Water Waves with vorticity

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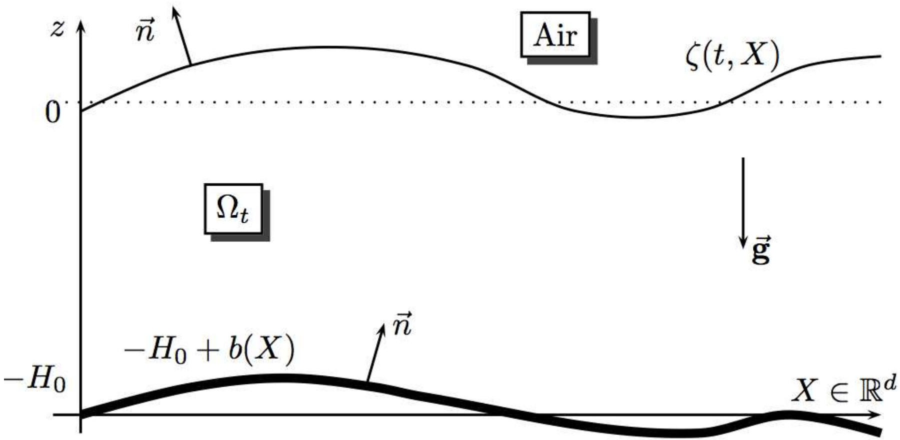
Hamiltonian PDEs: Analysis, Computations and Applications


z
$\vec{n}$
Air
$\zeta(t, X)$
0
Ω_t
$\vec{\mathbf{g}}$
$\vec{n}$
$-H_0 + b(X)$
$-H_0$
$X \in \mathbb{R}^d$

(H1) The fluid is homogeneous and inviscid
(H2) The fluid is incompressible
(H3) The flow is irrotational
(H4) The surface and the bottom can be parametrized as graphs
(H5) The fluid particles do not cross the bottom
(H6) The fluid particles do not cross the surface
(H7) There is no surface tension and the external pressure is constant.
(H8) The fluid is at rest at infinity
(H9) The water depth does not vanish

(H1) $\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla_{X,z})\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}P - g\mathbf{e_z}$ in Ω_t

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3. The equations can be put under the canonical Hamiltonian form

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{grad}_{\zeta,\psi} H$$

   with the Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \int_{\Omega} |\mathbf{U}|^2$$

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Remark. One has the exact relation

$$G[\zeta,b]\psi = -\nabla\cdot(h\overline{V}) \quad \text{with } h = H_0+\zeta-b \text{ and } \overline{V} = \frac{1}{h}\int_{-H_0+b}^{\zeta} V(X,z)dz$$

- Equation on ψ. We use (H1)" and (H7)"

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$$\boxed{\partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0.}$$

- Equation on ψ. We use (H1)" and (H7)"

$$\partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) \qquad \text{AND} \qquad P_{|_{z=\zeta}} = P_{atm}$$

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The Zakharov-Craig-Sulem equations

$$\begin{cases} \partial_t\zeta - G[\zeta,b]\psi = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{cases}$$

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Derive simpler asymptotic models describing the solutions to the water waves equations in shallow water.

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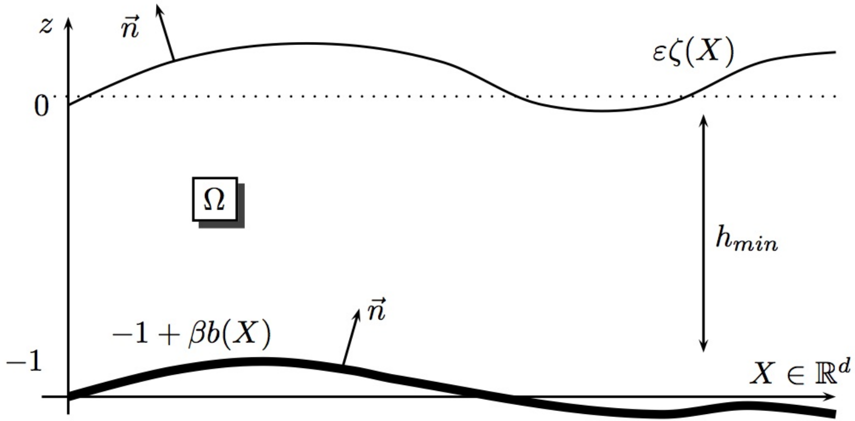
- For the sake of simplicity, we consider here a flat bottom ($b = 0$).
- We introduce three characteristic scales
  1. The characteristic water depth H_0
  2. The characteristic horizontal scale L
  3. The order of the free surface amplitude a
- Two independent dimensionless parameters can be formed from these three scales. We choose:

$$\frac{a}{H_0} = \varepsilon \quad \text{(amplitude parameter)},$$

$$\frac{H_0^2}{L^2} = \mu \quad \text{(shallowness parameter)}.$$

We proceed to the simple nondimensionalizations

$$X' = \frac{X}{L}, \qquad z' = \frac{z}{H_0}, \qquad \zeta' = \frac{\zeta}{a}, \quad \text{etc.}$$



$$\begin{cases} \partial_t \zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t \nabla\psi + \nabla\zeta + \frac{\varepsilon}{2}\nabla|\nabla\psi|^2 - \varepsilon\mu\nabla\frac{(-\nabla\cdot(h\overline{V}) + \nabla(\varepsilon\zeta)\cdot\nabla\psi)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = 0, \end{cases}$$

where in dimensionless form

$$h = 1 + \varepsilon\zeta \quad \text{and} \quad \overline{V} = \frac{1}{h}\int_{-1}^{\varepsilon\zeta} V(x,z)dz.$$

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$$\text{Saint-Venant} \qquad \begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t \overline{V} + \nabla\zeta + \varepsilon \overline{V} \cdot \nabla \overline{V} = 0. \end{cases}$$

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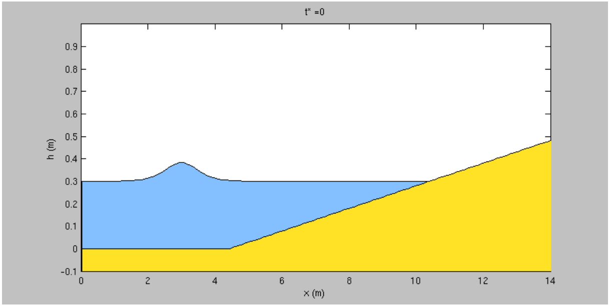
- All this procedure can be fully justified (cf Walter Craig for KdV !)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x (m)
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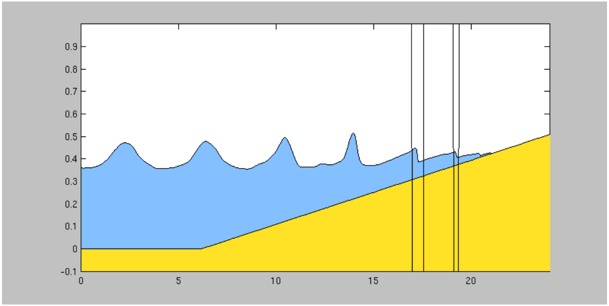
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
2
4
6
8
10
12
14

"


0.9
0.8
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0.2
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0
-0.1
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5
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15
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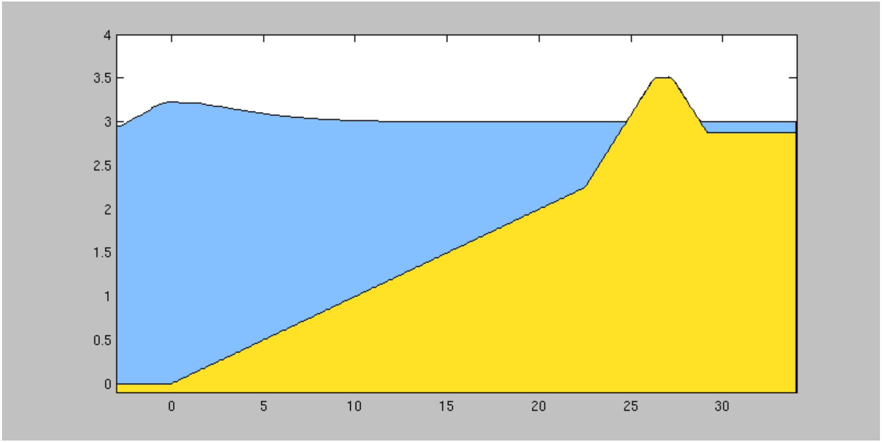

"

Bonneton, Chazel, L. , Marche, Tissier 2011-2012

2D configurations can also be handled (D.L. & F. Marche, 2014):

- Tsunami island

2D configurations can also be handled (D.L. & F. Marche, 2014):

- Beach

2D configurations can also be handled (D.L. & F. Marche, 2014):

- Overtopping


Bonneton (CD2013) – UMR EPOC – Photo. Y. Lavigne


Bonneton (CD2013) – UMR EPOC – Photo. Y. Lavigne

- (H1) The fluid is homogeneous and inviscid
- (H2) The fluid is incompressible
- (H3) ~~The flow is irrotational~~
- (H4) The surface and the bottom can be parametrized as graphs above the still water level
- (H5) The fluid particles do not cross the bottom
- (H6) The fluid particles do not cross the surface
- (H7) There is no surface tension and the external pressure is constant.
- (H8) The fluid is at rest at infinity
- (H9) The water depth is always bounded from below by a nonnegative constant

Refs: Lindblad, Coutand-Shkoller, Shatah-Zeng, Zhang-Zhang, Masmoudi-Rousset, ...

$$
\begin{aligned}
\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} &= -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z, \\
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\end{aligned}
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Irrotational case

- We get from curl $\mathbf{U} = 0$ that $\mathbf{U} = \nabla_{X,z} \Phi$

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- We get from curl $\mathbf{U} = 0$ that $\mathbf{U} = \nabla_{X,z}\Phi$
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$$\partial_t \Phi + \frac{1}{2} |\nabla_{X,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm})$$

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- We eliminate the pressure by taking the trace on the interface
- We reduce the problem to an equation on ζ and $\psi(t,X) = \Phi(t,X,\zeta(t,x))$.

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Rotational case

One has curl $\mathbf{U} = \omega \neq 0$ and

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  $\rightsquigarrow$ How can we use the boundary condition on the pressure P?

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- One cannot work with the Benouilli equation
  $\rightsquigarrow$ How can we use the boundary condition on the pressure P?
- One can remark that

$$\begin{aligned} (\nabla_{X,z}P)_{|_{z=\zeta}} &= \begin{pmatrix} \nabla(P_{|_{z=\zeta}}) \\ 0 \end{pmatrix} + N\partial_z P_{|_{z=\zeta}} \\ &= 0 + N\partial_z P_{|_{z=\zeta}} \end{aligned}$$

One has

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⇝ This leads to an equation on the tangential part of the velocity at the surface

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Notation

With $\underline{U} = (\underline{V}, \underline{w}) = \mathbf{U}_{|_{z=\zeta}}$, we write

$$U_{\parallel} = \underline{V} + \underline{w} \nabla\zeta \qquad \text{so that} \qquad \underline{U} \times N = \begin{pmatrix} -U_{\parallel}^{\perp} \\ -U_{\parallel}^{\perp} \cdot \nabla\zeta \end{pmatrix}$$

$$\left\{ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \right\}_{|z=\zeta} \times N$$

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$\Downarrow$ (with some computations)

$$\partial_t U_\parallel + g \nabla \zeta + \frac{1}{2} \nabla |U_\parallel|^2 - \frac{1}{2} \nabla \big((1 + |\nabla \zeta|^2) \underline{w}^2 \big) + \underline{\omega} \cdot N \underline{V}^\perp = 0.$$

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$$\Downarrow \quad \text{(with some computations)}$$

$$\partial_t U_{\|} + g\nabla\zeta + \frac{1}{2}\nabla|U_{\|}|^2 - \frac{1}{2}\nabla\big((1+|\nabla\zeta|^2)\underline{w}^2\big) + \underline{\omega}\cdot N\underline{V}^{\perp} = 0.$$

What does it give in the irrotational case?

$$\left\{ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \right\}_{|_{z=\zeta}} \times N$$

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In the irrotational case, one has

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What does it give in the irrotational case?

In the irrotational case, one has

$$U_{\parallel} = \nabla\psi.$$

How do we generalize to the rotational case?

We decompose $U_{\parallel}$ into

$$\boxed{U_{\parallel} = \nabla\psi + \nabla^{\perp}\widetilde{\psi}}$$

We have found

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- This is done by applying $\frac{\text{div}}{\Delta}$ and $\frac{\text{div}^\perp}{\Delta}$ to the equation
- The "orthogonal gradient" component yields

$$\boxed{\partial_t(\underline{\omega}\cdot N - \nabla^\perp\cdot U_\parallel) = 0},$$

which is trivially true and does not bring any information

$$\partial_t U_\parallel + g\nabla\zeta + \frac{1}{2}\nabla|U_\parallel|^2 - \frac{1}{2}\nabla\big((1+|\nabla\zeta|^2)\underline{w}^2\big) + \underline{\omega}\cdot N\underline{V}^\perp = 0.$$

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- The "gradient" component of the equation is obtained by applying $\frac{\text{div}}{\Delta}$. After remarking that

$$\frac{\text{div}}{\Delta}U_\parallel = \frac{\text{div}}{\Delta}(\nabla\psi + \nabla^\perp\widetilde{\psi})$$

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$$\partial_t U_\parallel + g\nabla\zeta + \frac{1}{2}\nabla|U_\parallel|^2 - \frac{1}{2}\nabla\big((1+|\nabla\zeta|^2)\underline{w}^2\big) + \underline{\omega}\cdot N\underline{V}^\perp = 0.$$

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(\mathrm{ZCS}) \quad \begin{cases} \partial_t \zeta - \underline{U} \cdot N = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\underline{U} \cdot N + \nabla\zeta \cdot \nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0 \\ \omega = 0. \end{cases}
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- We are therefore led to solve

$$\begin{cases} \text{curl } \mathbf{U} = & \omega & \text{in} \quad \Omega \\ \text{div } \mathbf{U} = & 0 & \text{in} \quad \Omega \\ U_\parallel = & \nabla\psi + \nabla^\perp\Delta^{-1}(\underline{\omega}\cdot N) & \text{at the surface} \\ \mathbf{U}_{|_{z=-H_0}}\cdot N_b = & 0 & \text{at the bottom.} \end{cases}$$

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Proposition

For all $\boldsymbol{\omega} \in H_b(\mathrm{div}_0, \Omega)$ and all $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$,
(1) There is a unique solution $\mathbf{U} \in H^1(\Omega)^{d+1}$ to the div-curl problem, and

$$\|\mathbf{U}\|_2 + \|\nabla_{X,z}\mathbf{U}\|_2 \leq C(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}})\big(\|\boldsymbol{\omega}\|_{2,b} + |\nabla\psi|_{H^{1/2}}\big).$$

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For all $\boldsymbol{\omega} \in H_b(\text{div}_0, \Omega)$ *and all* $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$*,*
(2) The solution $\mathbf{U}$ *can be written* $\mathbf{U} = \text{curl } \mathbf{A} + \nabla_{X,z}\Phi$ *with*

$$\begin{cases} \text{curl curl } \mathbf{A} & = \boldsymbol{\omega} & \text{in } \Omega, \\ \text{div } \mathbf{A} & = 0 & \text{in } \Omega, \\ N_b \times \mathbf{A}_{|_{\rm bott}} & = 0 & \\ N \cdot \mathbf{A}_{|\rm surf} & = 0 & \\ (\text{curl } \mathbf{A})_{\parallel} & = \nabla^{\perp}\Delta^{-1}\underline{\omega}\cdot N, & \\ N \cdot \text{curl } \mathbf{A}_{|_{\rm bott}} & = 0, & \end{cases}$$

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(2) [...] while $\Phi \in \dot{H}^1(\Omega)$ *solves*

$$\begin{cases} \Delta_{X,z}\Phi = 0 \quad \text{in} \quad \Omega, \\ \Phi_{|_{z=\varepsilon\zeta}} = \psi, \qquad \partial_n\Phi_{|_{z=-1+\beta b}} = 0. \end{cases}$$

Proof.

$$\begin{cases} \text{curl curl } \mathbf{A} &= \boldsymbol{\omega} \\ N_b \times \mathbf{A}_{|_{z=-H_0}} &= 0 \\ N \cdot \mathbf{A}_{|_{z=\zeta}} &= 0 \\ \big(\text{curl } \mathbf{A}_{|_{z=\zeta}}\big)_{\parallel} &= \nabla^{\perp}\widetilde{\psi}. \end{cases}$$

Step 4. Solving $\Delta\tilde{\psi} = \underline{\omega} \cdot N$ in $\dot{H}^{1/2}(\mathbb{R}^d)$.

- Use Lax-Milgram in $\dot{H}^1(\mathbb{R}^d)$ to solve the variational formulation of the equation: for all $v \in \dot{H}^1(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla v \cdot \nabla\tilde{\psi} &= \int_{\mathbb{R}^d} \underline{\omega} \cdot N v \\ &= \int_{\mathbb{R}^d} \omega_b \cdot N v^{\mathrm{ext}}_{|_{z=-1+\beta b}} - \int_{\Omega} \boldsymbol{\omega} \cdot \nabla_{X,z} v^{\mathrm{ext}} \\ &\leq \underbrace{\big(\big||D|^{-1}\omega_b \cdot N_b\big|_{H^{1/2}} + \|\boldsymbol{\omega}\|_2\big)}_{:=\|\boldsymbol{\omega}\|_{2,b}} |\nabla v|_2 \end{aligned}$$

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Corollary

This is a closed system of equations in $(\zeta, \psi, \boldsymbol{\omega})$.

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Quasilinearization

- The “good unknown” is natural for the study of free boundary problems (Alinhac). Here

$$\partial_k \partial^\beta \psi \rightsquigarrow \underline{U}_{(\beta)\|} \cdot \mathbf{e}_k \quad \text{with} \quad U_{(\beta)} = \partial^\beta U - \text{"}\partial^\alpha \zeta \partial_z U\text{"}$$

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- Differentiating the equations we get

$$\begin{aligned}
(\partial_t + \underline{V} \cdot \nabla)\partial^\alpha \zeta - \partial_k \underline{U}_{(\beta)} \cdot N &\sim 0, \\
(\partial_t + \underline{V} \cdot \nabla)(\underline{U}_{(\beta)\|} \cdot \mathbf{e}_k) + \mathfrak{a} \partial^\alpha \zeta &\sim 0, \\
(\partial_t^\sigma + U \cdot \nabla_{X,z}^\sigma)\partial^\beta \omega &\sim 0
\end{aligned}$$

with $\mathfrak{a} = g + (\partial_t + \underline{V} \cdot \nabla)\underline{w}$ and $\partial^\alpha = \partial_k \partial^\beta$.

A priori estimates

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For all $|\alpha| \leq N$ ($N \geq 5$), we get

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with

$$\mathcal{E}^N(\zeta,\psi,\omega)\sim|\zeta|_{H^N}^2 + \sum_{0<|\alpha|\leq N}|\nabla\psi_{(\alpha)}|_{H^{-1/2}}^2 + \|\omega\|_{H^{N-1}}^2 + |\omega_b\cdot N_b|_{H_0^{-1/2}}.$$

and $\psi_{(\alpha)} = \partial^\alpha\psi - \underline{w}\partial^\alpha\zeta$.

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Theorem (Angel Castro, D. L. 2014)

The $(ZCS)_{\rm gen}$ equations are locally well posed in the energy space associated to $\mathcal{E}^N$ with $N \geq 5$.

In the irrotational case, we know since Zakharov,

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = J \mathrm{grad}_{\zeta,\psi} H \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and with the Hamiltonian

$$H = \frac{1}{2} \int_{\mathbb{R}^d} g\zeta^2 + \int_{\Omega} |\mathbf{U}|^2.$$

Can this be generalized to our new formulation with vorticity?

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$$H = \frac{1}{2} \int_{\mathbb{R}^d} g\zeta^2 + \int_\Omega |\mathbf{U}|^2.$$

Theorem (Angel Castro, D. L. 2014)

Let us define the Fréchet manifold

$$\mathcal{M} = \{(\zeta, \psi, \boldsymbol{\omega}), H_0 + \zeta > h_{\min}, \operatorname{div} \boldsymbol{\omega} = 0 \textit{ in } \Omega_\zeta, \ |D|^{-1}\omega_b \cdot N_b \in H^\infty\}.$$

In the irrotational case, we know since Zakharov,

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There exists a mapping $J : T^*\mathcal{M} \to T\mathcal{M}$, antisymmetric for the $T^*\mathcal{M} - T\mathcal{M}$ duality product, and such that the equations can be written

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Corollary

The equations are equivalent to the Hamiltonian equation

$$\forall F \in \mathcal{A}, \qquad \dot{F} = \{F, H\},$$

where the Poisson bracket $\{\cdot,\cdot\}$ *is defined as*

$$\{F,G\} = \int_{\mathbb{R}^d} \frac{\delta F}{\delta \zeta}\frac{\delta G}{\delta \psi} - \frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \zeta} - \int_{\mathbb{R}^d} \underline{\omega}_h \cdot \Big[\frac{\delta F}{\delta \psi}\frac{\nabla^\perp}{\Delta}\frac{\delta G}{\delta \psi} - \frac{\delta G}{\delta \psi}\frac{\nabla^\perp}{\Delta}\frac{\delta F}{\delta \psi}\Big]$$
$$+ \int_\Omega (\mathrm{curl}\ \frac{\delta F}{\delta \boldsymbol{\omega}}) \cdot (\boldsymbol{\omega} \times \mathrm{curl}\ \frac{\delta G}{\delta \boldsymbol{\omega}}).$$

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6. $(h, \overline{V})$ satisfy the same equations as in the irrotational case
7. One finds an equation for Q

$$(\partial_t + \overline{V}\cdot\nabla)Q + \overline{V}\cdot\nabla Q = 0.$$

$$\begin{cases} \partial_t \zeta + \nabla(h\overline{V}) = 0, \\ \partial_t(h\overline{V}) + \nabla\cdot(h\overline{V}\otimes\overline{V}) + h\nabla\zeta = 0 \end{cases}$$

The velocity at the surface if then given by

$$V_{|_{z=\varepsilon\zeta}} = \overline{V} + \sqrt{\mu}Q, \qquad \text{with} \qquad (\partial_t + \overline{V}\cdot\nabla)Q + \overline{V}\cdot\nabla Q = 0.$$

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To do list:

- Higher order model (Green-Naghdi)
- Horizontal vorticity generation
- Vorticity generation by shocks
- Numerical implementation and experimental check

Happy birthday Walter!