High-frequency instabilities of small-amplitude solutions of Hamiltonian PDEs

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Hamiltonian PDEs: Analysis, Computations and Applications
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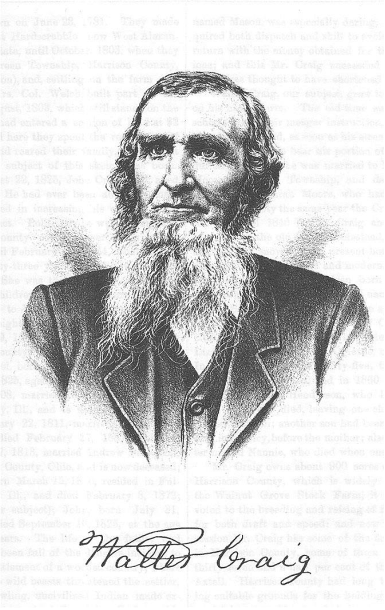
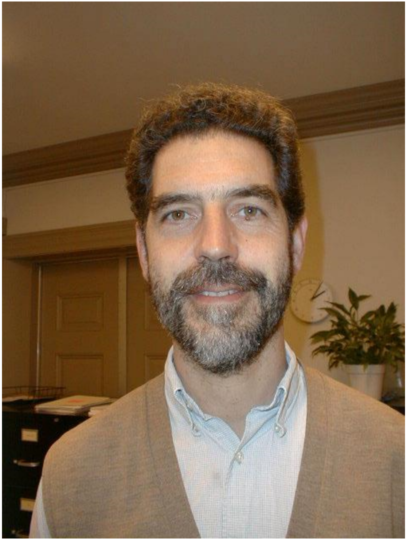
Acknowledgements 1



I have known Walter
a long time

Acknowledgements 1



Actually,

for about 15 years

Acknowledgements 2

- Joint work with Olga Trichtchenko (UW)
- BD and Olga Trichtchenko, *High-frequency instabilities of small-amplitude solutions of Hamiltonian PDEs*, To be submitted, 2014
- Support from the National Science Foundation (NSF-DMS-1008001)

The big picture

Consider the Hamiltonian PDE

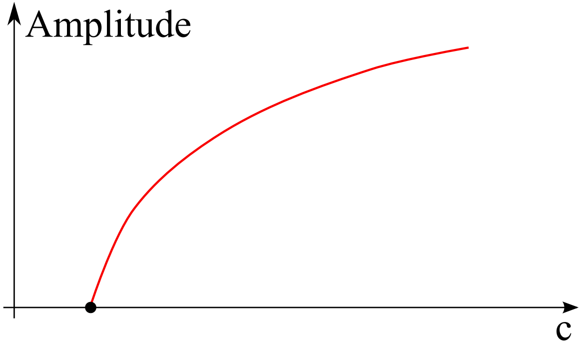
$$u_t = J \frac{\delta H}{\delta u}, \tag{1}$$

posed in a suitable function space of periodic functions. We examine traveling-wave solutions $u(x, t) = U(x - ct)$ of this system. These satisfy

$$-cU_x = J \frac{\delta H}{\delta U}. \tag{2}$$

Assumptions

1. For a range of c values $U = 0$ is a solution of (2).



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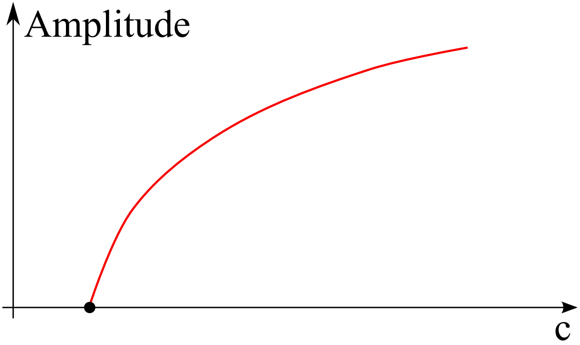

2. The linearization around $u = 0$ of (1) is dispersive.

Digression

- It is possible for linear, constant coefficient Hamiltonian PDEs to be non-dispersive.

  Example.

$$H = \int_0^{2\pi} q_x p_x dx, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}:$$

$$q_t = q_{xx}, \quad p_t = -p_{xx}.$$

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- Is it possible for linear, constant coefficient, dispersive PDEs to be non-Hamiltonian?

The big picture, continued

- As we will see, the $u = 0$ solution is spectrally (neutrally) stable.

The big picture, continued

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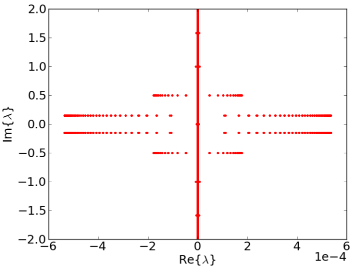
- As we increase the amplitude of the solution, the eigenvalues of the spectral stability problem move continuously in $\mathbb{C}$.

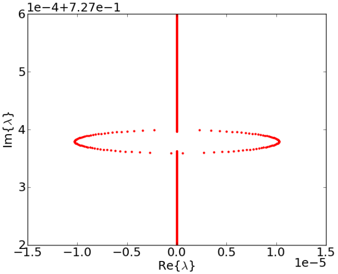
The big picture, continued

- Due to the quadrufold symmetry of the problem, the only way for eigenvalues to leave the imaginary axis is by collision.

The big picture, continued

- Given J and H, we shall establish necessary conditions for eigenvalue collisions to result in eigenvalues off the imaginary axis, resulting in spectral instabilities of small-amplitude traveling wave solutions.





The big picture, continued

- The goal is to obtain conditions that are easily used and verified, at the expense of the precision of the conclusions reached. In other words, the goal is usability over rigor.

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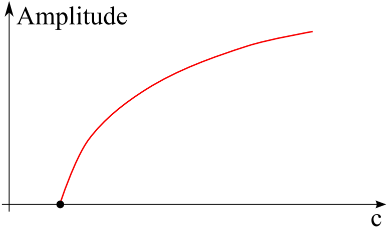
- The goal is to obtain conditions that are easily used and verified, at the expense of the precision of the conclusions reached. In other words, the goal is usability over rigor. Almost all conclusions are formulated in terms of the dispersion relation of the linear problem.

The big picture, continued

- All calculations take place at the bifurcation point of the trivial solution branch. By continuity, any stability conclusion holds for solutions on the bifurcation branch of small, but nonzero amplitude.



The big picture, continued

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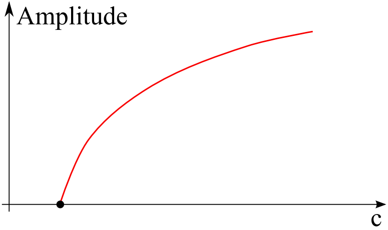

- In effect, the theory is finite dimensional, as only a finite number of eigenvalues participate in a collision.

Some literature

- MacKay & Saffman (1986): a criterion for the onset of instability through the collision of eigenvalues in the water wave problem.
- MacKay (1987): the finite-dimensional case.

Scalar Hamiltonian PDES with $J = \partial_x$

(Examples: KdV, Whitham, ...)

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We consider equations whose linearization is of the form

$$u_t = -i\omega(-i\partial_x)u,$$

where $\omega(k)$ (real valued) is the dispersion relation:

$$\omega(k) = \sum_{n=0}^{\infty} \alpha_n k^{2n+1}, \quad \alpha_j \in \mathbb{R},$$

and

$$H = -\frac{1}{2}\int_0^{2\pi} \sum_{n=0}^{\infty} \alpha_n u_{nx}^2 dx.$$

Note that $\int_0^{2\pi} u dx$ is a Casimir.

Scalar Hamiltonian PDES with $J = \partial_x$

In a moving coordinate frame,

$$\begin{aligned} & & u_t - c u_x &= -i\omega(-i\partial_x)u \\ \Rightarrow & & u_t &= -i\Omega(-i\partial_x)u, \end{aligned}$$

with $\Omega(k) = \omega(k) - kc$.

Scalar Hamiltonian PDES with $J = \partial_x$

Step 1. Bifurcation point. We need a singular Jacobian, requiring

$$\Omega(k) = 0 \quad \Rightarrow \quad c = \frac{\omega(k)}{k},$$

the phase speed.

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Step 1. Bifurcation point. We need a singular Jacobian, requiring

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For periodic solutions, we need $k = N$, integer, so that

$$c = \frac{\omega(N)}{N}.$$

Typically, we choose $N = 1$.

Scalar Hamiltonian PDES with $J = \partial_x$

Step 2. Stability analysis. Let $u(x,t) = e^{\lambda t} U(x) + c.c.$, with

$$U(x) = \sum_{n=-\infty}^{\infty} a_n e^{i(n+\mu)x},$$

with $\mu \in [-1/2, 1/2)$. We get

$$\lambda_n^{\mu} = -i\Omega(n+\mu).$$

- All $\lambda_n^{(\mu)}$ are imaginary. Thus the zero solution is neutrally spectrally stable.

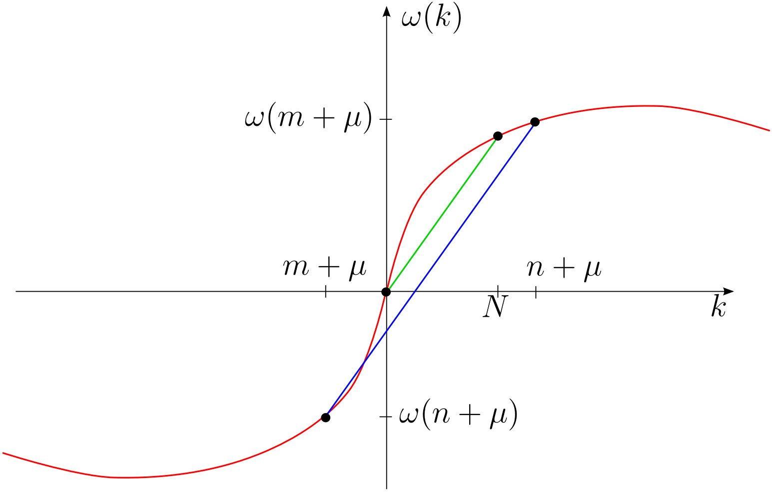
Scalar Hamiltonian PDES with $J = \partial_x$

Step 3. Eigenvalue collisions. We need

$$\lambda_n^{(\mu)} = \lambda_m^{(\mu)}$$

$$\Rightarrow \qquad \frac{\omega(n+\mu) - \omega(m+\mu)}{n-m} = \frac{\omega(N)}{N}.$$

Graphically, this is a condition expressing the equality of two slopes.



$\omega(k)$
$\omega(m+\mu)$
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$n+\mu$
N
k
$\omega(n+\mu)$

Scalar Hamiltonian PDES with $J = \partial_x$

Step 4. Krein signature.

- The contribution to the Hamiltonian from a single mode is $\sim |a_n|^2 \Omega(n+\mu)/(n+\mu)$. The Krein signature of this mode is the sign of this contribution.

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- In order for two colliding eigenvalues to leave the imaginary axis, it is necessary that they have opposite Krein signature.
- After simplification, this requires $mn < 0$.

Scalar Hamiltonian PDES with $J = \partial_x$: Summary

Consider a Hamiltonian PDEs with $J = \partial_x$, whose linearization has the real-valued dispersion relation $\omega(k)$. In order for small-amplitude solutions of period $2\pi N$ to be susceptible to high-frequency instabilities, it is necessary that there exist $m, n \in \mathbb{Z}$ and $\mu \in [-1/2, 1/2)$ such that

- $\lambda_n^{(\mu)} = i(n+\mu)\frac{\omega(N)}{N} - i\omega(n+\mu) \neq 0$.
- (Collision condition)

$$\frac{\omega(n+\mu) - \omega(m+\mu)}{n-m} = \frac{\omega(N)}{N}.$$

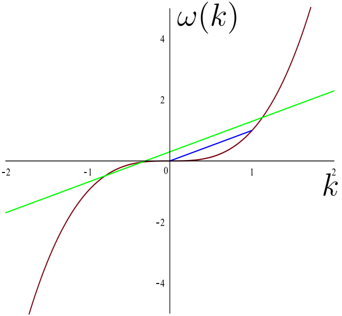
- (Krein signature condition) $mn < 0$.

Example. KdV-like equations.

Consider equations of the form

$$u_t = \partial_x(u_{xx} + N(u)),$$

where $\lim_{\epsilon\to 0} N(\epsilon u)/\epsilon = 0$. Then $\omega = k^3$.



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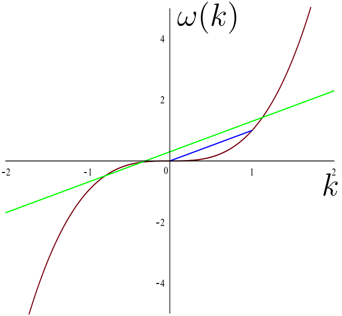
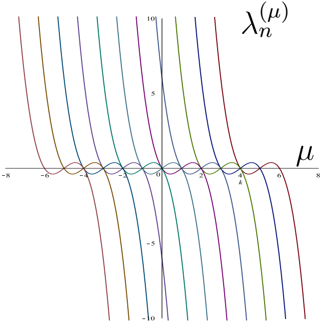




Example. KdV-like equations.

- There are no collisions away from $\lambda = 0$. Thus small-amplitude periodic solutions of KdV-like equations are not susceptible to high-frequency instabilities.
- This result includes KdV, mKdV, generalized KdV, *etc.*

Example. KdV-like equations.

- There are no collisions away from $\lambda = 0$. Thus small-amplitude periodic solutions of KdV-like equations are not susceptible to high-frequency instabilities.
- This result includes KdV, mKdV, generalized KdV, *etc.*
- Solutions of superKdV-like equations **are** susceptible to high-frequency instabilities.

$$u_t = u_{xxx} + \alpha u_{xxxxx} + \text{nonlinear}.$$

2. Two-dimensional Hamiltonian PDEs with canonical J

(Examples: Sine-Gordon, the water wave problem, ...)

Here

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we consider equations of the form

$$q_t = \frac{\delta H}{\delta p}, \quad p_t = -\frac{\delta H}{\delta q}.$$

2-D Hamiltonian PDEs with canonical J

The Hamiltonian of their linearization can be written as

$$H = \int_0^{2\pi} \left(\frac{1}{2} \sum_{j=0}^{\infty} \beta_j p_{jx}^2 + \frac{1}{2} \sum_{j=0}^{\infty} \gamma_j q_{jx}^2 + p \sum_{j=0}^{\infty} \alpha_j q_{jx} \right) dx,$$

so that

$$q_t = \sum_{j=0}^{\infty} \alpha_j q_{jx} + \sum_{j=0}^{\infty} (-1)^j \beta_j p_{2jx},$$

$$p_t = -\sum_{j=0}^{\infty} (-1)^j \gamma_j q_{2jx} - \sum_{j=0}^{\infty} (-1)^j \alpha_j p_{jx}.$$

2-D Hamiltonian PDEs with canonical J

The dispersion relation is given by

$$\det \begin{pmatrix} i\omega + \sum_{j=0}^{\infty} \alpha_j (ik)^j & \sum_{j=0}^{\infty} \beta_j k^{2j} \\ -\sum_{j=0}^{\infty} \gamma_j k^{2j} & i\omega - \sum_{j=0}^{\infty} \alpha_j (-1)^j (ik)^j \end{pmatrix} = 0,$$

which gives $\omega_1(k)$ and $\omega_2(k)$, both real for real k.

In a moving coordinate frame, the Hamiltonian has the extra term $c\int_0^{2\pi} pq_x dx$.

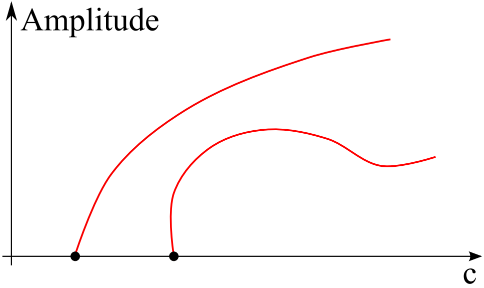
2-D Hamiltonian PDEs with canonical J

Step 1. Bifurcation point.

As before, the bifurcation points from the trivial solution are found by finding for which value of c the Jacobian is singular. This time, there are two solutions.

$$c_{1,2} = \frac{\omega_{1,2}(k)}{k} = \frac{\omega_{1,2}(N)}{N}.$$

since $k \in \mathbb{Z}$, for periodic solutions.



2-D Hamiltonian PDEs with canonical J

Step 2. Stability analysis. Working with the first branch of solutions, we obtain

$$\lambda_{n,j}^{(\mu)} = i(n+\mu)c_1 - i\omega_j(n+\mu),$$

for $j = 1, 2$, $\mu \in [-1/2, 1/2)$, $n \in \mathbb{Z}$.

- All $\lambda_{n,j}^{(\mu)}$ are imaginary. Thus the zero solution is neutrally spectrally stable.

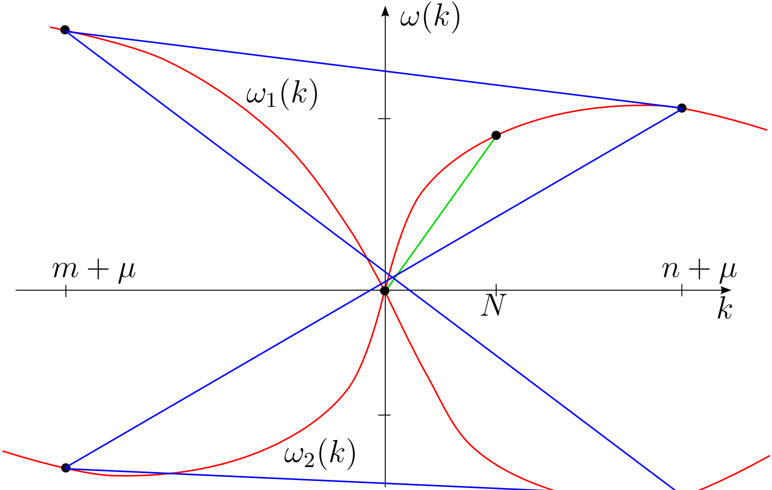
2-D Hamiltonian PDEs with canonical J

Step 3. Eigenvalue collisions. We need

$$\lambda_{n,j_1}^{(\mu)} = \lambda_{m,j_2}^{(\mu)}$$

$$\Rightarrow \qquad \frac{\omega_{j_1}(n+\mu) - \omega_{j_2}(m+\mu)}{n-m} = \frac{\omega_1(N)}{N}.$$

Once more, this is a condition expressing the equality of two slopes.



$\omega(k)$
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2-D Hamiltonian PDEs with canonical J

Step 4. Krein signature.

- The linear system may be written as $u_t = JLu$, where L is the second variation of H. The Krein signature of the v mode may also be computed as the sign of $v^* L v$.

2-D Hamiltonian PDEs with canonical J

Step 4. Krein signature.

- The linear system may be written as $u_t = JLu$, where L is the second variation of H. The Krein signature of the v mode may also be computed as the sign of $v^* L v$.
- For our setting, one finds that the signature of the eigenmode $(Q_{n,j}^{(\mu)}, P_{n,j}^{(\mu)})^T$ is the sign of

$$\lambda_{n,j}^{(\mu)} \det \begin{pmatrix} Q_{n,j}^{(\mu)} & P_{n,j}^{(\mu)} \\ {Q_{n,j}^{(\mu)}}^* & {P_{n,j}^{(\mu)}}^* \end{pmatrix}.$$

2-D Hamiltonian PDEs with canonical J

- Explicitly, the necessary condition for opposite Krein signatures is

$$\sum_{j=0}^{\infty} \gamma_j (n+\mu)^{2j} \sum_{j=0}^{\infty} \gamma_j (m+\mu)^{2j} \times$$
$$\left(\omega_{j_1}(n+\mu) + \sum_{j=0}^{\infty} \alpha_{2j+1} (-1)^j (n+\mu)^{2j+1} \right) \times$$
$$\left(\omega_{j_2}(m+\mu) + \sum_{j=0}^{\infty} \alpha_{2j+1} (-1)^j (m+\mu)^{2j+1} \right) < 0.$$

2-D Hamiltonian PDEs with canonical J: Summary

Consider a Hamiltonian PDEs with canonical J, whose linearization has the quadratic Hamiltonian
$H = \int_0^{2\pi} \left(\frac{1}{2} \sum_{j=0}^{\infty} \beta_j p_{jx}^2 + \frac{1}{2} \sum_{j=0}^{\infty} \gamma_j q_{jx}^2 + p \sum_{j=0}^{\infty} \alpha_j q_{jx} \right) dx$
with real-valued dispersion relations $\omega_{1,2}(k)$. In order for small-amplitude solutions of period $2\pi N$ to be susceptible to high-frequency instabilities, it is necessary that there exist $j_{1,2} \in (1,2)$, $m, n \in \mathbb{Z}$ and $\mu \in [-1/2, 1/2)$ such that

- (Collision condition)

$$\frac{\omega_{j_1}(n+\mu) - \omega_{j_2}(m+\mu)}{n-m} = \frac{\omega(N)}{N}.$$

- (Krein signature condition) See previous slide.

Example. The water wave problem.

The linearized water wave problem is

$$\begin{aligned}\eta_t &= -i\tanh(-ih\partial_x)q_x,\\ q_t &= -g\eta,\end{aligned}$$

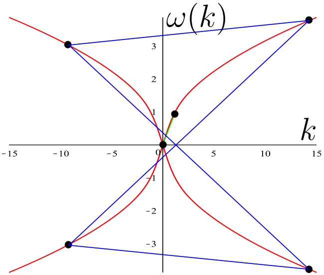
with

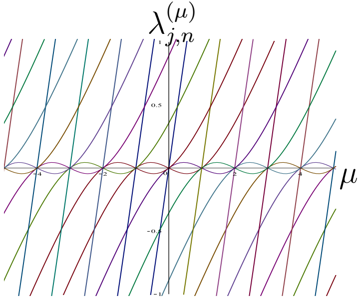
$$H = \int_0^{2\pi}\left(\frac{1}{2}q(-i\tanh(-ih\partial_x)q_x) + \frac{1}{2}g\eta^2\right)dx,$$

and

$$\omega^2 = gk\tanh(kh).$$

Example. The water wave problem.





$\rightarrow$ there are collisions!

Example. The water wave problem.

- The Krein condition gives $\omega_{j_1}\omega_{j_2}g^2 < 0$, which is always satisfied.

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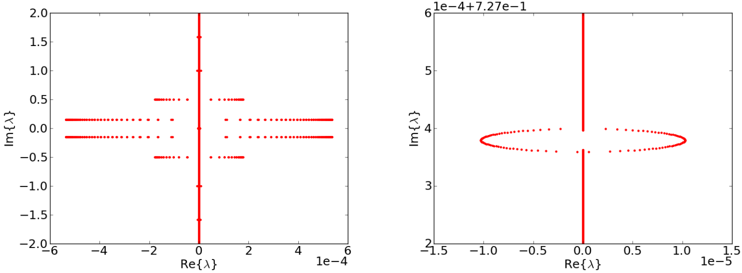

Example. The Whitham equation vs. the water wave problem.

Consider

$$u_t + \partial_x N(u) + \int_{-\infty}^{\infty} K(x-y) u_y(y,t)\,dy = 0$$

$$\Rightarrow \quad u_t + \partial_x \left(N(u) + \partial_x \int_{-\infty}^{\infty} K(x-y) u(y,t)\,dy \right) = 0,$$

where

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx}\,dk,$$

with $c(k) = \omega(k)/k = \sqrt{g \tanh(kh)/k}$.

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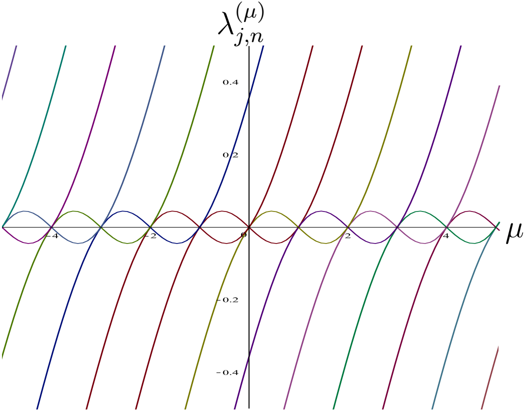
where

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dk,$$

with $c(k) = \omega(k)/k = \sqrt{g \tanh(kh)/k}$. The Hamiltonian of the linearized equation is

$$H = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x-y) u(x,t) u(y,t) dx dy.$$

Example. The Whitham equation



- There are no collisions (except at $\lambda = 0$).
- The Whitham equation does not capture the high-frequency instabilities of small-amplitude solutions of the water wave problem.

Thank you!

Questions?

Happy Birthday, Walter!