Tracial Rokhlin property for actions of amenable groups

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June 18, 2014

In this talk, all C*-algebras are assumed to be simple, unital and separable. All groups are assumed to be countable discrete and amenable.

Notation

Let A be a C*-algebra, and G be a group.

- A^∞ —— $\ell^\infty(\mathbb{N}, A)/c_0(A)$.
- A_∞ —— $A^\infty \cap A'$.
- $\mathrm{Aut}_G(A)$ —— the set of all actions $\alpha\colon G \to \mathrm{Aut}(A)$.
- For a finite subset $K \subset G$ and $\varepsilon > 0$, we say a finite subset $T \subset G$ is (K, ε)-invariant if

$$|T \cap \bigcap_{g \in K} gT| \geq (1 - \varepsilon)|T|.$$

Definition

Let $f \in (A^\infty)_+$ and $a \in A_+$. We say f is pointwisely Cuntz subequivalent to a and write $f \precsim_{\mathrm{p.w.}} a$, if f has a representative $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$, such that each f_n is positive and $f_n \precsim a$ in A, for all $n \in \mathbb{N}$.

Definition

Let $\alpha \in \mathrm{Aut}_G(A)$. We say α has the weak tracial Rokhlin property, if for any finite subset K of G and any $\varepsilon > 0$, there exists an (K, ε)-invariant finite subset $T \subset G$, such that for any nonzero positive contraction $z \in A$, there is a positive contraction f in A_∞ satisfying:

(1) $\alpha_g(f)\alpha_h(f) = 0$, for any $g, h \in T$ such that $g \neq h$.

(2) With $e = \sum_{g\in T} \alpha_g(f)$, $1 - e \precsim_{\mathrm{p.w.}} z$.

If in addition, we can always choose f to be a projection, then we say that α has the tracial Rokhlin property.

When A has strict comparison, Condition (2) could be replaced by:

(2′)

$$\lim_{n\to\infty}\max_{\tau\in T(A)}\tau(1-\sum_{g\in T}\alpha_g(f_n))=0,\ \text{where } f=(f_n)_{n\in\mathbb{N}}.$$

Matui and Sato also have a definition of (weak) tracial Rokhlin property for actions of amenable groups, where Condition (2) is replaced by:

(2″)

$$\lim_{n\to\infty}\max_{\tau\in T(A)}|\tau(\alpha_g(f_n))-\frac{1}{|T|}|=0.$$

Matui and Sato's definition is formally stronger than ours, we shall see that the two definition coincide in good situations.

We say an action $\alpha \in \mathrm{Aut}_G(A)$ is strongly outer, if and only if for any $g \neq 1$ and any $\tau \in \mathrm{T}^{\alpha_g}(A)$, the weak extension of α_g on $\pi_\tau(A)''$ is not weakly inner.

Proposition

Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the weak tracial Rokhlin property. Suppose that the tracial state space $\mathrm{T}(A)$ has finitely many extreme points. Then α is strongly outer.

Theorem

(Matui & Sato) If A is nuclear, stably finite, infinite dimensional and has finitely many extreme tracial states, and G has property (Q), then

$$\text{strongly outer} \iff \text{weak tracial Rokhlin property.}$$

Definition

(Matui & Sato) Let $\alpha \in \mathrm{Aut}_G(R)$ be an outer action on the AFD II_1 factor R. We say that G has the property (Q) if the following holds: For any finite subset $K \subset G$ and $\varepsilon > 0$, there exists an (K, ε)-invariant finite subset $T \in G$ and a sequence of projections $(p_n)_n$ in R such that

$$\|1 - \sum_{g \in T} \alpha_g(p_n)\|_2 \to 0 \quad \text{and} \quad \|[x, p_n]\|_2 \to 0, \ \forall x \in R.$$

as $n \to \infty$.

Matui and Sato have shown that all elementary amenable groups have property (Q).

Corollary

If G has property (Q) and A is nuclear, stably finite, infinite dimensional with only finitely many extreme tracial states, then our definition coincide with Matui and Sato's.

The concept of tracially $\mathcal{Z}$-stable for C*-algebras was introduced by Ilan Hirshberg and Joav Oravitz as a "tracial version" of $\mathcal{Z}$-stable. We can take the following reformulation as the definition

Proposition

Let A be a unital simple separable C*-algebra. Then A is tracially $\mathcal{Z}$-stable if and only if for any $n \in \mathbb{N}$ and any $z \in A_+ \setminus \{0\}$, there exists a c.p.c. order zero map $\phi \colon M_n \to A_\infty$, such that $1 - \phi(1) \precsim_{\mathrm{p.w.}} z$.

Hirshberg and Oravitz proved that a simple $\mathcal{Z}$-stale C*-algebra is always tracially $\mathcal{Z}$-stable, and the converse holds if A is moreover nuclear. They have also shown the following:

Theorem

(Hirshberg & Oravitz) Let A be a simple tracially $\mathcal{Z}$-absorbing C*-algebra. Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the weak tracial Rokhlin property. Suppose G is finite, or $G = \mathbb{Z}$ and A has only finitely many extremal tracial states, then $A \rtimes_\alpha G$ is also tracially $\mathcal{Z}$-absorbing.

Theorem

(Matui & Sato) Let A be nuclear, stably finite, infinite dimensional with finitely many extremal tracial states. Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the weak tracial Rokhlin property. Then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable.

Following the ideas by Hirshberg and Oravitz, we could show:

Theorem

Let A be a tracially $\mathcal{Z}$-absorbing C*-algebra. Suppose that the Cuntz semigroup $\mathrm{W}(A)$ is almost divisible. Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the weak tracial Rokhlin property. Then $A \rtimes_\alpha G$ is also tracially $\mathcal{Z}$-absorbing.

It is probably true that the Cuntz semigroup of a tracially $\mathcal{Z}$-stable C*-algebra is automatically almost divisible.

In the proof, we need to somehow control the size of the Rokhlin contractions. This is why Oravitz and Hirshberg need to assume finite extremal tracial states. The following proposition enables us to drop this assumption.

Proposition

Let A be a simple unital infinite dimensional C*-algebra. Suppose the Cuntz semigroup $\mathrm{W}(A)$ (Murray-Von Neumann semigroup $\mathrm{V}(A)$, respectively) is almost unperforated and almost divisible. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of countable amenable group. Let $a, b \in \mathrm{M}_\infty(A)$ be two nonzero positive elements (projections, respectively) such that $d(a) < d(b)$ for any α-invariant dimension function in $\mathrm{DF}(A)$ (α-invariant traces, respectively). Then $a \precsim b$ in $\mathrm{M}_\infty(A \rtimes_\alpha G)$.

The orginal version concerning projections is due to Phillips and Osaka.

Theorem

(Phillips & Osaka) Let A be a simple unital C*-algebra with real rank zero and has strict comparison for projections. Let $\alpha \in \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property. Then $A \rtimes_\alpha \mathbb{Z}$ has real rank zero and strict comparison for projections. If furthermore, A has stable rank one, then so is $A \rtimes_\alpha \mathbb{Z}$.

Based on their work we can show:

Theorem

The same is true if $\mathbb{Z}$ is replaced by general amenable groups.

A slightly better result is:

Theorem

Let A have Property (SP) and strict comparison for projections, and suppose $V(A)$ is almost divisible. Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ also has strict comparison for projections.

Lemma

Suppose A has property (SP) and strict comparison for projections, and $\mathrm{V}(A)$ is almost divisible. Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the tracial Rokhlin property. Then for every finite $F \subset_f A \rtimes_\alpha G$, every $\varepsilon > 0$, and every nonzero $z \in (A \rtimes_\alpha G)_+$, there exist a projection $f \in A$, a finite set $T \subset G$, an embedding $\phi\colon \mathrm{M}_{|T|} \otimes fAf \to A \rtimes_\alpha G$ whose image shall be called D, and a projection $p \in D$, where $\{e_{g,h}\}_{g,h\in T}$ are the standard system of matrix units for $\mathrm{M}_{|T|}$, such that

(1) $\|\phi(e_{g,h} \otimes a) - u_g a u_h^*\| \leq \varepsilon \|a\|$, for any $g \in T$ and $a \in fAf$.

(2) there exists some $g_0 \in T$ such that $\phi(e_{g_0,g_0} \otimes a) = a$, for any $a \in fAf$.

(3) $pb \subset_\varepsilon D$ and $bp \subset_\varepsilon D$, for any $b \in F$.

(4) $1 - p \precsim z$.

Lemma

Let A be a simple unital C*-algebra with real rank zero and has strict comparison for projections. Let $\alpha \in \mathrm{Aut}_G(A)$ have the tracial Rokhlin property. Then for any self-adjoint element $a \in A \rtimes_\alpha G$, any $\varepsilon > 0$ and any nonzero positive element $z \in A \rtimes_\alpha G$, there is a C*-subalgebra D of $A \rtimes_\alpha G$ with real rank zero and a projection $p \in D$ such that:

(1) $\|pa - ap\| < \varepsilon$.

(2) $pap \in_\varepsilon D$.

(3) $1 - p \precsim z$.

The above lemma says that any single self-adjoint element of the crossed product could be 'tracially' approximated by subalgebras with real rank zero. It is weaker than tracial approximation formulated by Elliott and Niu, but is good enough to deduce that the crossed product has real rank zero.

Proposition

Let A be a unital simple C*-algebra with strict comparison for projections. Suppose for any self-adjoint element $a \in A$, any $\varepsilon > 0$ and any nonzero positive element $z \in A$, there is a unital C*-subalgebra D of A with real rank zero and $1_D = p$ such that:

1. $\|pa - ap\| < \varepsilon$,
2. $pap \in_\varepsilon D$,
3. $1 - p \precsim z$.

Then A has real rank zero.

Lemma

Let A be a simple C*-algebra with real rank zero and strict comparison for projections. Let $\alpha \in \mathrm{Aut}_G(A)$ with the tracial Rokhlin property. Then for any projections $p_1, \ldots, p_n \in A \rtimes_\alpha G$ and arbitrary elements $a_1, \ldots, a_m \in A \rtimes_\alpha G$, any $\varepsilon > 0$, there exist a unital subalgebra $A_0 \subset A \rtimes_\alpha G$ which is stably isomorphic to A, a projection $p \in A_0$ and subprojections $r_1, \ldots, r_n$ of p such that:

(1) $pa_i \in_\varepsilon A_0$, $a_i p \in_\varepsilon A_0$, for $i = 1, 2, \ldots, m$

(2) $p_k r_k =_\varepsilon r_k$, for any k.

(3) $1 - p \precsim r_k$, for any k.

Proposition

Let A be a unital simple stably finite C*-algebra with Property (SP). If for any $x \in A$, any $\varepsilon > 0$ and any projection $p_1, \ldots, p_n$, there is a unital simple subalgebra D with stable rank one and Property (SP), a projection $p \in D$ and subprojections $r_1, \ldots, r_n$ of p such that:

(1) $pap \in_{\varepsilon} D$.

(2) $r_k p_k =_{\varepsilon} r_k$.

(3) $1 - p \precsim r_k$.

Then A has stable rank one.

Corollary

Let A be a unital simple nuclear C*-algebra with tracial rank zero and a unique trace. Suppose $\alpha \in \mathrm{Aut}_G(A)$ is an action with the tracial Rokhlin property. Suppose, in addition, that $A \rtimes_\alpha G$ is quasidiagonal. Then $A \rtimes_\alpha G$ has tracial rank zero.

Remark

We should mention here that it is still open whether the crossed product by actions of $\mathbb{Z}$ with tracial Rokhlin property preserve the class of unital simple C*-algebras with tracial rank zero. Important partial results have been obtained by Lin.
The requirement that $A \rtimes_\alpha G$ is quasidiagonal is automatic in many situations, even without the assumption that α has the tracial Rokhlin property. For example, if A is an simple AH algebra and G is finitely generated abelian group, quasidiagonality follows from Lin's result (he also dealt with non-simple AH algebra).

Definition

(Weiss) A *monotile* T in a discrete group G is a finite set for which one can find a set C such that $Tc\colon c \in C$ is a covering of G by disjoint sets.

Definition

An amenable group is called *Weiss-tileable* if for any finite T in G and any $\varepsilon > 0$, there is a monotile T in G which is (T, ε)-invariant.

Remark

There was no name for such group in literature. Although it has been studied in the paper *monotileable group* by Weiss, the term *monotileable group* was defined to mean somethings else.

Theorem

(Weiss) All residually finite amenable group and all elementary amenable groups are Weiss-tileable.

Theorem

Let G be a Weiss-tileable group, and let $\mathcal{Z}$ be the Jiang-Su algebra. Suppose σ is an action on $\mathbb{Z}$ (viewed as a set), then there is an induced action α on $\otimes_{i\in\mathbb{Z}}\mathcal{Z}$ defined by

$$\alpha_g(\otimes_i z_i) = \otimes_i z_{\sigma_g^{-1}(i)}.$$

If σ is faithful, the action α has the weak tracial Rokhlin property. In particular, the Benoulli shift has weak tracial Rokhlin property.

Corollary

Any Weiss-tileable group has Property (Q).

Definition

Let $A = \otimes_{i=1}^{\infty} \mathrm{B}(H_i)$, where H_i is a finite dimensional Hilbert space for each i. An action $\alpha\colon G \mapsto \mathrm{Aut}(A)$ is called a *product-type action* if and only if for each i, there exists a unitary representation $\pi_i\colon G \to \mathrm{B}(H_i)$, which induces an inner action $\alpha_i\colon g \mapsto \mathrm{Ad}(\pi_i(g))$, such that $\alpha = \otimes_{i=1}^{\infty} \alpha_i$.

Theorem

Let $\alpha \in \mathrm{Aut}_G(A)$ be a product-type action where G has property (Q) and A is UHF. Let H_i, π_i and α_i be defined as above. Let d_i be the dimension of H_i and χ_i be the character of π_i. Define $\chi\colon G \mapsto \mathbb{C}$ to be the characteristic function on 1_G. Then the action α has the tracial Rokhlin property if and only if there exists a telescope, such that for any $n \in \mathbb{N}$, the infinite product

$$\prod_{n \le i < \infty} \frac{1}{d_i} \chi_i = \chi. \tag{1}$$

Another types of example comes from actions on non-commutative tori. Let θ be a non-degenerate anti-symmetric bicharacter on $\mathbb{Z}^d$. We identify it with its matrix under the canonical basis of $\mathbb{Z}^d$. Then the associated non-commutative tori A_θ is simple, unital **AT** algebra with a unique trace. A_θ is generated by unitaries $\{U_x \mid x \in \mathbb{Z}^d\}$ subject to the relation

$$U_y U_x = exp(\pi i < x, \theta y >) U_{x+y}, \forall x, y \in \mathbb{Z}^d.$$

Proposition

(Phillips) For any $\lambda \in \mathbb{T}^n$ and $T \in M_n(\mathbb{Z})$, the map $U_x \to \lambda^x U_{Tx}$ give rises to an endomorphism $\alpha_{\lambda,T}$ of A_θ if and only if

$$\frac{1}{2}(T^t \theta T - \theta) \in M_d(\mathbb{Z})$$

It is an automorphism if and only if T is invertible. For any $T \neq I$, $\alpha_{\lambda,T}$ is not weakly inner.
If $d = 2$, the condition is satisfied for any $T \in GL_2(\mathbb{Z})$.

If we can find one example of actions with (weak) tracial Rokhlin property, we can actually find a family of them by forming inner tensors. More specifically, we have the following:

Proposition

Let $\alpha \in \mathrm{Aut}_G(A)$ be an action with the weak tracial Rokhlin property and $\beta \in \mathrm{Aut}_G(B)$ be arbitrary, where A, B are both simple. Then the inner tensor of these two actions $\gamma = \alpha \otimes \beta \colon G \to \mathrm{Aut}(A \otimes_{\mathrm{min}} B)$ has the the weak tracial Rokhlin property. If α has the tracial Rokhlin property, then γ has the tracial Rokhlin property.