LONG TIME BEHAVIOR OF PERIODIC SOLUTIONS TO SCALAR CONSERVATION LAWS IN SEVERAL SPACE DIMENSIONS

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CAUCHY PROBLEM

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{n} \dfrac{\partial f_i(u)}{\partial x_i} = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \end{cases}$$

Periodicity:

$u_0 \in L^\infty(\mathbb{R}^n)$, $u_0(x+e_i) = u_0(x)$, $x \in \mathbb{R}^n$, $e_i = (0,\dots,1,\dots,0)$, $i = 1,\dots,n$

Zero mean:

$$\int_{T^n} u_0(x)\,dx = 0$$

NONLINEARITY

$$Q_\xi = \{ u \in \mathbb{R} : \xi \cdot f''(u) = 0 \}, \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

$0 \in \operatorname{Int} Q_{\xi'}$, $\xi \in \mathbb{Z}^n \setminus \{0\} \Rightarrow \xi \cdot f'(u) = \lambda$, $u \in (-\varepsilon, \varepsilon)$

$\Rightarrow$ space-time periodic solutions $u(x,t) = \varepsilon \sin[2\pi(\xi \cdot x - \lambda t)]$

THEOREM: $0 \notin \operatorname{Int} Q_\xi$, for all $\xi \in \mathbb{Z}^n \setminus \{0\}$, implies

$u(\cdot, t) \to 0$ in $L^p(T^n)$, $1 \le p < \infty$, as $t \to \infty$

EARLIER WORK : $n=1$

LAX (1957) : $Q = \emptyset$

DAFERMOS (1972) : Q contains at most one accumulation point

ENGQUIST & E (1993) : $0 \notin \mathrm{Int}\, Q$

EARLIER WORK: $n>1$

ENGQUIST & E (1993), $n=2$, $\mathrm{Int}\, Q_\xi = \phi$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $f^{(k)}(u) \neq 0$, $k \geq 2$

use supersolutions and subsolutions

CHEN & FRID (1999), $\mathrm{meas}\, Q_\xi = 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$

need $\{u(\alpha x, \alpha t) : \alpha \geq 0\}$ compact in L^1_{loc}

LIONS-PERTHAME-TADMOR (1994)

PANOV (2012), $\mathrm{Int}\, Q_\xi = \phi$, $\xi \in \mathbb{Z}^n \setminus \{0\}$

needs $\{u(\alpha x, \alpha t) : \alpha \geq 0\}$ compact in L^1_{loc}

uses H-measure TARTAR (1990), GERARD (1991)

THEORY OF CAUCHY PROBLEM : KRUZKOV (1970)

$u(x,t)$: space-periodic L^∞ solution with initial value $u_0(x)$

$\bar{u}(x,t)$: space-periodic L^∞ solution with initial value $\bar{u}_0(x)$

$$\int_{T^n} [u(x,t)-\bar{u}(x,t)]^+ \, dx \le \int_{T^n} [u_0(x)-\bar{u}_0(x)]^+ \, dx , \quad t \in \mathbb{R}^+$$

$$\int_{T^n} |u(x,t)-\bar{u}(x,t)| \, dx \le \int_{T^n} |u_0(x)-\bar{u}_0(x)| \, dx , \quad t \in \mathbb{R}^+$$

$$\| u(\cdot,t) \|_{L^\infty(T^n)} \le \| u_0(\cdot) \|_{L^\infty(T^n)} , \quad t \in \mathbb{R}^+$$

$$V(u(\cdot,t), T^n) \le V(u_0(\cdot), T^n) , \quad t \in \mathbb{R}^+$$

MINIMALITY OF ω-LIMIT SET

$$u(\cdot, t_k) \longrightarrow v_0(\cdot) \in \omega(u_0), \text{ in } L^1(T^n), \ k \to \infty$$

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(v)}{\partial x_i} = 0, \quad v(x,0) = v_0(x)$$

$$t_{j_k} > t_k + k, \quad \tau_k = t_{j_k} - t_k$$

$$\int_{T^n} |v(x,\tau_k) - v_0(x)|\, dx \le \int_{T_n} |v(x,\tau_k) - u(x,t_{j_k})|\, dx + \int_{T^n} |v_0(x) - u(x,t_{j_k})|\, dx$$

$$\int_{T^n} |v(x,\tau_k) - u(x, t_k + \tau_k)|\, dx \le \int_{T^n} |v_0(x) - u(x,t_k)|\, dx$$

$$v(\cdot, \tau_k) \longrightarrow v_0(\cdot), \text{ in } L^1(T^n), \ k \to \infty$$

USE OF INTEGRAL GEOMETRY

$$u(\cdot, t_k) \longrightarrow v_0(\cdot), \text{ in } L^1(T^n), \ k \to \infty$$

$$v_0 = 0 \quad \text{a.e.}$$

$$\Leftrightarrow \quad \text{meas}\, V = 0, \quad V = \{x \in T^n : v_0(x) > 0\}$$

$$\Leftrightarrow \quad \varphi = 0 \ \text{ a.e.}, \quad \varphi(x) = \chi_V(x) - \text{meas}\, V$$

$$\Leftrightarrow \quad \int_{P_{\xi,p}} \varphi = 0, \quad P_{\xi,p} = \{x : \xi \cdot x = p\}, \ \xi \in \mathbb{Z}^n \setminus \{0\}, \ p \in \mathbb{R}$$

Radon Transform on Torus

STRICHARTZ (1982), GELFAND-GINDIKIN-GRAEV (2003)

COORDINATES CHANGE

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(v)}{\partial x_i} = 0, \qquad v(x,0) = v_0(x)$$

$\xi \in \mathbb{Z}^n \setminus \{0\}$, $\xi_n \neq 0$, $x \mapsto (y,z)$, $y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$

$$y_i = \frac{1}{\xi_n} x_i, \quad i = 1, \dots, n-1, \qquad z = \frac{1}{\xi_n} x \cdot \xi$$

$$v(x,t) = w(y,z,t), \qquad v_0(x) = w_0(y,z)$$

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial g_i(w)}{\partial y_i} + \frac{\partial h(w)}{\partial z} = 0, \qquad w(y,z,0) = w_0(y,z)$$

$$g_i = \frac{1}{\xi_n} f_i, \quad i = 1, \dots, n-1, \qquad h = \frac{1}{\xi_n} \xi \cdot f$$

THE INTEGRAL OF φ IN THE NEW COORDINATES

$$\int_{P_{\xi,\rho}} \varphi = \xi_n^{n-2} |\xi| \int_{T^{n-1}} \psi\left(y, \frac{1}{\xi_n}\rho\right) dy$$

where: $\psi(y,z) = \chi_W(y,z) - \operatorname{meas} W$

$$W = \{ (y,z) : y \in T^{n-1}, z \in T, w_0(y,z) > 0 \}$$

COMPARISON SOLUTION

$0 \notin \operatorname{Int} Q_{\xi} \Rightarrow h''(u) \neq 0$ for $u \in [\hat{u}-\varepsilon, \hat{u}+\varepsilon] \subset (0,s)$

$$\frac{\partial v}{\partial t} + \frac{\partial h(v)}{\partial z} = 0, \quad v(z,0) = v_0(z)$$

$$v_0(z) = \begin{cases} \hat{u}+\varepsilon, & z \in (a-\delta, a+\delta) \\ \hat{u}-\varepsilon, & z \in (b-\delta, b+\delta) \\ \hat{u}, & \text{otherwise} \end{cases}$$

Define: $\bar{w}(y,z,t) = v(z,t), \quad \bar{w}_0(y,z) = v_0(z)$

$$\bar{w}(y,z,t) \to \hat{u}, \quad \text{as } t \to \infty$$

COMPLETION OF PROOF OF THE THEOREM

$$\int_T \int_{T^{n-1}} [w(y,z,t) - \bar{w}(y,z,t)]^+ \, dy\, dz \le \int_T \int_{T^{n-1}} [w_0(y,z) - \bar{w}_0(y,z)]^+ \, dy\, dz$$

$$\int_T \int_{T^{n-1}} [w_0(y,z) - \hat{u}]^+ \, dy\, dz \le \int_T \int_{T^{n-1}} [w_0(y,z) - v_0(z)]^+ \, dy\, dz$$

$$F_s = \{(y,z) : y \in T^{n-1},\ z \in (a-\delta, a+\delta),\ w_0(y,z) > s\},\quad s \ge 0$$

$$G_s = \{(y,z) : y \in T^{n-1},\ z \in (b-\delta, b+\delta),\ w_0(y,z) > s\} \quad s \ge 0$$

$$\varepsilon \operatorname{meas} F_s \le \varepsilon \operatorname{meas} G_0 \implies \operatorname{meas} F_0 = \operatorname{meas} G_0$$

$$\int_{a-\delta}^{a+\delta} \int_{T^{n-1}} \chi_W(y,z)\, dy\, dz = \int_{b-\delta}^{b+\delta} \int_{T^{n-1}} \chi_W(y,z)\, dy\, dz$$

$$\int_{T^{n-1}} \chi_W(y,a)\, dy = \int_{T^{n-1}} \chi_W(y,b)\, dy$$