Kinematic Vortices in a Thin Film Driven by an Applied Current

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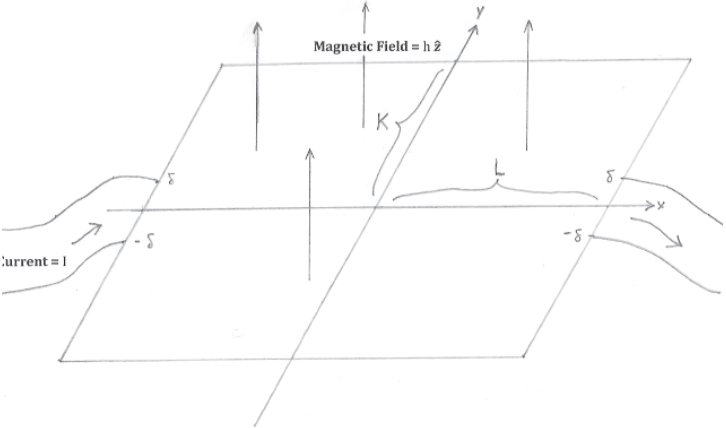
Joint work with Lydia Peres Hari and Jacob Rubinstein
Technion



Consider a thin film superconductor subjected to an applied current of magnitude I (fed through the sides) and a perpendicular applied magnetic field of magnitude h.

Goal: Understanding anomalous vortex behavior

Standard magnetic vortex: localized region of trapped magnetic flux. Within Ginzburg-Landau theory: zero of complex-valued order parameter carrying non-zero degree.

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-entering the sample on opposite sides and ultimately colliding and annihilating each other in the middle.

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Andronov, Gordion, Kurin, Nefedov, Shereshevsky ’93,
Berdiyorov, Elmurodov, Peeters, Vodolazov, Milosevic ’09, Du ’03

Ginzburg-Landau formulation of problem

$$\Psi_t + i\phi\Psi = (\nabla - ihA_0)^2\,\Psi + (\Gamma - |\Psi|^2)\Psi \text{ for } (x,y) \in \mathcal{R},\;\; t > 0,$$

$$\Delta\phi = \nabla \cdot \left(\frac{i}{2}\{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\} - |\Psi|^2\, hA_0 \right) \text{ for } (x,y) \in \mathcal{R},\;\; t > 0,$$

where $\mathcal{R} = [-L, L] \times [-K, K]$, $A_0 = (-y, 0)$ and $\Gamma > 0$ prop. to $T_c - T$.

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Boundary conditions for Ψ:

$$\Psi(\pm L, y, t) = 0 \text{ for } |y| < \delta,$$

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$$\phi_x(\pm L, y, t) = \begin{cases} -\mathrm{I} & \text{for } |y| < \delta, \\ 0 & \text{for } \delta < |y| < K, \end{cases}$$

$$\phi_y(x, \pm K, t) = 0 \text{ for } |x| \le L.$$

Rigorous bifurcation from normal state

Normal State: At high temp. (Γ small) and/or large magnetic field or electric current, expect to see no superconductivity:

$$\Psi \equiv 0, \quad \phi = \mathrm{I}\,\phi^0$$

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Note: One easily checks that ϕ^0 is odd in x and even in y:

$$\phi^0(-x, y) = -\phi^0(x, y) \quad \text{and} \quad \phi^0(x, -y) = \phi^0(x, y).$$

Linearization about Normal State:

$$\Psi_t = \mathcal{L}[\Psi] + \Gamma\Psi \quad \text{in } \mathcal{R},$$

where

$$\mathcal{L}[\Psi] := (\nabla - ihA_0)^2\Psi - i\mathrm{I}\phi^0\Psi.$$

subject to boundary conditions

$$\Psi(\pm L, y, t) = 0 \text{ for } |y| < \delta,$$

$$(\nabla - ihA_0)\,\Psi \cdot \mathbf{n} = 0 \quad \text{elsewhere on } \partial\mathcal{R},$$

$\mathcal{L}$ = Imaginary perturbation of (self-adjoint) magnetic Schrödinger operator.

Spectral Properties of $\mathcal{L}$

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$$\text{with} \;\; 0 < \mathrm{Re}\,\lambda_1 \leq \mathrm{Re}\,\lambda_2 \leq \ldots, \quad \text{and} \quad |\mathrm{Im}\,\lambda_j| < \left\|\phi^0\right\|_{L^\infty} \mathrm{I}$$

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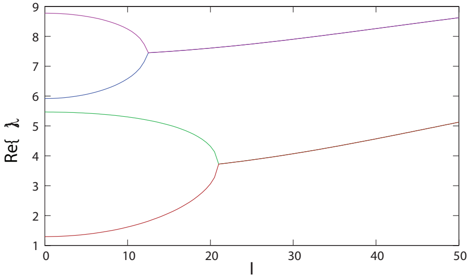
Hence, if (λ_j, u_j) is an eigenpair then so is $(\lambda_j^*, u_j^\dagger)$ where

$$u_j^\dagger(x, y) := u_j^*(-x, y).$$

If λ_j is real, then $u_j = u_j^\dagger$, and indeed each λ_j *is* real for I small.

Eigenvalue collisions $\implies$ Complexification of spectrum



Collisions of first 4 eigenvalues for $L = 1$, $K = 2/3$, $\delta = 1/6$, $h = 0$.

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Set $\mathcal{L}_1 := \mathcal{L} + \mathrm{Re}\,\lambda_1$, so that bottom of spectrum of $\mathcal{L}_1$ consists of purely imaginary eigenvalues:

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To capture this (Hopf) bifurcation we take

$$\Gamma = \operatorname{Re} \lambda_1 + \varepsilon \quad \text{for } 0 < \varepsilon \ll 1.$$

Formulation as a single nonlocal PDE:

With the choice $\Gamma = \mathrm{Re}\,\lambda_1 + \varepsilon$ for $0 < \varepsilon \ll 1$, full problem then takes the form of a single nonlinear, nonlocal PDE:

$$\Psi_t = \mathcal{L}_1[\Psi] + \varepsilon\Psi + \mathcal{N}(\Psi),$$

where

$$\mathcal{N}(\Psi) := -\left|\Psi\right|^2 \Psi - i\tilde{\phi}[\Psi]\Psi,$$

with $\tilde{\phi} = \tilde{\phi}[\Psi]$ solving

$$\Delta\tilde{\phi} = \nabla \cdot \left(\frac{i}{2}\{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\} - \left|\Psi\right|^2 hA_0\right) \quad \text{in } \mathcal{R}$$

along with homogeneous boundary conditions on Ψ and $\tilde{\phi}$.

Existence of periodic solutions via Center Manifold Theory

There exists a value $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$, the system undergoes a supercritical Hopf bifurcation to a periodic state $(\psi_\varepsilon, \phi_\varepsilon)$.

One has the estimate

$$\left\| \psi_\varepsilon - \left(a^\varepsilon(t) u_1 + a^\varepsilon(t)^* u_1^\dagger \right) \right\|_{H^2(\mathcal{R})} \leq C \varepsilon^{3/2}$$

with

$$a^\varepsilon(t) := C_0 \varepsilon^{1/2} e^{-i\chi t} \quad \text{where } \chi = \mathrm{Im}\,\lambda_1 + \gamma\varepsilon$$

and C_0 and γ are constants depending on certain integrals of u_1.

Generalization of techniques from 1d problem by J.R., S. and K. Zumbrun.

A key element of the proof: Exploiting PT symmetry on center manifold.

• For each ε small, there exists a graph $\Phi^{\varepsilon} : \mathcal{S} \to H^2(\mathcal{R}; \mathbb{C})$ over center subspace $\mathcal{S} := \mathrm{Span}\{u_1, u_2\}$ and complex-valued functions $\alpha_1(t)$, $\alpha_2(t)$ such that (for small initial data) solution to TDGL ψ_{ε} describable as

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Four real equations in four unknowns.
• One proves exponential attraction to PT-symmetric subset of center manifold.

$$\alpha_1(t)u_1 + \alpha_2(t)u_2 = (\alpha_1(t)u_1 + \alpha_2(t)u_2)^\dagger \iff \alpha_2 = \alpha_1^*.$$

Easy system for α_1–explicitly solvable.

A kinematic vortex motion law

According to theorem, the leading order term $\left(O(\varepsilon^{1/2})\right)$ is:

$$\psi = a^{\varepsilon}(t) u_1 + a^{\varepsilon}(t)^* u_1^{\dagger} \quad \text{with } a^{\varepsilon}(t) = C_0 \varepsilon^{1/2} e^{-i\chi t}.$$

Focusing our attention along the center line $x = 0$ and writing

$$u_1(0, y) = |u_1(0, y)|\, e^{i\beta(y)} \quad \text{for some phase } \beta(y)$$

we find that

$$\psi(0, y, t) = 2 C_0 \varepsilon^{1/2} \, |u_1(0, y)| \cos\left(\beta(y) - \chi t\right).$$

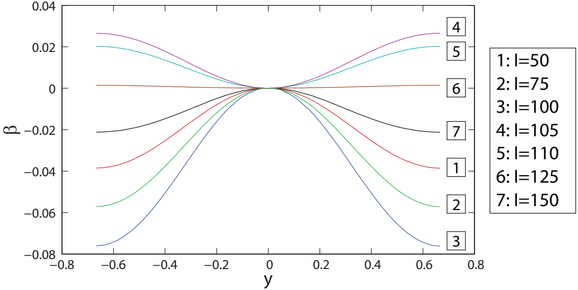
Hence, the order parameter vanishes on the center line $x = 0$ whenever the equation

$$\chi t = \beta(y) + \pi/2 + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

is satisfied. Recall that $\chi = \mathrm{Im}\, \lambda_1 + o(1)$.

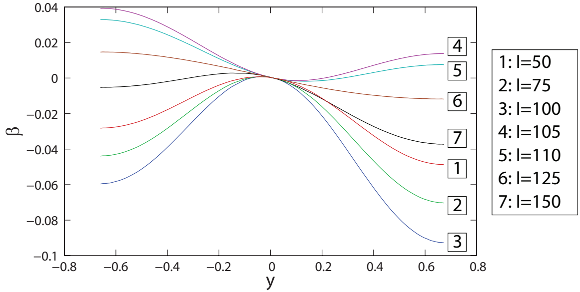
Using shape of β to explain anomalous vortex behavior

Case 1: No magnetic field, $h = 0$. Recall that β = phase of $u_1(0, y)$. Numerical computations reveal sensitive dependence on I.



Here $L = 1, K = 2/3, \delta = 4/15$. Note symmetry of β.

Case 2: Graphs of β when magnetic field present: $h > 0$.



Symmetry broken so vortices enter/exit boundaries $y = K$ and $y = -K$ at different times. Here we have taken $h = 0.05$.

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(iii) 'magnetic vortices' appear far from center line, presumably associated with vortices of ground-state u_1 of perturbed magnetic Schrödinger operator

$$(\nabla - ihA_0)^2 u_1 - i\mathrm{I}\phi^0 u_1 = -\lambda_1 u_1$$

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• What happens deep in the nonlinear regime? (No longer small amplitude)