

- **Theorem.** (Nica, Speicher 1996)

$$\mathrm{cf}_n(R_{\mu\boxtimes\nu}) = \sum_{\pi\in NC(n)} \mathrm{cf}_{n;\pi}(R_\mu)\,\mathrm{cf}_{n;K(\pi)}(R_\nu)$$

There is also a 'k-tuples' analogue of this formula.

- **Theorem.** The above theorem translates into the language of Hopf algebras.

Why translate anything into Hopf algebras?

- Organize information
- Use universal properties such as free, cofree ...
- Algebraic structure theory, e.g., Milnor-Moore Theorem.
- Duality: interplay between B and B^*.
- Calculus in the convolution algebra: in graded connected seeting all formal power series are locally finite on certain elements of the convolution algebra, we can define functions such as exp and log by their usual power series expansion, they still have the usual properties.
- Universality of *Qsym* and *sym*; expose connections with other areas of mathematics.

Symmetric functions as a graded connected Hopf algebra

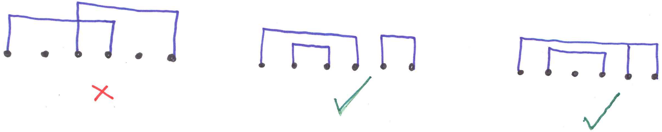
- $sym \subseteq \mathbb{C}[[x_1, x_2, \ldots, x_n, \ldots]]_b$.
- $\Delta\colon sym \to sym \otimes sym$ is an algebra homomorphism given by $\Delta(f) = \sum_i f_i' \otimes f_i''$, where f_i' and f_i'' are such that $f(x_1, x_2, \ldots; y_1, y_2, \ldots) = \sum_i f_i'(x_1, x_2, \ldots) f_i''(y_1, y_2, \ldots)$.
- $\varepsilon\colon sym \to \mathbb{C}$ is given by $\varepsilon(f) = f(1, 0, 0, \ldots)$.
- graded by the obvious notion of degree
- $sym = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots]$
  $\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$, $\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$.
- $\mathbb{X}(sym) = \{f\colon sym \to \mathbb{C} : f \text{ an algebra map }\}$ is a group under $f * g = \mathrm{mult}(f \otimes g)\Delta$.

Sym encodes the representation theory of S_n

- $B = \bigoplus_{n=0}^{\infty} Rep(S_n)$
- $\rho \cdot \tau = \mathrm{Ind}_{S_i \times S_j}^{S_{i+j}} (\rho \otimes \tau)$
- $\Delta(\rho) = \sum_{i+j=n} \mathrm{Res}_{S_i \times S_j}^{S_n}(\rho).$
- The isomorphism $B \to sym$ is given by
  Specht's module corresponding to partition $\lambda \mapsto$ Schur function s_λ.
- Character values $\chi_\lambda(\mu)$ can be recovered by expressing s_λ's as linear combinations of products of p_n's.
- Expressing $s_\lambda s_\mu$ as a linear combination of s_ν's encodes the Littlewod-Richardson Rule.

Noncrossing partitions

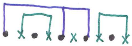
$NC(n) =$ noncrossing partitions of $[n] = \{1, 2, \ldots, n\}$

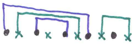


$|NC(n)| = \frac{1}{n+1}\binom{2n}{n} =$ Catalan number

$NC(n)$ is a lattice

Kreweras complement

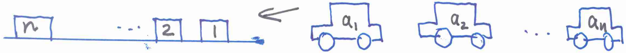
$\pi =$  ⇝  ⇝ $K(\pi) =$ 

$\rho =$  ⇝  ⇝ $K(\rho) =$ 

Symmetric functions y_n

- For $\pi = \{B_1, \ldots, B_k\} \in NC(n)$ define $e_\pi = e_{|B_1|} \ldots e_{|B_k|}$
- Define $y_n = \sum_{\pi \in NC(n-1)} e_\pi \in sym$
- $y_1 = 1, y_2 = e_1, y_3 = e_1^2 + e_2, y_4 = e_1^3 + 3e_1 e_2 + e_3, \ldots$
- $sym = \mathbb{C}[y_2, y_3, \ldots]$
- **Theorem.** (Gessel, 1996) To every poset there corresponds a quasi-symmetric function. (More precisely, Gessel explicitely describes a Hopf morphism from Rota's Hopf algebra of posets to *Qsym*; this morphism can now be seen as a special case of the universal Aguiar-Bergeron-Sottile morphism)
- y_n are the symmetric functions corresponding to $NC(n)$.

Parking functions



Car C_i prefrers space a_i. If a_i is occupied, then C_i takes next available space. We say that $(a_1, a_2, \dots, a_n)$ is a parking function if all cars can park.

Fact : $(a_1, a_2, \dots, a_n)$ is a parking function if and only if the increasing rearrangement $b_1 \le b_2 \le \dots \le b_n$ has the property that $b_i \le i$ for all i.

$n = 2$ 11 12 21

$n = 3$ 111 112 121 211 113 131 311 122
212 221 123 132 213 231 312 321

Parking functions

- $|PF(n)| = (n+1)^{n-1} = |\text{maximal chains in } NC(n)|$
- Parking function representation of S_n: S_n acts on $\mathbb{C}PF(n)$ by permuting the indices
- Twisted parking function representation of S_n: twist above by the sign representation, that is $\sigma(a_1, \ldots, a_n) = \tau(\sigma)(a_{\sigma_1}, \ldots, a_{\sigma n})$.
- **Theorem.** (Stanley 1997)
  Symmetric function of the twisted parking representation of S_{n-1}
  $=$ symmetric function of the poset $NC(n)$
  $= y_n$

Diagonal Harmonics

- S_n acts diagonally on $\mathbb{F}[\mathbf{x}, \mathbf{y}] := \mathbb{F}[x_1, \ldots, x_n; y_1, \ldots, y_n]$
- Invariants $= \mathbb{F}[\mathbf{x}; \mathbf{y}]^{S_n}$
- Coinvariants $= \mathbb{F}[\mathbf{x}; \mathbf{y}]_{S_n} := \mathbb{F}[\mathbf{x}; \mathbf{y}] / \langle \mathbb{F}[\mathbf{x}; \mathbf{y}]_+^{S_n} \rangle$
- Problem: find a nice basis for coinvariants!
- Diagonal harmonics $=$
  $DH_n = \{p : \sum_{i=1}^n \partial_{x_i}^r \partial_{y_i}^s p = 0$ for all $0 \leq r, s$ with $1 \leq r + s \leq n\}$
- Fact: DH_n is an S_n-invariant subspace of $\mathbb{F}[\mathbf{x}; \mathbf{y}]$ whose projection to coinvariants is an S_n-equivariant isomorphism.
- Problem: find an explicit description for DH_n.
- **Theorem.**(Haiman 2001) DH_n is isomorphic to the twisted parking function representation. In particular $|DH_n| = (n+1)^{n-1}$.

y_n and free multiplicative convolution

- **Proposition.**

$$\Delta(y_n) = \sum_{\pi \in NC(n)} y_\pi \otimes y_{K(\pi)}$$

- For $\mu \in \mathcal{G}$ define $\chi_\mu \in \mathbb{X}(sym)$ by $\chi_\mu(y_n) = \mathrm{cf}_n(R_\mu)$.

- **Theorem.** There is an isomorphism between $\mathcal{G}$ and $\mathbb{X}(sym)$ given by $\mu \mapsto \chi_\mu$. There is also a 'k-tuples' analogue.

- **LS transform.** Using calculus in convolution algebra we see that $\log(\chi_\mu)$ gives rise to a transform $LS\mu(z) = -z \log S_\mu(z) \in \mathbb{C}[[z]]$ with the property that $LS_{\mu \boxtimes \nu} = LS_\mu + LS_\nu$. LS has a k-tuples analogue with the property of linearizing products of commuting distributions.

Free probabilistic definition of y_n

We have the following dictionary:

$$\begin{aligned}
\chi_\mu(y_n) &= \mathrm{cf}_n(R_\mu), \\
\chi_\mu(h_n) &= (-1)^n \,\mathrm{cf}_n(S_\mu), \\
\chi_\mu(e_n) &= \mathrm{cf}_n(1/S_\mu), \\
\chi_\mu(p_n) &= (-1)^{n+1} n \,\mathrm{cf}_n(\log S_\mu).
\end{aligned}$$

An alternative way to define y_n is to define them to be the unique symmetric functions that make this dictionary work.

Questions

- **Question.** Can you find a nice combinatorial proof for

$$\Delta(y_n) = \sum_{\pi \in NC(n)} y_\pi \otimes y_{K(\pi)}?$$

- In our paper we present a failry short, but indirect proof using Minor-Moore Theorem, 'dictionary' definition of y_n and the multiplicativity property of the S-transform.

- We have a lengthy and cumbersome induction proof using the recursion

$$y_n = \sum_{m=2}^{n} \left(e_{m-1} \cdot \sum_{1=i_1<i_2<\cdots<i_m=n} y_{i_2-i_1}\, y_{i_3-i_2} \cdots y_{i_m-i_{m-1}} \right).$$

Questions.

- **Question.** Can we transfer any results/tools between diagonal harmonics and free probability? For example, can the formula for $\Delta(y_n)$ tell you anything about diagonal harmonics?
- **Question.** There are results encoding some classical probability into *sym*. Can we use these results to find even more connections between classical and free probability?
- **Question.** In the paper where Stanley proves that y_n are twisted parking symmetric functions, he also also looks at parking symmetric functions in type B. Is there an analogous result connecting them with noncrossing partitions of type B? Does this analogous result extend to cover the coproduct in the Hopf algebra translation of the free-cummulant formula for $\boxtimes$ in type B?

Thank You!