About the Vlasov-Dirac-Benney Equation

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Vlasov-Dirac and Vlasov

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) - \partial_x \rho_f(t,x)\partial_v f(t,x,v) = 0\,,$$
$$\rho_f(t,x) = \int_{\mathbf{R}} f(t,x,v)dv\,.$$

Similarities: Liouville, Energy, Hamiltonian.

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0\,,$$
$$E = -\nabla_x \int_{\mathbf{R}^d} V(x-y)\Big(\int_{\mathbf{R}^d} f(t,y,v)dv - 1\Big)dy\,,$$
$$\dot{x}(t) = v(t)\,, \quad \dot{v}(t) = -\int_{\mathbf{R}^d} \nabla_x V(x(t)-y)\Big(\int_{\mathbf{R}^d} f(t,y,w)dw - 1\Big)dy\,.$$
$$\mathcal{E}(f) = \int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{|v|^2}{2} f(t,x,v)dxdv$$
$$+\frac{1}{2}\int_{\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d} V(x-y)f(t,x,v)f(t,y,w)dwdydxdv\,.$$
$$\partial_t f + \{\mathcal{E}, f\} = 0\,.$$

Differences

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0 \quad \rho_f(x,t) = \int_{\mathbf{R}_v} f(t,x,v)dv.$$

•The mapping $f \mapsto \rho_f \mapsto E = -\nabla_x \rho_f$ is an operator of degree 1 while for the original Vlasov–Poisson equation it is an operator of degree -1.

•The effect of the instabilities will be much more drastic and while for the original Vlasov–Poisson equation the issue is the large time asymptotic behavior, here the issue is that the Cauchy problem may be badly posed even for regular initial data and for arbitrarily small time. That will be one of the main issue.

Comments on the VDB

•Focus on the one-dimensional ($d = 1$) version of the problem ∂ instead of ∇

•The interest of a one-dimensional space model justified by physical reasons, particularly in the quasineutral-limit when the Debye length vanishes. It is in one dimension that the spectral analysis of the linearized problem is, by an adaptation of the method of Penrose , the most explicit.

•There is a natural connection between the properties of the linearized and the fully nonlinear model.

•This connection emphasizes the role of "bumps" in the initial profile. In particular in the case of the one-bump profile the connection with the Benney equation gives a new stability theorem for the full nonlinear problem.

•The stability results are in full agreement with what is known concerning the WKB limit of the Non-Linear Schrödinger equation.

Plane waves for the Vlasov equation

$$f \mapsto f + G(v)$$

$$\partial_t f + v\partial_x f - \partial_x \left(\int_{\mathbf{R}_y} V(x-y) \int_{\mathbf{R}_w} f(y,w,t)dwdy \right) G'(v) = 0\,.$$

$$e_k(t,x,v) = A(k,v)e^{i(kx-\omega(k)t)},$$

$$(-i\omega(k) + ikv)A(k,v) - ik\widehat{V}(k)\hat{\rho}_A(k)G'(v) = 0,$$

$$A(k,v) - \widehat{V}(k)\frac{G'(v)}{v-\omega(k)/k}\hat{\rho}_A(k) = 0\,,$$

$$\left(1 - \widehat{V}(k)\int_{\mathbf{R}} \frac{G'(v)}{v-\omega(k)/k}dv\right)\hat{\rho}(k) = 0.$$

Synthesis of plane waves

$$\text{With}\omega(k) \quad (1 - \int_{\mathbf{R}} \frac{\widehat{V}(k)G'(v)}{v - \omega(k)/k} dv)\hat{\rho}_A(k) = 0$$

$$f(x, v, t) = \int e^{i(kx - \omega(k)t)} (\frac{\widehat{V}(k)G'(v)}{v - \omega(k)/k})\hat{\rho}(k)dk(\text{whenever they exist })$$

are the unique solutions of the Cauchy problem with initial data

$$f(x, v, 0) = \int e^{ikx} (\frac{\widehat{V}(k)G'(v)}{v - \omega(k)/k})\hat{\rho}(k)dk$$

$$\omega(k) \text{ with } \Im\omega(k) > 0\,.$$

For Vlasov Poisson The unstable spectra is in a “band"

$$1 = \frac{1}{k^2}\int_{\mathbf{R}} \frac{G'(v)}{v-\omega(k)/k}dv)$$
$$\Rightarrow |\Im\omega(k)| \le |\hat{v}(k)||k| \int |G'(v)|dv = O(|k|^{-1})$$

For Vlasov Dirac the dispersion relation in homogeneous in k

$$1 = \int_{\mathbf{R}} \frac{G'(v)}{v-\omega^*}dv$$

With a solution ω^* with $\Im\omega^* \neq 0$ all the modes $\omega^* k$ are unstable!
The Cauchy problem is ill posed in any Sobolev space!

Theorem and examples

For the existence of unstable plane waves for the genuine $1d$ Vlasov Poisson a criteria was proposed by Penrose. This criteria can be partly adapted to the present case (even if the consequences are different).

Theorem

Assume that the original profile:

$$v \mapsto G(v) \geq 0 \quad \int G(v)dv = 1$$

as a unique maximum then there are no unstable modes.

A direct proof will be given below.

2. $G(v)$ even with $G(0) = G'(0) = 0$, then for ϵ small enough, there exist unstable modes for the profile $G_\epsilon(v) = \frac{1}{\epsilon} G\left(\frac{v}{\epsilon}\right)$.

$$0 = 1 - \int_{\mathbf{R}} \frac{G'_\epsilon(v)}{v - \omega^*} dv = 1 - \int_{\mathbf{R}} \frac{G'_\epsilon(v) v}{v^2 + \sigma^2} dv - i \int_{\mathbf{R}} \frac{G'_\epsilon(v)\sigma}{v^2 + \sigma^2} dv$$

$$= 1 - \int_{\mathbf{R}} \frac{G'_\epsilon(v) v}{v^2 + \sigma^2} dv \,.$$

$$I(\infty) = 0 \quad \text{and} \quad I(0) = \frac{2}{\epsilon^2} \int_0^\infty \frac{G(v)}{v^2} dv \,.$$

3. $G(v) = \delta_v$ is a Dirac mass a limit case of above .

$$G(v) = \delta_v \Longrightarrow \int_{\mathbf{R}} \frac{G'(v)}{v-\omega} dv = \int_{\mathbf{R}} \frac{\delta_v}{(v-\omega)^2} dv = \frac{1}{\omega^2},$$

2. For $G(v) = \frac{1}{2}(\delta_{v-a} + \delta_{v+a})$ the existence of unstable modes depends on the size of a. Dirac masses generate unstable modes, if and only if they are close enough, according to the formula

$$1 - \int_{\mathbf{R}} \frac{G'(v)}{v-\omega} dv = 1 - \frac{1}{(a-\omega)^2} + \frac{1}{(a+\omega)^2},$$

which has non real solutions if and only if $a^2 < 2$.

Uniform stability of the linearized problem near single bump profile

Proposition $x \mapsto V(x)$ even and one bump $G(v)$:

$$G'(v) := -H(v)(v-a) \text{ with } \quad H(v) > 0\,.$$

Then any smooth solution $f(t,x,v)$ of the linearized Vlasov equation with potential V:

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) - G'(v)\,\partial_x \int_{\mathbf{R}} V(x-y)(\int_{\mathbf{R}} f(t,y,w)dw)dy = 0$$

satisfies the energy identity,

$$\frac{1}{2}\frac{d}{dt}(\int_{\mathbf{R}\times\mathbf{R}} H^{-1}(v)(f(t,x,v))^2 dxdv + \int_{\mathbf{R}\times\mathbf{R}} V(x-y)\rho_f(x,t)\rho_f(x,t)dxdy) = 0\,.$$

Proof

With $a = 0$. $f(t,x,v) = H(v)\tilde{f}(t,x,v)$, multiply by $\tilde{f}$ and integrate over the phase-space (x,v)

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) - G'(v)\,\partial_x \int_{\mathbf{R}} V(x-y)\rho_f(y)dy = 0$$

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbf{R}\times\mathbf{R}} H^{-1}(v)(f(t,x,v))^2 dxdv\Big) + \int_{\mathbf{R}\times\mathbf{R}} \partial_x V(x-y)\rho_f(t,y)\int_{\mathbf{R}} H(v)v\tilde{f}(t,x,v)dvdydx = 0\,.$$

$$\partial_t \rho_f(t,x) + \partial_x \int_{\mathbf{R}} vH(v)\tilde{f}(t,x,v)dv = 0\,.$$

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbf{R}\times\mathbf{R}} H^{-1}(v)(f(t,x,v))^2 dxdv\Big) + \int_{\mathbf{R}\times\mathbf{R}} V(x-y)\rho_f(t,y)\partial_t\rho_f(t,x)dydx = 0\,.$$

Consequence of Energy Conservation for the Linearized equation:

• $x \mapsto V(x)$ positive semi-definite even potential, $G(v)$ one bum profile $G'(v) = -H(v)(v-a)$ and $\mathcal{H}_V = \{f\}$ such that:

$$\int_{\mathbf{R}\times\mathbf{R}} H^{-1}(v)(f(x,v))^2 dvdx + \int_{\mathbf{R}\times\mathbf{R}} V(x-y)\rho_f(x)\rho_g(y)dxdy < \infty,$$

• The dynamic of the linearized problem with initial data in $\mathcal{H}_V$ is described by a strongly continuous unitary group.
• This evolution is “stable " with respect to perturbations in V and $G(v)$.
• Hypothesis valid for $V(x) = \delta_x$ and and also approximations $V(x) \to \delta_x$.

Consequences 1 For the original V-D-B problem with general initial data

- **Theorem** $\dot{H}^m$ the space of functions $f \in L^\infty(\mathbf{R}_x, L^1(\mathbf{R}_v))$ with, for $1 \leq l \leq m$, derivatives $\partial_x^l f \in L^2(\mathbf{R}_x; L^1(\mathbf{R}_v))$. For every m, the Cauchy problem for the dynamics $S(t)$ defined by the V−D−B equation is not locally ($\dot{H}^m \mapsto \dot{H}^1$) well-posed.

- **Theorem** Jabin-Nouri (2011) : For any (x, v) analytic function $f_0(x, v)$ with

$$\forall \alpha, \; m\,, \; n \quad \sup_x |\partial_{x^m} \partial_{v^n} f_0(x, v)|(1 + |v|)^\alpha = C(m, n) o(|v|)$$

there exists, for a finite time T, an analytic solution of the Cauchy problem.

Consequences for the V-D-B equation in relation with fluid mechanic: Examples 1.

The phase space density: mono kinetic solution:

$$f(t,x,v)=\rho(t,x)\delta(v-u(t,x))$$

is a distributional solution of the V−D−B equation if and only if its moments

$$\rho(t,x)=\int_{\mathbf{R}} f(t,x,v)dv \quad \text{and} \quad \rho(t,x)u(t,x)=\int_{\mathbf{R}} vf(t,x,v)dv$$

are solutions of the system

$$\partial_t\rho+\partial_x(\rho u)=0\,,\quad \partial_t(\rho u)+\partial_x\left(\rho u^2+\frac{\rho^2}{2}\right)=0.$$

For $(\rho,u)\in\mathbf{R}_+\times\mathbf{R}$ *it is strictly hyperbolic* $\Rightarrow$ *existence of a local in time (near* $(\tilde\rho_0+\alpha,u_0)$ *with* $\alpha>0$ *and* $(\tilde\rho_0,u_0)\in H^2(\mathbf{R})$*) of smooth solutions is ensured. In full agreement with the stability results of the modal analysis.*

Consequences for the V-D-B equation in relation with fluid mechanic: Examples 2.

Multi-kinetic densities :

$$f(t,x,v)=\sum_{1\leq n\leq N}\rho_n(t,x)\delta(v-u_n(t,x))$$

are solutions of the V-D-B equation if and only if:

$$\partial_t\rho_n+\partial_x(\rho_n u_n)=0\,,$$
$$\partial_t(\rho_n u_n)+\partial_x\left(\rho_n u_n^2\right)+\rho_n\partial_x\Big(\sum_{1\leq\ell\leq N}\rho_\ell\Big)=0.$$

This system is not always hyperbolic ; the Cauchy problem is not always locally in time well posed. In particular for $N=2$ *and* $(\rho_1,\rho_2,u_1,u_2)=(1,1,a,-a)$ *direct computations show that the system is hyperbolic (hence the Cauchy problem is well posed) if and only if* $a^2>2$. *Once again this is in full agreement with the "modal examples".*

As long as $v \mapsto f(t,x,v)$ remains (for (t,x) given a.e.) a one-bump profile, with maximum equal to 1 for simplicity, i.e.

$$\sup_{v\in\mathbf{R}} f(t,x,v)=1, \quad (t,x) \text{ a.e.},$$

one defines a.e. in $(x,a)\in\mathbf{R}\times[0,1]$ $v_\pm(t,x,a)$:

$$v_-(t,x,a)\leq v_+(t,x,a) \quad f(t,x,v_\pm(t,x,a))=a,$$

and recover the one-bump profile $f(t,x,v)$ by:

$$f(t,x,v)=\int_0^1 \mathrm{Y}(v_+(t,x,a)-v)-\mathrm{Y}(v_-(t,x,a)-v))da$$

f is a distributional solution of the V−D−B equation if and only if contours $v_\pm(t,x,a)$ are solutions of the system

$$\partial_t v_\pm + v_\pm\partial_x v_\pm + \partial_x\rho = 0, \quad \rho(t,x)=\int_0^1 (v_+(t,x,a)-v_-(t,x,a))da.$$

The Benney equation-at last!

With mean density and a dependent velocity

$$\varrho(t,x,a) = v_+(t,x,a) - v_-(t,x,a),\; u(t,x,a) = \frac{1}{2}(v_+(t,x,a) + v_-(t,x,a))$$

the (v_-, v_+) system is equivalent to the fluid type system

$$\partial_t \varrho(t,x,a) + \partial_x(\varrho(t,x,a)u(t,x,a)) = 0,$$
$$\partial_t u(t,x,a) + \partial_x \left(\frac{1}{2}u^2(t,x,a) + \frac{1}{8}\varrho^2(t,x,a)\right) + \partial_x \int_0^1 \varrho(t,x,b)db = 0,$$

Derived by Benney as a model for water-waves (This the reason for the name Vlasov-Dirac-Benney).

Benney equation and energy-entropy

Without the integral term $\partial_x \int_0^1 \varrho(t,x,a)da$ the infinite dimensional system $\rho(x,t), u(x,a,t))$ would be an infinite system of isentropic Euler equations: On the other hand it still have an energy-entropy.

$$\begin{aligned}\mathcal{E}(\varrho,u) =&\frac{1}{2}\int_{\mathbf{R}}\int_0^1 \left(\varrho(t,x,a)u^2(t,x,a)+\frac{1}{12}\varrho^3(t,x,a)\right)dadx\\ &+\frac{1}{2}\int_{\mathbf{R}}\left(\int_0^1 \varrho(t,x,a)da\right)^2 dx,\end{aligned}$$

Therefore one should have a local in time stability result.. Proven below in the **V** variable.

Entropy for the Benney equation

• For $\mathbf{V} = (\mathbf{v}_-, \mathbf{v}_+)^{\mathbf{t}}$ the system is of the form:

$$\partial_t \mathbf{V} + \partial_{\mathbf{x}} \mathbf{F}(\mathbf{V}) = \mathbf{0} \text{ with } \mathbf{F}(\mathbf{V}) = \begin{cases} \frac{1}{2} v_-^2 + \int_0^1 (v_+(t,x,a) - v_-(t,x,a))da. \\ \frac{1}{2} v_+^2 + \int_0^1 (v_+(t,x,a) - v_-(t,x,a))da \end{cases}$$

• $\mathbf{V} \mapsto \mathbf{F}'(\mathbf{V})$ is a linear continuous operator in $L^2(0,1)$

• The system has an entropy:

$$\eta(f) = \int_{\mathbf{R}_x \times \mathbf{R}_v} \frac{|v|^2}{2} f(t,x,v)dxdv + \frac{1}{2} \int_{\mathbf{R}_x} (\int_{\mathbf{R}_v} f(t,x,v)dv)^2 dx$$

$$= \int_{\mathbf{R}_x} [\frac{1}{6} \int_0^1 (v_+^3(t,x,a) - v_-^3(t,x,a))da$$

$$+ \frac{1}{2} \left(\int_0^1 (v_+(t,x,a) - v_-(t,x,a))da \right)^2]dx$$

Proposition H Hilbert space, $F : H \mapsto H$ and $\eta : H \mapsto \mathbf{R}$ assume that F is differentiable (Gateaux) and η twice differentiable the if η is an entropy for

$$\partial_t \mathbf{V} + \partial_x (F(\mathbf{V})) = 0$$

Then the operator $\mathbf{V} \mapsto \eta"(\mathbf{V})F'(\mathbf{V})$ is symmetric (self adjoint).
Proof Observe that the formula

$$(\eta"(\mathbf{V})F'(\mathbf{V})U, W) = (\eta"(\mathbf{V})F'(\mathbf{V})W, U)$$

is noting more that the Schwarz lemma on the $2d$ affine space $(\gamma, \sigma) \mapsto \mathbf{V} + \gamma U + \sigma W$.
Therefore if η is a convex entropy one should have local existence and stability for smooth solutions.
Explicit computations given on the next slide

Same but explicit

$$\partial_t v_\pm + v_\pm \partial_x v_\pm + \partial_x \rho = 0, \quad \rho(t,x) = \int_0^1 (v_+(t,x,a) - v_-(t,x,a))da.$$

$$\partial_t \mathbf{V} + F'(\mathbf{V})\partial_x \mathbf{V} = 0$$

$$F'(\mathbf{V}) = \begin{pmatrix} v_-(t,x,a) - \int_0^1 da & \int_0^1 da \\ -\int_0^1 da & v_+(t,x,a) + \int_0^1 da \end{pmatrix}$$

$$\eta" = \begin{pmatrix} -v_-(t,x,a) + \int_0^1 da & -\int_0^1 da \\ -\int_0^1 da & v_+(t,x,a) + \int_0^1 da \end{pmatrix}$$

$$\eta" F' = \begin{pmatrix} -v_-^2 + \int_0^1 da \cdot v_- + v_- \cdot \int_0^1 da & -v_- \cdot \int_0^1 da - \int_0^1 da \cdot v_+ \\ -v_+ \cdot \int_0^1 da - \int_0^1 da \cdot v_- & v_+^2 + \int_0^1 da \cdot v_+ + v_+ \cdot \int_0^1 da \end{pmatrix}$$

Explicit computations

Proposition A priori estimate Any smooth solution $\mathbf{V} = (v_-, v_+)^t$, satisfies the a priori nonlinear Gronwall estimate

$$\frac{d}{dt}\left(\|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \int (\eta''(\mathbf{V})\partial_{\mathbf{x}}^{\mathbf{3}}\mathbf{V}, \partial_{\mathbf{x}}^{\mathbf{3}}\mathbf{V})\mathbf{dx}\right)$$
$$\leq C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbf{R}\times(0,1))}\right)^2 .$$

First

$$\|\partial_x^2 \rho\|^2_{L^\infty(\mathbf{R})} \leq C\left(\|\partial_x^3 \rho\|^2_{L^2(\mathbf{R})} + \|\rho\|^2_{L^\infty(\mathbf{R})}\right) \leq C\left(\|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbf{R}\times(0,1))} + \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))}\right).$$

Then from $\partial_t v_\pm + v_\pm \partial_x v_\pm + \partial_x \rho_f = 0$

$$\partial_t \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} \leq C\left(\|\partial_x \rho\|^2_{L^\infty(\mathbf{R})} + \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))}\right)$$
$$\leq C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbf{R}\times(0,1))}\right)^2.$$

$$\partial_t \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))}$$
$$\leq C\left(1 + \|\mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x \mathbf{V}\|^2_{L^\infty(\mathbf{R}\times(0,1))} + \|\partial_x^3 \mathbf{V}\|^2_{L^2(\mathbf{R}\times(0,1))}\right)^2.$$

Symmetrization and Main stability Theorem

Second Consider :

$$\partial_x^3(\partial_t \mathbf{V} + \partial_x^3 F'(\mathbf{V})\partial_x \mathbf{V} = 0)$$

multiply by the symmetrizer η"$(\mathbf{V})$ and proceed as in the classical case.

Theorem

Introduce:

$$\mathcal{B}(T^*)=\{\mathbf{V}\in C(0,T^*;L^\infty(\mathbf{R}_x\times(0,1)))\cap L^\infty(0,T^*;L^2((0,1);H^3(\mathbf{R}_x)))\}$$

and the open subset $\mathcal{O}(m,M,T^*)\subset\mathcal{B}(T^*)=\{\mathbf{V}\in\mathcal{B}(T^*)$
$-M<v_-(t,x,a)<-m<0<m<v_+(t,x,a)<M<\infty\}$.

Then for or initial data such that

$$\partial_x^3\mathbf{V}(0)\in L^2(\mathbf{R}\times(0,1))$$
$$-M<-v_-(0,x,a)<-m<0<mv_+(0,x,a)<M$$

there is for T^* small enough a solution $\mathbf{V}(x,t,)\in\mathcal{O}(m,M,T^*)$.

Remarks

Biggest constraint : The functions $a \mapsto v_\pm(0,x,a)$ have to be defined on a fixed interval (say $a \in [0,1]$) and bounded above and below. Implies for the initial profiles $v \mapsto f_0(x,v)$ the following x independent properties.

(H1) There exist an x independent constant $0 < M < \infty$ such that

$$|v| \geq M \Rightarrow f_0(x,v) = 0.$$

(H2) There exist an x independent constant $0 < m < \infty$ constant such that

$$|v| \leq m \Rightarrow f_0(x,v) = 1.$$

(H3) The map $v \mapsto f_0(x,v)$ is non-decreasing on the interval $]-\infty,-m]$ and non-increasing on the interval $[m,+\infty[$. In short it is a “plateau "profile near $v=0$.

No other regularity with respect to v is needed and the introduction of the $v_\pm^N$ satisfying the hypothesis of the proposition shows the validity of the waterbag model as a convenient approximation for the continuous model.

With Weyl calculus and Wigner transform Vlasov equations are formally WKB limit of the Schrodinger or Von-Neumann dynamic. However for the non linear Schrodinger equation which corresponds to the V-D-B such formal semi-classical limits turn out to be "rigorously proven limits " only in cases which also correspond to the stability near one-bump profile (and also are in agreement with the analysis of the linearized problem).

Consider the self consistent Schrödinger equation

$$i\hbar\partial_t\psi = \mathcal{H}(\hbar, V(t))\psi = -\frac{\hbar^2}{2}\Delta\psi + V(t,x)\psi,$$

with a time-dependent potential

$$V(t,x) = \int_{\mathbf{R}^d} \mathcal{V}(x-y)|\psi(t,y)|^2 dy$$

and a normalized solution $\int_{\mathbf{R}^d} |\psi_\hbar(t,x)|^2 dx = 1$
Whenever $\psi_\hbar(t)$ is solution of the self consistent Schrödinger equation $K_\hbar(t,x,y) = \psi_\hbar(t,x) \otimes \overline{\psi_\hbar(t,y)}$ is a solution of the Von-Neumann equation:

$$i\hbar\partial_t K_\hbar(t) = [K_\hbar(t), \mathcal{H}(\hbar, V(t))]$$

$$\text{with} \quad V(t,x) = \int_{\mathbf{R}^d} \mathcal{V}(x-y) K_\hbar(t,y,y) dy,$$

The formal $\hbar \to 0$ WKB limit of the Wigner transform of the operator $K_\hbar(t)$

$$W_\hbar(t,x,v) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{-iy\cdot v} K_\hbar\Big(t, x+\frac{\hbar}{2}y, x-\frac{\hbar}{2}y\Big)dy$$

is a solution of the Vlasov equation

$$\partial_t W(t,x,v) + v\cdot\nabla_x W(t,x,v) \\ -\nabla_x\left(\int_{\mathbf{R}^d}\mathcal{V}(x-y)\int_{\mathbf{R}^d} W(t,y,w)dwdy\right)\cdot\nabla_v W(t,x,v) = 0,$$

with

$$W_0(x,v) := W(0,x,v) = \lim_{\hbar\to 0} W_\hbar(0,x,v).$$

About Convergence with $\mathcal{V} = \delta$

Proven when the potential $\mathcal{V}$ is smooth enough. For the Non-Linear Schrödinger equation and for its formal limit the V–D–B equation the situation is completely different.

Since the Cauchy problem may be ill posed. No chances of such convergence (even for $\mathscr{C}^\infty$ data and small time).

Two situations where one may have convergence

i) When the initial data $W_\hbar(0,x,v)$ is uniformly (in $\hbar$) analytic.

ii) When the initial data converges to a one bum profile..This includes the WKB approximation.

$$\psi_h(0,x) = \sum_{1\le k\le N} \rho_k(x) e^{i\frac{S_k(x)}{\hbar}}$$

$$W_\hbar(0,x,v) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iyv}\psi_\hbar\Big(0,x+\frac{\hbar}{2}y\Big)\overline{\psi}_\hbar\Big(0,x-\frac{\hbar}{2}y\Big)dy$$

$$W_\hbar(0,x,v) \to \sum_{1\le k\le N} \rho_k(x)\delta(v-\nabla S_k(x)), \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

For $N = 1$, this corresponds to a mono-kinetic initial data. In this setting o the Wigner transform of $\psi_\hbar(t,x)\otimes\overline{\psi_\hbar(t,y)}$ converges to the solution of V−D−B equation. Gerard (analytic) , Grenier (Modification of the Madelung transform) and Jin, Levermore and McLaughlin (Inverse scattering).
Multikinetic: $N > 1$ been considered by Zakharov (with formal proofs of convergence.) These proofs should completely work in the analytic case. In less regular cases for example with $N = 2$ convergence may hold in some cases but not in every cases.

$$f(x,v,t)=\rho(x,t)\delta(v-u(x,t))\,,$$

$$\rho(x,t)=\int f(x,v,t)dv \quad \rho(x,t)u(x,t)=\int f(x,v,t)dxdv$$

$$\partial_t\rho+\partial_x(\rho u)=0\,,\quad \partial_t(\rho u)+\partial_x\left(\rho u^2+\frac{\rho^2}{2}\right)=0.$$

It this case $f(x,v,t)$ is the semi classical limit of the NLSE. The system for (ρ,u) is hyperbolic with an infinite set of conserved quantities and flux: Lax Entropy! The NLSE being integrable has also infinite set of conserved quantities (in the phase of regularity) they converge to Lax Entropies.

$$\eta"(\rho,u)F'(\rho,u)=F'(\rho,u)\eta"(\rho,u) \tag{1}$$

Question ? Are all Lax-Entropies limit of conserved quantities for the NLSE

$$f(t,x,v) = \int_0^1 \mathrm{Y}(v_+(t,x,a)-v) - \mathrm{Y}(v_-(t,x,a)-v))da$$

$$\partial_t v_\pm + v_\pm \partial_x v_\pm + \partial_x \rho = 0, \quad \rho(t,x) = \int_0^1 (v_+(t,x,a) - v_-(t,x,a))da.$$

Several authors Benney Zakharov Miura have found an infinite set of conserved quantities. They must satisfy the operational equation:

$$(\eta"(\mathbf{V})F'(\mathbf{V})U, W) = (\eta"(\mathbf{V})F'(\mathbf{V})W, U) \qquad (2)$$

which is the counterpart of (1). Solutions of (1) are in fact solutions of an hyperbolique equation.

Is there a generalization for the solutions of (2) ???
In fact with

$$H(V)=\int(\frac{1}{6}(v_+^3+v_-^3)+\frac{1}{2}(\int_0^1(v_+-v_-)da)^2)dx$$

The Benney equation can be written has an Hamiltonian system and the characterization of the formula (2) is equivalent to the fact that the functions $(H(V),\eta(V))$ are in involution. With convenient hypothesis on the “plateau type " profile the Wigner transform of the solution of NLSE should converge to the solution of Benney equation and the convergence of conserved quantities should follow (not proven to the best of my knowledge). Then does this process describes all the conserved quantities for the Benney equation.

Thanks for the invitation !

Thanks for the attention !

Happy Birthday Walter!!