3-wave resonant interactions (an integrable Hamiltonian system)

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A. What are the 3-wave equations?

ODEs: $u_1(\tau), u_2(\tau), u_3(\tau)$ are complex-valued

$$\frac{du_1}{d\tau} = i\, u_2^* u_3^*, \quad \frac{du_2}{d\tau} = i\, u_3^* u_1^*, \quad \frac{du_3}{d\tau} = i\, u_1^* u_2^*.$$

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Simplest nonlinear interaction for dispersive waves without dissipation.

$$\vec{k}_1 = \vec{k}_2 \pm \vec{k}_3 \quad \omega_1 = \omega_2 \pm \omega_3$$

Explosive case – solutions blow up in finite time.

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PDEs: $u_1(\mathrm{x},\tau), u_2(\mathrm{x},\tau), u_3(\mathrm{x},\tau)$ are complex-valued

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*, \quad \partial_\tau u_2 + c_2 \partial_x u_2 = i\, u_3^* u_1^*,$$

$$\partial_\tau u_3 + c_3 \partial_x u_3 = i\, u_3^* u_1^*.$$

Hamiltonian structure

ODEs: $$\frac{du_j}{d\tau} = i\, u_k^* u_l^*, \quad \frac{du_j^*}{d\tau} = -i\, u_k u_l, \quad \{j,k,l\} \text{ cyclic}$$

$$j = 1,2,3,\;\; p_{\mathrm{j}} = u_{\mathrm{j}},\; q_{\mathrm{j}} = u_{\mathrm{j}}^*,\;\; H = i(u_1 u_2 u_3 + u_1{}^* u_2{}^* u_3{}^*)$$

PDEs: $$\partial_\tau u_j + c_j \partial_x u_j = i\, u_k^* u_l^*,$$

$$H = \int_a^b \left[\frac{1}{2} \sum_{j=1}^{3} c_j (u_j \partial_x u_j^* - u_j^* \partial_x u_j) + i(u_1 u_2 u_3 + u_1^* u_2^* u_3^*) \right] dx$$

B. Completely integrable

1. Liouville: A Hamiltonian system of ODEs is *completely integrable* if there exists a canonical transformation: $\{p_j, q_j, H\}$ –> $\{P_j, Q_j, K\}$, with $K = K(P_j$, only). Then

$$\frac{dQ_j}{d\tau} = \frac{\partial K}{\partial P_j}, \quad \frac{dP_j}{d\tau} = -\frac{\partial K}{\partial Q_j} = 0.$$

$P_j(\tau)$ = const. ("action" variables),

$Q_j(\tau) = Q_j(0) + \omega_j \tau$ ("angle" variables).

Completely integrable

2 .Zakharov & Faddeev (1971): A Hamiltonian set of PDEs in *completely integrable* under the same conditions, but now in an infinite-dimensional phase space.

Solving the PDE by the method of inverse scattering amounts to discovering a canonical transformation to action-angle variables

Completely integrable

For both ODEs and PDEs, some Hamiltonian systems are completely integrable, and some are not.

Q: How to identify which systems are integrable, and which are not?

A1: Find the transformation to action/angle variables (if one exists)

A2: Find the appropriate Lax pair (if one exists)

A3: Test for the Painlevé property

C. Painlevé property

1. ***Def'n***: A system of ODEs (Hamiltonian or not) has the *Painlevé property* if in every solution, every singularity in the complex τ-plane is a pole .

2. Kovalevskaya (1888) used this property (before Painlevé) to find **all** known integrable cases of the eq'ns for the motion of a frictionless top about a fixed point.

(This suggests relation between the Painlevé property and integrability.)

C. Painlevé property

3.The *Painlevé conjecture* (1978)

A system of nonlinear PDEs is solvable by the method of inverse scattering only if every set of ODEs obtain from the PDE by an exact reduction has the Painlevé property, perhaps after a change of variables.

4. *Current status*: I know of no general proof that the Painlevé property picks out **all** integrable systems. It is known to be a very effective tool.

D. What's actually new in this talk?

Proposal: A Painlevé-type singularity analysis can be used to find the (almost) general solution of an integrable PDE.

If the proposal works out, then this is an alternative to finding Lax pairs and doing inverse scattering.

(What follows is a progress report, in two parts, on implementing this proposal.)

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Prototype problems: 3-wave ODEs and PDEs

- ODEs: Integrability is well known. The Laurent series for the solution reproduces the known results.
- PDEs: The Laurent series for the ODEs generalizes naturally, and produce the Laurent series for the PDEs, and its (almost) general solution.

E. Integrability of 3-wave ODEs

ODEs: 3 complex-valued (<–> 6 real equations)

$$\frac{du_1}{d\tau} = i\, u_2^* u_3^*, \quad \frac{du_2}{d\tau} = i\, u_3^* u_1^*, \quad \frac{du_3}{d\tau} = i\, u_1^* u_2^*.$$

We want 3 "action variables", and 3 "angles".

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We want 3 "action variables", and 3 "angles".

Find 2 Manley-Rowe constants:

$$K_2 = |u_1|^2 - |u_2|^2 = const, \quad K_3 = |u_1|^2 - |u_3|^2 = const$$

Hamiltonian:

$$H = i\, u_1 u_2 u_3 + i\, u_1^* u_2^* u_3^* = const$$

E. Integrability of 3-wave ODEs

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3 actions:

$$K_1 = |u_1|^2 - |u_3|^2 = const, \quad K_2 = |u_2|^2 - |u_3|^2 = const$$

$$H = i\, u_1 u_2 u_3 + i\, u_1^* u_2^* u_3^* = const$$

3 angles:

$$\arg(u_1) = \theta_1, \quad \arg(u_2) = \theta_2, \quad \tau \to (\tau - \tau_0)$$

Painlevé analysis of 3-wave ODEs

$$\frac{du_1}{d\tau} = i\, u_2^* u_3^*, \quad \frac{du_2}{d\tau} = i\, u_3^* u_1^*, \quad \frac{du_3}{d\tau} = i\, u_1^* u_2^*.$$

Hypothesis: Any singularity of these ODEs is a pole (of order 1). The complete Laurent series of the solution should provide an expression with 6 free constants.

$$u_1(\tau) = \frac{e^{i\theta_1}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_1 + (\tau_0 - \tau)^2 \beta_1 + (\tau_0 - \tau)^3 \delta_1 + \ldots\right]$$

$$u_2(\tau) = \frac{e^{i\theta_2}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_2 + (\tau_0 - \tau)^2 \beta_2 + (\tau_0 - \tau)^3 \delta_2 + \ldots\right]$$

$$u_3(\tau) = \frac{e^{i\theta_3}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_3 + (\tau_0 - \tau)^2 \beta_3 + (\tau_0 - \tau)^3 \delta_3 + \ldots\right]$$

Painlevé analysis of 3-wave ODEs

$$\frac{du_1}{d\tau} = i\, u_2^* u_3^*, \quad \frac{du_2}{d\tau} = i\, u_3^* u_1^*, \quad \frac{du_3}{d\tau} = i\, u_1^* u_2^*.$$

$$u_j(\tau) = \frac{e^{i\theta_j}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_j + (\tau_0 - \tau)^2 \beta_j + (\tau_0 - \tau)^3 \delta_j + \dots\right] \; j = 1,2,3$$

- Eqn's are autonomous => (τ_0) is a free constant
- $(\tau_0 - \tau)^{-2}$: need $\theta_1 + \theta_2 + \theta_3 = \pi/2$.

=> 3 arbitrary angle variables $\{\tau_0, \theta_1, \theta_2\}$.

Painlevé analysis of 3-wave ODEs

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- $(\tau_0 - \tau)^{-2}$: need $\theta_1 + \theta_2 + \theta_3 = \pi/2$.

  => 3 arbitrary angle variables $\{\tau_0, \theta_1, \theta_2\}$.
- $(\tau_0 - \tau)^{-1}$: need $\alpha_1 = \alpha_2 = \alpha_3 = 0$
- $(\tau_0 - \tau)^{0}$: need $Re(\beta_1 + \beta_2 + \beta_3) = 0$

  => 2 arbitrary action variables $\{Re(\beta_2), Re(\beta_3)\}$

Painlevé analysis of 3-wave ODEs

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$$u_j(\tau) = \frac{e^{i\theta_j}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_j + (\tau_0 - \tau)^2 \beta_j + (\tau_0 - \tau)^3 \delta_j + \ldots\right] \; j = 1,2,3$$

=> 3 arbitrary angle variables $\{\tau_0, \theta_1, \theta_2\}$.

- $(\tau_0 - \tau)^0$: need $Re(\beta_1 + \beta_2 + \beta_3) = 0$

  => 2 arbitrary action variables $\{Re(\beta_2), Re(\beta_3)\}$

  $$Im(\beta_1) = Im(\beta_2) = Im(\beta_3) = 0$$

- $(\tau_0 - \tau)^1$: need $Im(\delta_1) = Im(\delta_2) = Im(\delta_3) = free$

Painlevé analysis of 3-wave ODEs

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$$u_j(\tau) = \frac{e^{i\theta_j}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_j + (\tau_0 - \tau)^2 \beta_j + (\tau_0 - \tau)^3 \delta_j + \ldots\right] \; j = 1,2,3$$

- 3 arbitrary angle variables $\{\tau_0, \theta_1, \theta_2\}$.
- 3 arbitrary action variables $\{Re(\beta_1), Re(\beta_2), Im(\delta_j)\}$
- $\{Re(\beta_2), \mathrm{Re}(\beta_3)\} \Leftrightarrow \{K_2, K_3\}, \quad Im(\delta_j) \Leftrightarrow H$
- At every order beyond $(\tau_0 - \tau)^1$, all coefficients are determined
- Convergence of series? (see Ruth Martin)

F. Integrability of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*, \quad \partial_\tau u_2 + c_2 \partial_x u_2 = i\, u_3^* u_1^*,$$

$$\partial_\tau u_3 + c_3 \partial_x u_3 = i\, u_3^* u_1^*.$$

Zakharov, Manakov, 1973 – found Lax pair

Zakharov, Manakov, 1976 – worked out inverse scattering on whole line

Kaup, 1976 – inverse scattering on whole line

Others…

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Others…

Our approach – no inverse scattering

G. Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*, \quad \partial_\tau u_2 + c_2 \partial_x u_2 = i\, u_3^* u_1^*,$$

$$\partial_\tau u_3 + c_3 \partial_x u_3 = i\, u_3^* u_1^*.$$

Procedure: Look for same kind of series, but now allow all free constants to become free functions of (x) – except for (τ_0).

$$u_1(x,\tau) = \frac{e^{i\theta_1(x)}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_1(x) + (\tau_0 - \tau)^2 \beta_1(x) + (\tau_0 - \tau)^3 \delta_1(x) + \ldots\right]$$

$$u_2(x,\tau) = \frac{e^{i\theta_2(x)}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_2(x) + (\tau_0 - \tau)^2 \beta_2(x) + (\tau_0 - \tau)^3 \delta_2(x) + \ldots\right]$$

$$u_3(x,\tau) = \frac{e^{i\theta_3(x)}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_3(x) + (\tau_0 - \tau)^2 \beta_3(x) + (\tau_0 - \tau)^3 \delta_3(x) + \ldots\right]$$

F. Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*$$

Procedure: Look for same kind of series, but now allow all free constants to become free functions of (x) – except for (τ_0).

Note: If $u_1(x,\tau) = \dfrac{e^{i\theta_1(x)}}{(\tau_0 - \tau)}[1 + \ldots]$,

then

$$\partial_\tau u_1 = \frac{e^{i\theta_1(x)}}{(\tau_0 - \tau)^2}[1 + \ldots],$$

$$i\, u_2^* u_3^* = \frac{e^{-i(\theta_2(x)+\theta_3(x))}}{(\tau_0 - \tau)^2}[1 + \ldots],$$

$$c_1 \partial_x u_1 = \frac{e^{i\theta_1(x)}}{(\tau_0 - \tau)}[i\theta_1'(x) + \ldots].$$

Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*, \ \ldots.$$

$$u_j(x,\tau) = \frac{e^{i\theta_j(x)}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_j(x) + (\tau_0 - \tau)^2 \beta_j(x) + (\tau_0 - \tau)^3 \delta_j(x) + \ldots\right]$$

- Eqn's are autonomous in τ => (τ_0) is a free constant
- $(\tau_0 - \tau)^{-2}$: need $\theta_1(x) + \theta_2(x) + \theta_3(x) = \pi/2$.

  => 2 free functions of $\{\theta_1(x),\ \theta_2(x)\}$

  No conditions on $\theta_j(x)$, except differentiability.

- $(\tau_0 - \tau)^{-1}$: $Re(\alpha_j(x)) = 0$, $Im(\alpha_j(x))$ fixed in terms of $\theta_n'(x)$

Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*, \ \ldots.$$

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- $(\tau_0 - \tau)^0$: $Im(\beta_j)$ fixed, $j = 1, 2, 3$,

  $Re(\beta_j) = \mathrm{B}(x) + K_j(x)$, with $\mathrm{B}(x)$ fixed, and

  $K_1(x) + K_2(x) + K_3(x) = 0.$

Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*,\ \ldots .$$

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Choose any two (differentiable) $\{K_2(x), K_3(x)\}$.

Painlevé analysis of 3-wave PDEs

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- $(\tau_0 - \tau)^0$: $Im(\beta_j)$ fixed, $j = 1, 2, 3$,

  $Re(\beta_j) = \mathrm{B}(x) + K_j(x)$, with $\mathrm{B}(x)$ fixed, and

  $K_1(x) + K_2(x) + K_3(x) = 0.$

Choose any two (differentiable) $\{K_1(x), K_2(x)\}$.

- $(\tau_0 - \tau)^1$: $Re(\delta_j)$ fixed, $j = 1, 2, 3$

  $Im(\delta_j) = H(x)$, $j = 1, 2, 3.$

Choose (differentiable) $H(x)$.

Painlevé analysis of 3-wave PDEs

$$\partial_\tau u_1 + c_1 \partial_x u_1 = i\, u_2^* u_3^*,\ \ldots.$$

$$u_j(x,\tau) = \frac{e^{i\theta_j(x)}}{(\tau_0 - \tau)}\left[1 + (\tau_0 - \tau)\alpha_j(x) + (\tau_0 - \tau)^2 \beta_j(x) + (\tau_0 - \tau)^3 \delta_j(x) + \ldots\right]$$

Results (so far):

(a) Painlevé analysis provides a (formal) Laurent series solution of the PDEs, which contains 5 arbitrary, differentiable functions of x.

(b) See Ruth Martin for convergence of series.

(c) This is one function short of the general solution of the PDEs, for ANY (sensible) boundary conditions.

Bonne fête, Walter