Rokhlin dimension for actions of residually finite groups

Workshop on C*-algebras and dynamical systems
Fields Institute, Toronto

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Let A be a C*-algebra. We say that A has nuclear dimension r, and write $\dim_{\mathrm{nuc}}(A) = r$, if r is the smallest natural number with the following property:
For all $F \subset\subset A$ and $\varepsilon > 0$, there exists a finite dimensional C^*-algebra $\mathcal{F}$ and a c.p.c. map $\psi : A \to \mathcal{F}$ and c.p.c. order zero maps $\varphi^{(0)}, \ldots, \varphi^{(r)} : \mathcal{F} \to A$ such that

$$\|a - \sum_{l=0}^{r} \varphi^{(l)} \circ \psi(a)\| \leq \varepsilon \quad \text{for all } a \in F.$$

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We will discuss a generalization to cocycle actions of residually finite groups:

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$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (A_\infty \cap A', \alpha_\infty) \quad (l = 0, \dots, d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.
If no number satisfies this condition, we set $\dim_{\mathrm{Rok}}(\alpha) := \infty$.

Remark

If G is finite or $\mathbb{Z}^m$, this agrees with the previous definition.

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Theorem (Hirshberg-Winter-Zacharias)

If $\alpha : G \curvearrowright A$ is a finite group action on a unital C^*-algebra, we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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Theorem (Hirshberg-Winter-Zacharias)

If A is a unital C^*-algebra and $\alpha \in \mathrm{Aut}(A)$, we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha \mathbb{Z}) \leq 2 \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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Theorem (S)

If $\alpha : \mathbb{Z}^m \curvearrowright A$ is an action on a unital C^*-algebra, we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq 2^m \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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Theorem (S-Wu-Zacharias)

Let G be a countable, discrete, residually finite group. Let A be any C^*-algebra and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Then we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\Box G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\square G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

The above constant denotes the asymptotic dimension of the box space of G. We shall elaborate:

Definition (Roe)

Let G be a countable, residually finite group. The box space $\Box G$ is the disjoint union of all finite quotients groups of G, equipped with its minimal connected G-invariant coarse structure for the left action of G by translation.

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Remark (Roe-Khukhro)

Take a decreasing sequence of normal subgroups $G_n \subset G$ with finite index, such that every finite index subgroup $H \subset G$ contains G_n for some n. Let $S \Subset G$ be a finite generating set, and equip G with the associated right-invariant word-length metric.

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$$\operatorname{dist}(G/G_n, G/G_m) \geq \max\{\operatorname{diam}(G/G_n), \operatorname{diam}(G/G_m)\}$$

for all $n, m \in \mathbb{N}$.

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Example

- The box space of a finite group is a one-point set, hence it has asymptotic dimension 0.
- $\operatorname{asdim}(\Box \mathbb{Z}^m) = m$.
- Probably all finitely generated, virtually nilpotent groups G satisfy $\operatorname{asdim}(\Box G) < \infty$. (Details still need to be checked!)

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This turns out to be true, and that is what makes the proof of the main result possible. Unfortunately, there is not enough time to get into details. Instead, we would like to look at the case of topological actions.

Let G be a countable, discrete group and $d \in \mathbb{N}$. Let $\triangle_d G \subset \ell^1(G)$ be the set of all probability measures of G supported on at most $d+1$ points. Let $\triangle G = \bigcup_{d\in\mathbb{N}} \triangle_d G$ be the set of all finitely supported probability measures of G.

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Definition (one of many equivalent versions)

Let $\alpha : G \curvearrowright X$ be an action on a compact metric space. Then α is amenable if there exist approximately equivariant maps

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That is, there exists a net of continuous maps $f_\lambda : X \to \triangle G$ such that $\|f_\lambda(\alpha_g(x)) - \beta_g(f_\lambda(x))\|_1 \to 0$ as $\lambda \to \infty$ for all $x \in X$ and $g \in G$.

Definition (Guentner-Willett-Yu)

Let $\alpha : G \curvearrowright X$ be an action on a compact metric space and $d \in \mathbb{N}$. α is said to have amenability dimension at most d, written $\dim_{\mathrm{am}}(\alpha) \leq d$, if there exist almost equivariant maps

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Theorem (Guentner-Willett-Yu)

For a free action $\alpha : G \curvearrowright X$, one has the estimate

$$\dim^{+1}_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha G) \leq \dim^{+1}_{\mathrm{am}}(\alpha) \cdot \dim^{+1}(X).$$

Let X be a compact metric space, G a countable, residually finite group, and $\alpha : G \curvearrowright X$ an action. Let $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$ denote the induced C^*-algebraic action.

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This can be answered as follows:

Theorem (S-Wu-Zacharias)

If $\alpha : G \curvearrowright X$ is free, one has the following estimates:

$$\dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}) \ \leq \ \dim_{\mathrm{am}}^{+1}(\alpha) \ \leq \ \mathrm{asdim}^{+1}(\Box G) \cdot \dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}).$$

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In particular, if $\mathrm{asdim}(\Box G) < \infty$, then α has finite amenability dimension if and only if $\bar{\alpha}$ has finite Rokhlin dimension.

Last year, the following result was obtained:

Theorem (S)

If $\alpha : \mathbb{Z}^m \curvearrowright X$ *is a free action on a compact metric space of finite covering dimension, then* $\dim_{\mathrm{Rok}}(\bar{\alpha}) < \infty$. *In particular,* $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m) < \infty$.

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Using the methods that were developed for this (e.g. marker property), and also using some additional ingredients, this extends to the following setting:

Theorem (S-Wu-Zacharias)

Let G be a finitely generated, nilpotent group. If $\alpha : G \curvearrowright X$ is a free action on a compact metric space of finite covering dimension, then both $\dim_{\mathrm{am}}(\alpha) < \infty$ and $\dim_{\mathrm{Rok}}(\bar{\alpha}) < \infty$. In particular, $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha G) < \infty$.

Thank you for your attention!