Weak amenability of Fourier algebras: old and new results

Yemon Choi
Lancaster University

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[Minor corrections made to slides after talk]

This talk is only about commutative Banach algebras

- a Banach A-bimodule X is called symmetric if $a \cdot x = x \cdot a$ for all $a \in A$ and all $x \in X$.
- a bounded linear map $D : A \to X$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

This talk is only about continuous derivations

$\mathsf{Der}(A, X) := \{\text{continuous derivations } A \to X\}$.

Remark

If A is a semisimple CBA then $\mathsf{Der}(A, A) = \{0\}$. (SINGER–WERMER, 1955.)

Given a character φ on A, let $\mathbb{C}_\varphi$ be the corresponding 1-dimensional A-bimodule.

Theorem

$\mathsf{Der}(A, \mathbb{C}_\varphi) \cong \left(\ker(\varphi) / \overline{\ker(\varphi)^2} \right)^*$.

Therefore, if $\ker(\varphi)^2$ is dense in $\ker(\varphi)$, $\mathsf{Der}(A, \mathbb{C}_\varphi) = \{0\}$. For example, this happens if $\{\varphi\}$ is a set of synthesis for A (when A is semisimple and regular).

Heuristic

If $\mathsf{Der}(A, \mathbb{C}_\varphi) \neq \{0\}$ then this may indicate one of the following:

- some kind of "analytic structure" in a suitable neighbourhood of φ;
- some kind of differentiability at φ.

Conversely, if you already know your algebra has analytic structure or smoothness, it is unsurprising to find $\mathsf{Der}(A, \mathbb{C}_\varphi) \neq \{0\}$ for some φ.

Definition (Bade–Curtis–Dales, 1987)

Let A be a **commutative** *Banach algebra. We say A is weakly amenable if* $\mathsf{Der}(A, X) = \{0\}$ *for every* **symmetric** *Banach A-bimodule X.*

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Remark

In fact, if A is commutative and $\mathrm{Der}(A, A^*) = \{0\}$ then A is weakly amenable.

In many examples where A is commutative and semisimple and $\mathrm{Der}(A, A^*) \neq \{0\}$, derivations arise from vestigial "analytic structure" or "smoothness". Today's talk is about the latter case.

Example 1. $C^1(\mathbb{T})$ with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$

Example 2. Given $\alpha \geq 0$, consider

$$\mathrm{A}_\alpha(\mathbb{T}) := \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|(1+|n|)^\alpha < \infty\}$$

with norm $\|f\|_{(\alpha)} = \sum_n |\widehat{f}(n)|(1+|n|^\alpha)$.

(The case $\alpha = 0$ is the usual Fourier algebra $\mathrm{A}(\mathbb{T})$.)

Folklore

$C^1(\mathbb{T})$ has non-zero point derivations, namely: $f \mapsto \dfrac{\partial f}{\partial \theta}(p)$ for some choice of $p \in \mathbb{T}$.

We then get derivations $C^1(\mathbb{T}) \to C^1(\mathbb{T})^*$ by e.g.

$$D(f)(g) := \int_{\mathbb{T}} \frac{\partial f}{\partial \theta}(p) g(p) \, d\mu(p)$$

where μ is normalized Lebesgue measure on the circle.

What about the algebras $A_\alpha(\mathbb{T})$, for $\alpha \geq 0$? When do they have point derivations? when are they weakly amenable?

Folklore

Let $p \in \mathbb{T}$. Then $\mathrm{Der}(A_\alpha(\mathbb{T}), \mathbb{C}_p) \neq \{0\}$ iff $\alpha \geq 1$.

Theorem (Bade–Curtis–Dales, 1987)

$\mathrm{Der}(A_\alpha(\mathbb{T}), A_\alpha(\mathbb{T})^*) \neq \{0\}$ *if and only if* $\alpha \geq 1/2$.

Proof of sufficiency: a direct calculation, using **orthonormality** of the standard monomials, shows

$$\left| \int_{\mathbb{T}} \frac{\partial f}{\partial \theta}(p) g(p)\, d\mu(p) \right| \leq \|f\|_{(1/2)} \|g\|_{(1/2)}$$

Informally: pointwise differentiation can be bad on a function algebra, but averaging can smooth it out.

Why was it so easy to show that $A_\alpha(\mathbb{T})$ is not weakly amenable when α is sufficiently large?

We had an explicit guess for what a derivation should look like: namely, a (partial) derivative of functions.

The norm on $A_\alpha(\mathbb{T})$ is defined in terms of Fourier coefficients; and the Fourier transform intertwines differentiation (hard) with multiplication (easy).

If G is LCA, with Pontryagin dual Γ, then $\mathrm{A}(G)$ is the range of the Fourier/Gelfand transform $L^1(\Gamma) \to C_0(G)$, equipped with the norm from $L^1(\Gamma)$.

If G is compact, there is a notion of matrix-valued Fourier transform:

$$f(x) \sim \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x)^*)$$

and

$$\mathrm{A}(G) = \left\{ f \in C(G) : \sum_{\pi} d_\pi \left\| \widehat{f}(\pi) \right\|_1 < \infty \right\}$$

For a general locally compact group G, EYMARD (1964) gave a definition of $\mathrm{A}(G)$ which generalizes both these cases.

If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a cts unitary rep, a coefficient function associated to π is a function of the form

$$\xi *_\pi \eta : p \mapsto \langle \pi(p)\xi, \eta \rangle \qquad (\xi, \eta \in \mathcal{H}_\pi).$$

Define A_π to be the coimage of the corresponding map $\theta_\pi : \mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}_\pi} \to C_b(G)$: that is, the range of θ_λ equipped with the **quotient** norm.

We have $\mathrm{A}_\pi + \mathrm{A}_\sigma \subseteq \mathrm{A}_{\pi \oplus \sigma}$ and $\mathrm{A}_\pi\, \mathrm{A}_\sigma \subseteq \mathrm{A}_{\pi \otimes \sigma}$.

Let $\lambda : G \to \mathcal{U}(L^2(G))$ be the left regular representation:

$$\lambda(p)\xi(s) = \xi(p^{-1}s) \qquad (\xi \in L^2(G); p, s \in G).$$

Define $\mathrm{A}(G)$, the Fourier algebra of G, to be the coefficient space A_λ. It is a subalgebra of $C_b(G)$ (by e.g. Fell's absorption principle).

Example 3. Suppose G is compact. Then: every cts unitary rep decomposes as a sum of irreps; and the left regular representation $\lambda : G \to \mathcal{U}(L^2(G))$ contains a copy of every irrep. It follows that

$$\mathrm{A}(G) = \bigoplus_{\pi \in \widehat{G}} \mathrm{A}_\pi$$

where the RHS is an ℓ^1-direct sum.

Theorem (folklore)

$\mathrm{Der}(\mathrm{A}(G), \mathrm{A}(G)) = \{0\}$.

Proof. $\mathrm{A}(G)$ is semisimple. Apply Singer–Wermer. □

Theorem (Forrest 1988)

Let $p \in G$. Then $\mathrm{Der}(\mathrm{A}(G), \mathbb{C}_p) = 0$.

Proof. $\{p\}$ is a set of synthesis, so $(J_p)^2$ is dense in J_p. □

So when is $\mathrm{A}(G)$ weakly amenable?

Note that if G is totally disconnected, the idempotents in $\mathrm{A}(G)$ have dense linear span, hence $\mathrm{A}(G)$ is WA. (Forrest, 1998)

As a special case of the results for $\mathrm{A}_\alpha(\mathbb{T})$ we know $\mathrm{A}(\mathbb{T})$ is weakly amenable.

In fact, for any LCA group G, $\mathrm{A}(G) = L^1(\widehat{G})$ is **amenable** and hence weakly amenable.

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In fact, for any LCA group G, $\mathrm{A}(G) = L^1(\widehat{G})$ is **amenable** and hence weakly amenable.

Theorem (Johnson, 1994)

Let G be either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. Then $\mathrm{A}(G)$ is not weakly amenable.

This theorem seems to have come as a surprise to people in the field.

A close reading of the last section in Johnson's paper shows that he has an **explicit construction** of a non-zero derivation $\mathrm{A}(\mathrm{SO}(3)) \to \mathrm{A}(\mathrm{SO}(3))^*$, not relying on abstract characterizations of WA.

Embed $\mathbb{T}$ in $\mathrm{SU}(2)$ as $e^{i\phi} \mapsto s_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$.

For $f \in C^1(\mathrm{SU}(2))$ define

$$\partial f(p) := \frac{\partial}{\partial \phi} f(p s_\phi)\Big|_{\phi=0}$$

then we get a derivation $C^1(\mathrm{SU}(2)) \to C(\mathrm{SU}(2))^*$

$$D(f)(g) = \int_{\mathrm{SU}(2)} (\partial f) g \, d\mu \qquad (f \in C^1(\mathrm{SU}(2)), g \in C(\mathrm{SU}(2)).$$

The part which needs work is to show that

$$\left| \int_{\mathrm{SU}(2)} (\partial f) g \, d\mu \right| \lesssim \|f\|_{\mathrm{A}} \, \|g\|_{\mathrm{A}}$$

but then, with some book-keeping, one gets a non-zero derivation $\mathrm{A}(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^*$.

One way to prove this estimate (not the approach in Johnson's paper, but probably known to him) is to use orthogonality relations for coefficient functions.

Schur orthogonality for compact groups

Let G be compact. If π and σ are irreps, ξ_1 and $\eta_1 \in \mathcal{H}_\pi$, ξ_2 and $\eta_2 \in \mathcal{H}_\sigma$:

$$\int_G \xi_1 *_\pi \eta_1 \, \overline{\xi_2 *_\sigma \eta_2} \, d\mu = \begin{cases} \dim(\mathcal{H}_\pi)^{-1} \langle \xi_1, \xi_2 \rangle \langle \eta_2, \eta_1 \rangle & \text{if } \pi = \sigma \\ 0 & \text{if } \pi \not\sim \sigma \end{cases}$$

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Remark

When $G = \mathbb{T}$ this is just the observation that $\{e^{in\theta} \ : \ n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2(\mathbb{T})$.

We return to SU(2) and the operator ∂. For any $\xi, \eta \in \mathcal{H}_\pi$

$$\partial(\xi *_\pi \eta)(p) = \frac{\partial}{\partial\phi}\langle \pi(ps_\phi)\xi, \eta\rangle = \langle \pi(p)F_\pi\xi, \eta\rangle$$

where

$$F_\pi = \left.\frac{\partial}{\partial\phi}\pi(s_\phi)\right|_{\phi=0} \in \mathcal{B}(\mathcal{H}_\pi).$$

So if f and g are coeff. fns of inequivalent irreps, $\int_{\mathrm{SU}(2)}(\partial f)\overline{g}d\mu = 0$.

If $f = \xi_1 *_\pi \eta_1$ and $g = \xi_2 *_\pi \eta_2$ are coeff. fns of the irrep π,

$$\left| \int_{\mathrm{SU}(2)} (\partial f)\overline{g}d\mu \right| \leq \dim(\mathcal{H}_\pi)^{-1} \|F_\pi\| \|\xi_1\| \|\xi_2\| \|\eta_1\| \|\eta_2\| \lesssim \|f\|_{\mathrm{A}} \|g\|_{\mathrm{A}}$$

(Use representation theory for SU(2) to get $\|F_\pi\| \lesssim \dim(\mathcal{H}_\pi)$.)

With some book keeping and the decomposition of A(SU(2)) in terms of the A_π, we obtain Johnson's inequality/result.

Theorem (Restriction theorem for Fourier algebras)

If G is a locally compact group and H is a closed subgroup, there is a quotient homomorphism of Banach algebras $\mathrm{A}(G) \to \mathrm{A}(H)$.

For compact G this is due to DUNKL (1969); the general case is due to HERZ (1973), see also ARSAC (1976).

Weak amenability of $\mathrm{A}(G)$, G compact

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Easy exercise

If A is a WA **commutative** Banach algebra, then so are all its quotient algebras.

Corollary

If G is locally compact and contains a closed subgroup isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$ then $\mathrm{A}(G)$ is not weakly amenable.

Theorem (PLYMEN, unpublished manuscript)

Let G be a compact, connected, non-abelian Lie group. Then $\mathrm{A}(G)$ is not weakly amenable.

Proof. By structure theory for compact groups, G contains a closed copy of either $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. □

Remark

It was observed in FORREST–SAMEI–SPRONK (2009) that the same holds for all compact connected groups (not just the Lie ones)

Theorem (FORREST–RUNDE, 2005)

If G_e is abelian then $\mathrm{A}(G)$ is weakly amenable.

It is an **open question** whether the converse holds.

Conjecture

If G is a connected, non-abelian Lie group then $\mathrm{A}(G)$ is not weakly amenable.

Impasse

The results that "just use Johnson" can tell us nothing about connected Lie groups where every compact connected subgroup is abelian, e.g. $\mathrm{SL}(2,\mathbb{R})$, the $ax+b$ group, or the Heisenberg group.

Conjecture

If G is a connected, non-abelian Lie group then $\mathrm{A}(G)$ is not weakly amenable.

Impasse

The results that "just use Johnson" can tell us nothing about connected Lie groups where every compact connected subgroup is abelian, e.g. $\mathrm{SL}(2,\mathbb{R})$, the $ax+b$ group, or the Heisenberg group.

Theorem (C.+Ghandehari, 2014)

The Fourier algebra of the $ax+b$ group is not weakly amenable.

The key insight which makes this example accessible: the $ax+b$ group is, like all compact groups, an AR group.

This group, which we denote by Aff, consists of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{R}_+^*$ and $b \in \mathbb{R}$.

For $f \in C^1(\mathsf{Aff})$, let $\mathrm{M}\partial f(b,a) = -\frac{1}{2\pi i} a \frac{\partial f}{\partial b}(b,a)$. Then set

$$D_0(f)(g) = \int_{\mathsf{Aff}} (\mathrm{M}\partial f) g \, d\mu$$

for all f and g in a suitable dense subalgebra $B \subset \mathrm{A}(\mathsf{Aff})$.

$\mathrm{A}(\mathsf{Aff})$ decomposes as $\mathrm{A}_{\pi_+} \oplus \mathrm{A}_{\pi_-}$ where the representations $\pi_\pm$ are **irreducible** and their coefficient functions satisfy generalized versions of the Schur orthogonality relations.

Explicitly: we can realize both π_+ and π_- on the same Hilbert space $L^2(\mathbb{R}_+^*, t^{-1}dt)$:

$$\pi_+(b,a)\xi(t) = e^{-2\pi ibt}\xi(at)$$

$$\pi_-(b,a)\xi(t) = e^{2\pi ibt}\xi(at)$$

We have a densely-defined, unbounded, self-adjoint operator K on $L^2(\mathbb{R}_+^*, t^{-1}dt)$:

$$(K\xi)(t) = t\xi(t) \qquad (t \in \mathbb{R}_+)$$

Provided ξ, η lie in the appopriate domains, we have:

Orthogonality relations

$$\langle \xi_1 *_{\pi_+} \eta_1, \xi_2 *_{\pi_+} \eta_2 \rangle_{L^2(G)} = \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle K^{-\frac{1}{2}}\xi_1, K^{-\frac{1}{2}}\xi_2 \rangle_{\mathcal{H}}.$$

$$\langle \xi_1 *_{\pi_-} \eta_1, \xi_2 *_{\pi_-} \eta_2 \rangle_{L^2(G)} = \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle K^{-\frac{1}{2}}\xi_1, K^{-\frac{1}{2}}\xi_2 \rangle_{\mathcal{H}}.$$

$$\langle \xi_1 *_{\pi_+} \eta_1, \xi_2 *_{\pi_-} \eta_2 \rangle_{L^2(G)} = 0.$$

The trick to our choice of M∂

Provided ξ and η are well-behaved,

$$\mathrm{M}\partial(\xi *_{\pi_+} \eta) = K\xi *_{\pi_+} \eta$$

for some densely defined self-adjoint operator K. Similarly for π_-.

The trick to our choice of $\mathrm{M}\partial$

Provided ξ and η are well-behaved,

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This turns out to make things similar enough to the **compact** case that we can push through (our version of) BEJ's methods, and get

$$|\int_{\mathsf{Aff}} (\mathrm{M}\partial f) g \, d\mu| \lesssim \|f\|_{\mathrm{A}} \, \|g\|_{\mathrm{A}}$$

for all f and g in some dense subspace of $\mathrm{A}(\mathsf{Aff})$.

Theorem (C.+GHANDEHARI, *ibid.*)

If G is a connected, **semisimple** *Lie group,* $\mathrm{A}(G)$ *is not weakly amenable.*

Proof. For compact connected Lie groups, this is Plymen's result. So we may WLOG assume G is non-compact and connected SSL. But then there is an Iwasawa decomposition $G = KAN$ where the closed subgroup AN contains a copy of the connected real $ax + b$ group. □

Theorem (C.+Ghandehari)

If G is connected, **simply connected**, *and non-solvable, then* $\mathrm{A}(G)$ *is not weakly amenable.*

Proof

More structure theory: the assumptions imply (Levi decomposition of Lie algebras and exponentiation) that G contains a closed, connected, semisimple subgroup.

Nilpotent examples?

Can use arguments similar to those for $ax + b$ to handle the reduced Heisenberg group. (Previously all the nilpotent examples had been out of reach.)

Can use different and more technical arguments (based around the Plancherel formula) to handle the Heisenberg group. (See the next talk.)