KAM for quasi-linear KdV

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KdV

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}$$

Quasi-linear Hamiltonian perturbation

$$\mathcal{N}_4 := -\partial_x\{(\partial_u f)(x, u, u_x)\} + \partial_{xx}\{(\partial_{u_x} f)(x, u, u_x)\}$$

$$\mathcal{N}_4 = a_0(x, u, u_x, u_{xx}) + a_1(x, u, u_x, u_{xx})u_{xxx}$$

$$\mathcal{N}_4(x, \varepsilon u, \varepsilon u_x, \varepsilon u_{xx}, \varepsilon u_{xxx}) = O(\varepsilon^4)\,, \quad \varepsilon \to 0$$

$$f(x, u, u_x) = O(|u|^5 + |u_x|^5),\ f \in C^q(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$$

Physically important for perturbative derivation from water-waves (that I learned from Walter Craig)

Hamiltonian PDE

$$u_t = X_H(u)\,, \quad X_H(u) := \partial_x \nabla_{L^2} H(u)$$

Hamiltonian KdV

$$H = \int_{\mathbb{T}} \frac{u_x^2}{2} + u^3 + f(x, u, u_x)dx$$

where the density $f(x, u, u_x) = O(|(u, u_x)|^5)$

Phase space

$$H_0^1(\mathbb{T}) := \left\{ u(x) \in H^1(\mathbb{T}, \mathbb{R}) \ : \ \int_{\mathbb{T}} u(x)dx = 0 \right\}$$

Non-degenerate symplectic form:

$$\Omega(u, v) := \int_{\mathbb{T}} (\partial_x^{-1} u)\, v \, dx$$

Goal: look for small amplitude quasi-periodic solutions

Definition: quasi-periodic solution with n frequencies

$u(t,x) = U(\omega t, x)$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,

$\omega \in \mathbb{R}^n (= \textit{frequency vector})$ is irrational $\omega \cdot k \neq 0\,,\ \forall k \in \mathbb{Z}^n \setminus \{0\}$

$\Longrightarrow$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^n$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto u(\varphi, x) \in \text{phase space}$$

is invariant under the flow evolution of the PDE

Linear Airy eq.

$$u_t + u_{xxx} = 0, \qquad x \in \mathbb{T}$$

Solutions: (superposition principle)

$$u(t,x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j e^{\mathrm{i} j^3 t} e^{\mathrm{i} j x}$$

Eigenvalues j^3 = "NORMAL FREQUENCIES"
Eigenfunctions: $e^{\mathrm{i} j x}$ = "NORMAL MODES"

All solutions are 2π- periodic in time: COMPLETELY RESONANT

$\Rightarrow$ Quasi-periodic solutions are a completely nonlinear phenomenon

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$\Rightarrow$ **Quasi-periodic solutions are a completely nonlinear phenomenon**

KdV is COMPLETELY INTEGRABLE

$$u_t + u_{xxx} - 3\partial_x u^2 = 0$$

All solutions are periodic, quasi-periodic, almost periodic

What happens under a small perturbation?

KAM theory

Kuksin '98, Kappeler-Pöschel '03: KAM for KdV

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0$$

1. SEMILINEAR PERTURBATION $\partial_x f(x, u)$
2. Also true for Hamiltonian perturbations

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x |\partial_x|^{1/2} f(x, |\partial_x|^{1/2} u) = 0$$

of order 2

$|j^3 - i^3| \geq i^2 + j^2$, $i \neq j \implies$ KdV gains up to 2 spatial derivatives

3. for QUASI-LINEAR KdV? OPEN PROBLEM

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Literature: KAM for "unbounded" perturbations

Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)

$$\mathrm{i}u_t - u_{xx} + M_\sigma u + \mathrm{i}\varepsilon f(u, \bar{u})u_x = 0$$

Zhang-Gao-Yuan '11 Reversible DNLS

$$\mathrm{i}u_t + u_{xx} = |u_x|^2 u$$

Craig-Wayne periodic solutions, Lyapunov-Schmidt + Nash-Moser

Bourgain '96, Derivative NLW

$$y_{tt} - y_{xx} + my + y_t^2 = 0\,, \qquad m \neq 0,$$

Craig '00, Hamiltonian DNLW

$$y_{tt} - y_{xx} + g(x)y = f(x, D^\beta y)\,, \quad D := \sqrt{-\partial_{xx} + g(x)},$$

quasi-periodic solutions

Berti-Biasco-Procesi '12, '13, reversible DNLW

$$u_{tt} - u_{xx} + mu = g(x, u, u_x, u_t)$$

For quasi-linear PDEs: Periodic solutions:

- Iooss-Plotinikov-Toland, Iooss-Plotnikov, '01-'10,
  Water waves: quasi-linear equation,
  new ideas for conjugation of linearized operator

quasi-periodic solutions

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Main results

Hamiltonian density:

$$f(x,u,u_x) = f_5(u,u_x) + f_{\geq 6}(x,u,u_x)$$

f_5 polynomial of order 5 in (u,u_x); $f_{\geq 6}(x,u,u_x) = O(|u| + |u_x|)^6$

Reversibility condition:

$$f(x,u,u_x) = f(-x,u,-u_x)$$

KdV-vector field $X_H(u) := \partial_x \nabla H(u)$ is **reversible** w.r.t the involution

$$\varrho u := u(-x)\,, \;\; \varrho^2 = I\,, \quad -\varrho X_H(u) = X_H(\varrho u)$$

Theorem ('13, P. Baldi, M. Berti, R. Montalto)

Let $f \in C^q$ (with $q := q(n)$ large enough). Then, for "generic" choice of the "TANGENTIAL SITES"

$$S := \{-\bar{\jmath}_n, \dots, -\bar{\jmath}_1, \bar{\jmath}_1, \dots, \bar{\jmath}_n\} \subset \mathbb{Z} \setminus \{0\},$$

the hamiltonian and reversible KdV equation

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T},$$

possesses small amplitude quasi-periodic solutions with Sobolev regularity H^s, $s \le q$, of the form

$$u = \sum_{j \in S} \sqrt{\xi_j}\, e^{\mathrm{i}\omega_j^\infty(\xi)\, t} e^{\mathrm{i}jx} + o(\sqrt{\xi}), \ \omega_j^\infty(\xi) = j^3 + O(|\xi|)$$

for a "Cantor-like" set of "initial conditions" $\xi \in \mathbb{R}^n$ with density 1 at $\xi = 0$. The linearized equations at these quasi-periodic solutions are reduced to constant coefficients and are **stable***.*
If $f = f_{\ge 7} = O(|(u, u_x)|^7)$ then **any** *choice of tangential sites*

Tangential sites

Explicit conditions:

- HYPOTHESIS (S_3) $j_1 + j_2 + j_3 \neq 0$ for all $j_1, j_2, j_3 \in S$
- HYPOTHESIS (S_4) $\nexists j_1, \ldots, j_4 \in S$ such that

$$j_1 + j_2 + j_3 + j_4 \neq 0, \quad j_1^3 + j_2^3 + j_3^3 + j_4^3 - (j_1 + j_2 + j_3 + j_4)^3 = 0$$

1. (S_3) used in the linearized operator. If $f_5 = 0$ then not needed
2. If also $f_6 = 0$ then (S_4) not needed (used in Birkhoff-normal-form)

"genericity":

After fixing $\{\bar{\jmath}_1, \ldots, \bar{\jmath}_n\}$, in the choice of $\bar{\jmath}_{n+1} \in \mathbb{N}$ there are only FINITELY MANY forbidden values

Comments

1. A similar result holds for

mKdV: focusing/defocusing

$$\partial_t u + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}$$

for all the tangential sites $S := \{-\bar{\jmath}_n, \ldots, -\bar{\jmath}_1, \bar{\jmath}_1, \ldots, \bar{\jmath}_n\}$ such that

$$\frac{2}{2n-1}\sum_{i=1}^{n} \bar{\jmath}_i^2 \notin \mathbb{N}$$

2. If $f = f(u, u_x)$ the result is **true** for **all** the tangential sites S
3. Also for generalized KdV (not integrable), with normal form techniques of Procesi-Procesi

Linear stability

(L): linearized equation $\partial_t h = \partial_x \partial_u \nabla H(u(\omega t, x))h$

$$h_t + a_3(\omega t, x)h_{xxx} + a_2(\omega t, x)h_{xx} + a_1(\omega t, x)h_x + a_0(\omega t, x)h = 0$$

There exists a quasi-periodic (Floquet) change of variable

$$h = \Phi(\omega t)(\psi, \eta, v), \quad \psi \in \mathbb{T}^\nu, \eta \in \mathbb{R}^\nu, v \in H^s_x \cap L^2_{S^\perp}$$

which transforms (L) into the **constant coefficients** system

$$\begin{cases} \dot\psi = b\eta \\ \dot\eta = 0 \\ \dot v_j = \mathrm{i}\mu_j v_j, \quad j \notin S, \ \mu_j \in \mathbb{R} \end{cases}$$

$$\Longrightarrow \eta(t) = \eta_0, v_j(t) = v_j(0)e^{\mathrm{i}\mu_j t} \Longrightarrow \|v(t)\|_s = \|v(0)\|_s \ : \ \text{stability}$$

Forced quasi-linear perturbations of Airy

Use $\omega = \lambda\vec{\omega} \in \mathbb{R}^n$ as 1-dim. parameter

Theorem (Baldi, Berti, Montalto , to appear Math. Annalen)

There exist $s := s(n) > 0$, $q := q(n) \in \mathbb{N}$, such that:

Let $\vec{\omega} \in \mathbb{R}^n$ diophantine. For every quasi-linear Hamiltonian nonlinearity $f \in C^q$ for all $\varepsilon \in (0, \varepsilon_0)$ small enough, there is a Cantor set $\mathcal{C}_\varepsilon \subset [1/2, 3/2]$ of asymptotically full measure, i.e.

$$|\mathcal{C}_\varepsilon| \to 1 \quad as \quad \varepsilon \to 0,$$

such that for all $\lambda \in \mathcal{C}_\varepsilon$ the perturbed Airy equation

$$\partial_t u + \partial_{xxx} u + \varepsilon f(\lambda\vec{\omega}t, x, u, u_x, u_{xx}, u_{xxx}) = 0$$

has a quasi-periodic solution $u(\varepsilon, \lambda) \in H^s$ (for some $s \leq q$) with frequency $\omega = \lambda\vec{\omega}$ and satisfying $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$.

Key: spectral analysis of quasi-periodic operator

$$\mathcal{L} = \omega \cdot \partial_\varphi + \partial_{xxx} + a_3(\varphi, x)\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x)$$

$a_i = O(\varepsilon)$, $i = 0, 1, 2, 3$

Main problem: the non constant coefficients term $a_3(\varphi, x)\partial_{xxx}$!

MAIN DIFFICULTIES:

1. The usual KAM iterative scheme is unbounded
2. We expect an estimate of perturbed eigenvalues

$$\mu_j(\varepsilon) = j^3 + O(\varepsilon j^3)$$

which is NOT sufficient for verifying second order Melnikov

$$|\omega \cdot \ell + \mu_j(\varepsilon) - \mu_i(\varepsilon)| \geq \frac{\gamma |j^3 - i^3|}{\langle \ell \rangle^\tau}, \quad \forall \ell, j, i$$

Idea to conjugate $\mathcal{L}$ to a diagonal operator

1. **"REDUCTION IN DECREASING SYMBOLS"**

$$\mathcal{L}_1 := \Phi^{-1}\mathcal{L}\Phi = \omega \cdot \partial_\varphi + m_3\partial_{xxx} + m_1\partial_x + \mathcal{R}_0$$

- $R_0(\varphi, x)$ pseudo-differential operator of **order** 0, $R_0(\varphi, x) : H^s_x \to H^s_x$, variable coefficients, $R_0 = O(\varepsilon)$,
- $m_3 = 1 + O(\varepsilon)$, $m_1 = O(\varepsilon)$, $m_1, m_3 \in \mathbb{R}$, CONSTANTS

Use suitable transformations "far" from the identity

2. **"REDUCTION OF THE SIZE of R_0"**

$$\mathcal{L}_\nu := \Phi_\nu^{-1}\mathcal{L}_1\Phi_\nu = \omega \cdot \partial_\varphi + m_3\partial_{xxx} + m_1\partial_x + r^{(\nu)} + \mathcal{R}_\nu$$

- $R_\nu = R_\nu(\varphi, x) = O(R_0^{2^\nu})$
- $r^{(\nu)} = \mathrm{diag}_{j\in\mathbb{Z}}(r_j^{(\nu)})$, $\sup_j |r_j^{(\nu)}| = O(\varepsilon)$,

KAM-type scheme, now transformations of $H^s_x \to H^s_x$

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KAM-type scheme, now transformations of $H^s_x \to H^s_x$

Higher order term

$$\mathcal{L} := \omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon a_3(\varphi, x)\partial_{xxx}$$

STEP 1: Under the **symplectic** change of variables

$$(Au) := (1 + \beta_x(\varphi, x))u(\varphi, x + \beta(\varphi, x))$$

we get

$$\begin{aligned} \mathcal{L}_1 := A^{-1}\mathcal{L}A &= \omega \cdot \partial_\varphi + (A^{-1}(1 + \varepsilon a_3)(1 + \beta_x)^3)\partial_{xxx} + O(\partial_{xx}) \\ &= \omega \cdot \partial_\varphi + c(\varphi)\partial_{xxx} + O(\partial_{xx}) \end{aligned}$$

imposing

$$(1 + \varepsilon a_3)(1 + \beta_x)^3 = c(\varphi),$$

There exist solution $c(\varphi) \approx 1$, $\beta = O(\varepsilon)$

STEP 2: Rescaling time

$$(Bu)(\varphi, x) = u(\varphi + \omega q(\varphi), x)$$

we have

$$\begin{aligned} B^{-1}\mathcal{L}_1 B &= B^{-1}(1+\omega\cdot\partial_\varphi q)(\omega\cdot\partial_\varphi) + B^{-1}c(\varphi)\partial_{xxx} + O(\partial_{xx}) \\ &= \mu(\varepsilon)B^{-1}c(\varphi)(\omega\cdot\partial_\varphi) + B^{-1}c(\varphi)\partial_{xxx} + O(\partial_{xx}) \end{aligned}$$

solving

$$1 + \omega\cdot\partial_\varphi q = \mu(\varepsilon)c(\varphi)\,, \quad q(\varphi) = O(\varepsilon)$$

Dividing for $\mu(\varepsilon)B^{-1}c(\varphi)$ we get

$$\mathcal{L}_2 := \omega\cdot\partial_\varphi + m_3(\varepsilon)\partial_{xxx} + O(\partial_x)\,,\ m_3(\varepsilon) := \mu^{-1}(\varepsilon) = 1 + O(\varepsilon)$$

which has the leading order with CONSTANT COEFFICIENTS

Autonomous KdV

New further difficulties:

- **No external parameters**. The frequency of the solutions is NOT fixed a-priori. Frequency-amplitude modulation.
- **KdV is completely resonant**
- **Construction of an approximate inverse**

Ideas:

- WEAK BIRKHOFF-NORMAL FORM
- General method to decouple the "*tangential dynamics*" from the "*normal dynamics*", developed with P. Bolle
  Procedure which reduces autonomous case to the forced one

Step 1. Bifurcation analysis: WEAK Birkhoff normal form

Fix the "tangential sites" $S := \{-\bar{\jmath}_n, \ldots, -\bar{\jmath}_1, \bar{\jmath}_1, \ldots, \bar{\jmath}_n\} \subset \mathbb{Z} \setminus \{0\}$

Split the dynamics:

$$u(x) = v(x) + z(x)$$

$$v(x) = \sum_{j \in S} u_j e^{\mathrm{i}jx} = \text{"tangential component"}$$

$$z(x) = \sum_{j \notin S} u_j e^{\mathrm{i}jx} = \text{"normal component"}$$

Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\int_{\mathbb{T}} v_x^2 + \frac{1}{2}\int_{\mathbb{T}} z_x^2 dx + \int_{\mathbb{T}} v^3 dx + 3\int_{\mathbb{T}} v^2 z dx \\ &+ \int_{\mathbb{T}} v^3 dx + 3\int_{\mathbb{T}} v^2 z dx + 3\int_{\mathbb{T}} v z^2 dx + \int_{\mathbb{T}} z^3 dx + \int_{\mathbb{T}} f(u, u_x) \end{aligned}$$

Goal: eliminate terms linear in $z \Longrightarrow \{z = 0\}$ is invariant manifold

Theorem (Weak Birkhoff normal form)

There is a symplectic transformation $\Phi_B : H^1_0(\mathbb{T}_x) \to H^1_0(\mathbb{T}_x)$

$$\Phi_B(u) = u + \Psi(u), \quad \Psi(u) = \Pi_E \Psi(\Pi_E u),$$

where $E := \mathrm{span}\{e^{\mathrm{i}jx}, 0 < |j| \le 6|S|\}$ *is* **finite-dimensional**, *s.t.*

$$\mathcal{H} := H \circ \Phi_B = H_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \mathcal{H}_{\ge 6},$$

$$\mathcal{H}_3 := \int_{\mathbb{T}} z^3 dx + 3 \int_{\mathbb{T}} v z^2 dx, \quad \mathcal{H}_4 := -\frac{3}{2} \sum_{j \in S} \frac{|u_j|^4}{j^2} + \mathcal{H}_{4,2} + \mathcal{H}_{4,3}$$

$$\mathcal{H}_{4,2} := 6 \int_{\mathbb{T}} v z \Pi_S\big((\partial_x^{-1} v)(\partial_x^{-1} z)\big)\, dx + 3 \int_{\mathbb{T}} z^2 \pi_0 (\partial_x^{-1} v)^2 dx,$$

$$\mathcal{H}_{4,3} := R(vz^3), \quad \mathcal{H}_5 := \sum_{q=2}^{5} R(v^{5-q} z^q),$$

and $\mathcal{H}_{\ge 6}$ *collects all the terms of order at least six in* (v, z).

Fourier representation

$$u(x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} u_j e^{\mathrm{i} jx} \,, \qquad u(x) \longleftrightarrow (u_j)_{j \in \mathbb{Z} \setminus \{0\}}$$

FIRST STEP. Eliminate the $u_{j_1} u_{j_2} u_{j_3}$ of H_3 with at **most one index outside** S. Since $j_1 + j_2 + j_3 = 0$ they are **finitely many**

$\Phi :=$ the time 1-flow map generated by

$$F(u) := \sum_{j_1 + j_2 + j_3 = 0} F_{j_1, j_2, j_3} u_{j_1} u_{j_2} u_{j_3}$$

The vector field X_F is supported on **finitely many** sites

$$X_F(u) = \Pi_{H_{2S}} X_F(\Pi_{H_{2S}} u)$$

$\Longrightarrow$ the flow is a **finite dimensional** perturbation of the identity

$$\Phi = Id + \Psi \,, \qquad \Psi = \Pi_{H_{2S}} \Psi \Pi_{H_{2S}}$$

For the other steps:

- Normalize the quartic monomials $u_{j_1} u_{j_2} u_{j_3} u_{j_4}$, $j_1, j_2, j_3, j_4 \in S$. The fourth order system $\mathcal{H}_4$ restricted to S turns out to be **integrable**, i.e.

$$-\frac{3}{2} \sum_{j \in S} \frac{|u_j|^4}{j^2} \quad (non - isochronous \ rotators)$$

Now $\{z = 0\}$ is an invariant manifold for $\mathcal{H}_4$ filled by quasi-periodic solutions with a frequency which varies with the amplitude

Difference w.r.t. other Birkhoff normal forms

1. Kappeler-Pöschel (KdV), Kuksin-Pöschel (NLS), **complete Birkhoff-normal form**: they remove/normalize also the terms $O(z^2), O(z^3), O(z^4)$
2. Pöschel (NLW), **semi normal Birkhoff normal form**: normalized only the term $O(z^2)$
3. Kappeler Global Birkhoff normal form for KdV, 1-d-cubic-NLS

The above transformations are

$$(1)\ I + bounded\,, \quad (2)\ I + O(\partial_x^{-1})\,, \quad (3)\ \Phi = \mathcal{F} + O(\partial_x^{-1})\,,$$

It is NOT enough for quasi-linear perturbations!

Our $\Phi = Id + finite\ dimensional \Longrightarrow$ it changes very little the third order differential perturbations in KdV

Rescaled action-angle variables:

$$u := \varepsilon v_\varepsilon(\theta, y) + \varepsilon z := \varepsilon \sum_{j \in S} \sqrt{\xi_j + |j| y_j}\, e^{\mathrm{i}\theta_j} e^{\mathrm{i}jx} + \varepsilon z$$

Hamiltonian:

$$H_\varepsilon = \mathcal{N} + P\,, \quad \mathcal{N}(\theta, y, z, \xi) = \alpha(\xi) \cdot y + \tfrac{1}{2}\big(N(\theta, \xi) z, z\big)_{L^2(\mathbb{T})}$$

where

Frequency-amplitude map:

$$\alpha(\xi) = \bar{\omega} + \varepsilon^2 A \xi$$

Variable coefficients normal form:

$$\tfrac{1}{2}\big(N(\theta, \xi) z, z\big)_{L^2(\mathbb{T})} = \tfrac{1}{2}\big((\partial_z \nabla H_\varepsilon)(\theta, 0, 0)[z], z\big)_{L^2(\mathbb{T})}$$

We look for quasi-periodic solutions of X_{H_ε} with

Diophantine frequencies:

$$\omega = \bar{\omega} + \varepsilon^2 A\xi$$

Embedded torus equation:

$$\partial_\omega i(\varphi) - X_{H_\varepsilon}(i(\varphi)) = 0$$

Functional setting

$$\mathcal{F}(\varepsilon, X) := \begin{pmatrix} \partial_\omega \theta(\varphi) - \partial_y H_\varepsilon(i(\varphi)) \\ \partial_\omega y(\varphi) + \partial_\theta H_\varepsilon(i(\varphi)) \\ \partial_\omega z(\varphi) - \partial_x \nabla_z H_\varepsilon(i(\varphi)) \end{pmatrix} = 0$$

unknown: $X := i(\varphi) := (\theta(\varphi), y(\varphi), z(\varphi))$

Main Difficulty

Invert linearized operator at approximate solution $i_0(\varphi)$:

$$D_i\mathcal{F}(i_0(\varphi))[\widehat{\imath}] =$$

$$\partial_\omega\widehat{\theta} - \partial_{\theta y}H_\varepsilon(i_0)[\widehat{\theta}] - \partial_{yy}H_\varepsilon(i_0)[\widehat{y}] - \partial_{zy}H_\varepsilon(i_0)[\widehat{z}]$$

$$\partial_\omega\widehat{y} + \partial_{\theta\theta}H_\varepsilon(i_0)[\widehat{\theta}] + \partial_{\theta y}H_\varepsilon(i_0)[\widehat{y}] + \partial_{\theta z}H_\varepsilon(i_0)[\widehat{z}]$$

$$\partial_\omega\widehat{z} - \partial_x\{\partial_\theta\nabla_z H_\varepsilon(i_0)[\widehat{\theta}] + \partial_y\nabla_z H_\varepsilon[\widehat{y}] + \partial_z\nabla_z H_\varepsilon[\widehat{z}]\}$$

Approximate inverse. Zehnder

A linear operator $T(X)$, $X := i(\varphi)$ is an APPROXIMATE INVERSE of $dF(X)$ if

$$\|dF(X)T(X) - Id\| \leq \|F(X)\|$$

1. $T(X)$ is an exact inverse of $dF(X)$ at a solution
2. It is sufficient to invert $dF(X)$ at a solution

Use the general method to construct an approximate inverse, reducing to the inversion of quasi-periodically forced systems, Berti-Bolle for autonomous NLS-NLW with multiplicative potential

How to take advantage that i_0 is a solution?

The invariant torus $i_0(\varphi) := (\theta_0(\varphi), y_0(\varphi), z_0(\varphi))$ is ISOTROPIC

$\Longrightarrow$

the transformation G of the phase space $\mathbb{T}^n \times \mathbb{R}^n \times H_{S^\perp}$

$$\begin{pmatrix} \theta \\ y \\ z \end{pmatrix} := G \begin{pmatrix} \psi \\ \eta \\ w \end{pmatrix} := \begin{pmatrix} \theta_0(\psi) \\ y_0(\psi) + D\theta_0(\psi)^{-T}\eta + D\tilde{z}_0(\theta_0(\psi))^T \partial_x^{-1} w \\ z_0(\psi) + w \end{pmatrix}$$

where $\tilde{z}_0(\theta) := z_0(\theta_0^{-1}(\theta))$, is SYMPLECTIC

In the new symplectic coordinates, i_0 is the trivial embedded torus

$$(\psi, \eta, w) = (\varphi, 0, 0)$$

Transformed Hamiltonian

$$K := H_\varepsilon \circ G = K_{00}(\psi) + K_{10}(\psi)\eta + (K_{01}(\psi), w)_{L^2_x} + \frac{1}{2} K_{20}(\psi)\eta \cdot \eta$$
$$+ (K_{11}(\psi)\eta, w)_{L^2_x} + \frac{1}{2}(K_{02}(\psi)w, w)_{L^2_x} + O(|\eta| + |w|)^3$$

Hamiltonian system in new coordinates:

$$\begin{cases} \dot\psi = K_{10}(\psi) + K_{20}(\psi)\eta + K_{11}^T(\psi)w + O(\eta^2 + w^2) \\ \dot\eta = -\partial_\psi K_{00}(\psi) - \partial_\psi K_{10}(\psi)\eta - \partial_\psi K_{01}(\psi)w + O(\eta^2 + w^2) \\ \dot w = \partial_x\left(K_{01}(\psi) + K_{11}(\psi)\eta + K_{02}(\psi)w\right) + O(\eta^2 + w^2) \end{cases}$$

Since $(\psi, \eta, w) = (\omega t, 0, 0)$ is a solution $\Longrightarrow$

$$\partial_\psi K_{00}(\psi) = 0\,, \quad K_{10}(\psi) = \omega\,, \quad K_{01}(\psi) = 0$$

$\Longrightarrow$

KAM (variable coefficients) normal-form

$$K := H_\varepsilon \circ G = const + \omega \cdot \eta + \frac{1}{2} K_{20}(\psi)\eta \cdot \eta + \big(K_{11}(\psi)\eta, w\big)_{L^2_x} + \frac{1}{2}\big(K_{02}(\psi)w, w\big)_{L^2_x} + O(|\eta| + |w|)^3$$

Hamiltonian system in new coordinates:

$$\begin{cases} \dot\psi = \omega + K_{20}(\psi)\eta + K_{11}^T(\psi)w + O(\eta^2 + w^2) \\ \dot\eta = O(\eta^2 + w^2) \\ \dot w = \partial_x \big(K_{11}(\psi)\eta + K_{02}(\psi)w\big) + O(\eta^2 + w^2) \end{cases}$$

$\Longrightarrow$ in the NEW variables the linearized equations at $(\varphi, 0, 0)$ simplify!

Linearized equations at the invariant torus $(\varphi, 0, 0)$

$$\begin{pmatrix} \partial_\omega \widehat{\psi} - K_{20}(\varphi)\widehat{\eta} - K_{11}^T(\varphi)\widehat{w} \\ \partial_\omega \widehat{\eta} \\ \partial_\omega \widehat{w} - \partial_x K_{11}(\varphi)\widehat{\eta} - \partial_x K_{02}(\varphi)\widehat{w} \end{pmatrix} = \begin{pmatrix} \Delta a \\ \Delta b \\ \Delta c \end{pmatrix}$$

may be solved in a TRIANGULAR way

Step 1: solve second equation

$$\widehat{\eta} = \partial_\omega^{-1} \Delta b + \eta_0 \,, \quad \eta_0 \in \mathbb{R}^\nu$$

Remark: Δb has zero average by reversibility, η_0 *fixed later*

Step 2: solve third equation

$$\mathcal{L}_\omega \widehat{w} = \Delta c + \partial_x K_{11}(\varphi)\widehat{\eta} \,, \quad \mathcal{L}_\omega := \omega \cdot \partial_\varphi - \partial_x K_{02}(\varphi) \,,$$

This is a quasi-periodically forced linear KdV operator!

Reduction of the linearized op. on the normal directions

$$\mathcal{L}_\omega h = \Pi_{S^\perp}\Big(\omega\cdot\partial_\varphi h + \partial_{xx}(a_1 \partial_x h) + \partial_x(a_0 h) - \varepsilon^2 \partial_x \mathcal{R}_2[h] - \partial_x \mathcal{R}_*[h]\Big)$$

$$a_1 - 1 := O(\varepsilon^3)\,, \quad a_0 := \varepsilon p_1 + \varepsilon^2 p_2 + \ldots$$

The remainders $\mathcal{R}_2, \mathcal{R}_*$ are finite range (very regularizing!)

Reduce $\mathcal{L}_\omega$ to constant coefficients as in forced case, hence invert it

1. Terms $O(\varepsilon)$, $O(\varepsilon^2)$ are NOT perturbative: $\varepsilon\gamma^{-1}$, $\varepsilon^2\gamma^{-1}$ is large! $\gamma = o(\varepsilon^2)$
2. These terms eliminated by **algebraic** arguments (integrability property of Birkhoff normal form)

Step 3: solve first equation

$$\partial_\omega \widehat{\psi} = K_{20}(\varphi)\widehat{\eta} + K_{11}^T(\varphi)\widehat{w} - \Delta a$$

Since

$$K_{20}(\varphi) = 3\varepsilon^2 Id + o(\varepsilon^2)$$

the matrix K_{20} is invertible and we choose η_0 (the average of $\widehat{\eta}$) so that the right hand side has zero average. Hence

$$\widehat{\psi} = \partial_\omega^{-1}\Big(K_{20}(\varphi)\widehat{\eta} + K_{11}^T(\varphi)\widehat{w} - \Delta a\Big)$$

This completes the construction of an approximate inverse

HAPPY BIRTHDAY !!