Recent advances on Arens (ir)regularity

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$\mathcal{A}$ comm.e $\Leftrightarrow X \Box Y = Y \vartriangle X \quad \forall \quad X, Y \in \mathcal{A}^{**}$

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$\leadsto$ How to measure the degree of non-regularity?

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Definition (Dales–Lau ’05)

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$\mathcal{A}$ comm.e $\Rightarrow$ $Z_\ell = Z_r =$ alg. centre of $\mathcal{A}^{**}$ (w.r.t. either product)

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Theorem (Lau–Losert ’88)

$L_1(\mathcal{G})$ is SAI for any locally compact group $\mathcal{G}$.

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Proposition (Dales–Lau '05; N)

$$LSAI \not\Leftrightarrow RSAI$$

Example *convolution algebra* $\mathcal{T}(\mathcal{G}) = (\mathcal{T}(L_2(\mathcal{G})), *)$

$$\rho * \tau := \int_{\mathcal{G}} L_x \rho L_{x^{-1}} \pi(\tau)(x)\ dx$$

$\mathcal{G}$ *non-compact, second countable* $\Rightarrow \mathcal{T}(\mathcal{G})$ *LSAI but not RSAI*

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Transfer of topological dynamics!

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Conjecture (Lau '94 & Ghahramani–Lau '95)

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First results

Theorem (N)

The conjecture holds for all non-compact groups $\mathcal{G}$ s.t.
$\mathcal{G}$ has non-measurable cardinality OR $k(\mathcal{G}) \geq 2^{\chi(\mathcal{G})}$

One cannot prove in ZFC the existence of measurable cardinals (Ulam '30).

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One cannot prove in ZFC the existence of measurable cardinals (Ulam '30).

Theorem (Losert '09)

The second condition can be weakened to $k(\mathcal{G}) \geq \chi(\mathcal{G})$.

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Theorem (N; Hu–N)

$M(\mathcal{G})$ has $F_{k(\mathcal{G})}$ (for non-compact $\mathcal{G}$) and $M_{|\mathcal{G}|}$.

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We only sketch the first case below (with $\mathcal{G}$ non-discrete).

Thinness: separation of singular measures

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Let κ be a cardinal. Then $\mu \in M(\mathcal{G})$ is κ-thin if $\exists\ P \subseteq \mathcal{G}$ s.t. $|P| = \kappa$ and $\mu * p \perp \mu * p'\ \forall\ p \neq p'$ in P.

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The following generalizes a result by Prokaj ('03) for $\mathcal{G} = \mathbb{R}$.

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The following generalizes a result by Prokaj ('03) for $\mathcal{G} = \mathbb{R}$.

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Every $\mu \in M_s(\mathcal{G})$ is $\mathfrak{c}$-thin.

Corollary (Separation)

$(F_\alpha)_{\alpha \in I}$ family of finite subsets of $M_s(\mathcal{G})$ with $|I| \leq \mathfrak{c}$
$\Rightarrow \exists\, (x_\alpha) \subseteq \mathcal{G}$ s.t. $(F_\alpha * x_\alpha) \perp (F_\beta * x_\beta)$ if $\alpha \neq \beta$ in I

Factorization in the dual of singular measures

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The key to construct h is the Separation Lemma.

The conclusion

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$\Rightarrow m_a \in Z_\ell(L_1(\mathcal{G})^{**}) = L_1(\mathcal{G})$, and $m = m_a + m_s \in M(\mathcal{G})$. □

Ghahramani–Lau beyond local compactness

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Theorem (L–N–P–S)

Let $\mathcal{G}$ be any Polish group. Then $M(\mathcal{G})$ is SAI.

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Ingredients of proof:

Theorem (Mycielski '64)

Let G be a Polish group and $\varnothing \neq Z \subseteq G$ a meagre subset. Then there is a perfect set $P \subseteq G$ s.t. $xy^{-1} \notin Z$ for all $x \neq y$ in P.

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If a Polish group G contains a non-meagre, σ-compact Borel set, then G is LC.

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Theorem (L–N–P–S)

If G is a Polish, non-LC group, every measure in $M(G)$ is $\mathfrak{c}$-thin.

Commercial Break 1

For further structural results on $M(\mathcal{G})^{**}$:

H.G. Dales, A.T.-M. Lau & D. Strauss

Second duals of measure algebras

Dissertationes Mathematicae (2011)

More on life beyond local compactness

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$$\langle X \Box Y, f \rangle := \langle X, Y \Box f \rangle$$

where $(Y \Box f)(x) := \langle Y, L_x f \rangle \quad (x \in \mathcal{G})$

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$$\mathcal{G}^{\mathrm{LUC}} = \text{spectrum of (commutative } C^*\text{-algebra) } \mathrm{LUC}(\mathcal{G})$$

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$$Z_t(\mathrm{LUC}(\mathcal{G})^*) \quad := \quad \{X \in \mathrm{LUC}(\mathcal{G})^* \mid \mathrm{LUC}(\mathcal{G})^* \ni Y \mapsto X \Box Y \ w^*\text{-cont.}\}$$

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Theorem (Ferri–N)

YES to Csiszár if $\mathcal{G}$ is separable

Pachl has generalized this to all ambitable groups.

DTC sets beyond local compactness

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Definition

A set $D \subseteq \mathrm{LUC}(\mathcal{G})^*$ is Determining for the Topological Centre if we have:
$m \in \mathrm{LUC}(\mathcal{G})^*$ lies in $Z_t(\mathrm{LUC}(\mathcal{G})^*)$
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Recall: $\kappa \geq \aleph_0$ cardinal; $\mathcal{G}$ is κ-bounded if for every open nhd. U of $e_{\mathcal{G}}$ there is a set $A \subseteq \mathcal{G}$ with $|A| \leq \kappa$ such that $\mathcal{G} = UA$.

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Theorem (Ferri–N–Pachl)

Assume that $\mathcal{G}$ is LC, or $\mathfrak{B}_{\mathcal{G}}$ is $\aleph_0$, or a successor cardinal. Then Csiszár's conjecture holds – with a 1 point DTC set!

Applications to $\mathcal{G}^{\mathsf{LUC}}$

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$\mathcal{G}$ *separable. TFAE:*

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Commercial Break 2

J. Pachl

Uniform Spaces and Measures

Fields Institute Monographs (2013)

The dual setting

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Problem (Cechini–Zappa '81)

Consider the Fourier algebra $A(\mathcal{G}) = \{\langle L_{(\cdot)}\xi, \eta\rangle \mid \xi, \eta \in L_2(\mathcal{G})\}$.
Is $A(\mathcal{G})$ SAI?

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Yes for any compact group that is sufficiently non-metrizable ($\chi(\mathcal{G})$ has uncountable cofinality); e.g., $SU(3)^{\aleph_1}$ and $SU(3)^{\mathfrak{c}}$

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Method of proof: Factorization & Mazur's property for $A(\mathcal{G})$

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Method of proof: Factorization & Mazur's property for $A(\mathcal{G})$

Theorem (Filali–Monfared–N)

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$A(\mathcal{G})$ has $M_{\chi(\mathcal{G})\cdot\aleph_0}$.

Lau–Wong's Conjecture, I

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Recall: $L_1(\mathcal{G})$ Arens regular $\Rightarrow$ $\mathcal{G}$ finite

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We know of no group outside of our class – is Olshanskii's group weakly amenable?

Topological centres and multipliers

Topological centres and multipliers

Problem (Lau–Ülger '96)

$\mathcal{A}$ Banach algebra with BAI s.t. $\mathcal{A}^*$ vN algebra. Let $X \in Z_r(\mathcal{A}^{**})$.
Consider $X_\square : \mathcal{A}^* \ni h \mapsto X \square h \in \mathcal{A}^*$.

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Proof: Combine Losert's result $Z(\mathcal{A}^{**}) \neq \mathcal{A}$ with the following

Theorem (Hu–N–Ruan)

Assume $\mathcal{A}$ *separable. Then, for* $X \in Z_r(\mathcal{A}^{**})$:

$$X \in \mathcal{A} \iff Ker(X_\Box) \text{ and } X_\Box(\mathrm{B}_1\mathcal{A}^*) \text{ are } w^*\text{-closed}$$

This uses work by Godefroy–Talagrand '89 and N

A natural new notion: metric Arens irregularity

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For any Banach algebra $\mathcal{A}$, consider

$$g(\mathcal{A}) := \sup_{m,n \in \mathrm{B}_1\mathcal{A}^{**}} \| m \Box n - m \triangle n \|$$

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Definition (Hu–N–Ruan)

We call a Banach algebra $\mathcal{A}$ with $g(\mathcal{A}) = 2$ metrically SAI.

Examples, I

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Corollary

If there is infinite discrete $\mathcal{G}$ with $g(\ell_1(\mathcal{G})) \neq 2$, then $\mathcal{G}$ is a counter-example to von Neumann's problem, such as Olshanskii's group (in fact, $\mathcal{G}$ admits no infinite amenable subgroups).

Examples, II

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Question: Which values can $g(\mathcal{A})$ take?
$\rightsquigarrow$ calculate $g(\mathcal{A})$ for Beurling algebras, $\mathcal{T}(\mathcal{G})$, ...

Group actions and invariant means

Group actions and invariant means

Solution to the Banach–Ruziewicz Problem (Banach '23; Margulis/Sullivan '80/'81; Drinfeld '84)

Except for $n = 1$, Lebesgue measure is the only invariant mean on $L_\infty(S^n)$ for the $O(n+1)$-action.

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What about general actions $\mathcal{G} \curvearrowright X$?

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$\mathcal{G} \curvearrowright X$ with $\mathcal{G}, X$ infinite countable.
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Arens type product and topological centre for group actions

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The topological centre of Foreman's group $\mathcal{F}$

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By using work of Erdös and Shelah, we even obtain:

Theorem (N–Pachl–Steprāns)

$CH \Rightarrow \ell_1(\mathcal{F}) \subsetneqq Z_t(\mathcal{F}, \mathbb{N}) \subsetneqq \ell_1(\mathcal{F})^{**}$

Ghahramani–Farhadi's multiplier problem

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1. *This fails for non-discrete groups.*
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Problem (Farhadi–Ghahramani '07)

Does every group $\mathcal{G}$ satisfy $(*)$?

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Remainder/Corona/Growth:

$$\mathcal{G}^* := \beta\mathcal{G} \setminus \mathcal{G}$$

$\Rightarrow \mathcal{G}^*$ compact right topological semigroup

Module maps

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$m \in \beta\mathcal{G}$ is called left cancellable if λ_m is injective on $\beta\mathcal{G}$

Proposition (Dales–Lau–Strauss '08)

$m \in \beta\mathcal{G}$ left cancellable $\Rightarrow$ $\lambda_m : \ell_1(\mathcal{G})^{**} \to \ell_1(\mathcal{G})^{**}$ isometry

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Write $\mathcal{A} := \ell_1(\mathcal{G})^{**}$.

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Proposition $\Rightarrow$ $\Phi := \lambda_m^* : \mathcal{A}^* \to \mathcal{A}^*$ (normal &) surjective

Need to show:

1. Φ is a right $\ell_1(\mathcal{G})$-module map
2. Φ is not a right $\ell_1(\mathcal{G})^{**}$-module map

$\Phi = \lambda_m^*$ is a right $\ell_1(\mathcal{G})$-module map

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Recall: $\mathcal{A} = \ell_1(\mathcal{G})^{**}$

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But $a \in \ell_1(\mathcal{G}) = Z(\mathcal{A})$, so a commutes with $m \in \mathcal{G}^* \subseteq \mathcal{A}$:

$$\langle \Phi(H \Box a), b \rangle = \langle H, m * a * b \rangle = \langle \Phi(H) \Box a, b \rangle$$

as desired.

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$\Rightarrow \ a * m * b = m * a * b \ \forall\ a, b \in \mathcal{A}$

$\Rightarrow \ (\text{with } b = \delta_e)\ m \in Z(\mathcal{A}) = \ell_1(\mathcal{G})$

This contradicts $m \in \mathcal{G}^*$.

Topological centres for quantum group algebras

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Definition

Hopf–von Neumann algebra (M, Γ)

- M von Neumann algebra
- $\Gamma : M \to M \bar{\otimes} M$ co-multiplication

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Locally compact quantum groups

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Non-commutative integration

N.s.f. weight $\lambda : M^+ \to [0, \infty]$

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LC Quantum Group $\mathbb{G} = (M, \Gamma, \lambda, \rho)$

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"Pontryagin duality" $\quad \widehat{\widehat{\mathbb{G}}} \cong \mathbb{G}$

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$$\mathrm{LUC}(\mathbb{G}) := \overline{\mathrm{lin}}\, L_\infty(\mathbb{G}) \square L_1(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$$

$$\mathrm{WAP}(\mathbb{G}) := \{\ T \in L_\infty(\mathbb{G}) \mid L_1(\mathbb{G}) \ni f \mapsto T \square f \text{ weakly compact } \}$$

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TFAE:

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If yes, then there is NO infinite $\mathcal{G}$ with $A(\mathcal{G})$ Arens regular!
Open for Olshanskii group . . .

Characterizations using invariant means on quantum groups

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$\mathbb{G}$ amenable $:\Leftrightarrow \exists$ mean on $L_\infty(\mathbb{G})$ s.t.

$$f \Box M = \langle f, 1 \rangle \; M \quad \forall\, f \in L_1(\mathbb{G})$$

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$\mathcal{G}$ LC group. Then $f \in L_\infty(\mathcal{G})$ is LUC $\Leftrightarrow \forall \varepsilon > 0\ \exists U \in \mathfrak{U}(e)$ s.t.

$$\|\ell_x f - f\|_\infty < \varepsilon \quad \forall x \in U$$

By Cohen: $LUC(\mathcal{G}) = L_\infty(\mathcal{G}) * L_1(\mathcal{G})$

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What about equi uniform continuity?

Recall: $(f_\alpha) \subseteq \mathrm{B}_1\mathrm{LUC}(\mathbb{G})$ is equi-LUC if $\forall \varepsilon > 0 \; \exists U \in \mathfrak{U}(e)$ s.t.

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General result: $\mathcal{A}$ Banach algebra with BAI for action on Banach A-module X; if $K \subseteq X$ norm-compact, then $\exists a \in \mathcal{A}$ s.t. $K \subseteq X * a$.
Apply this with $\mathcal{A} = L_1(\mathcal{G})$, $X = \mathrm{LUC}(\mathcal{G})$, $K = \overline{\{f_\alpha\}}$. □

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From quantum groups back to groups

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Theorem (Kalantar–N)

The functor $\mathbb{G} \to \widetilde{\mathbb{G}}$ *preserves*

- *local compactness*
- *compactness*
- *discreteness*
- *(hence) finiteness*

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The embedding $\widetilde{\mathbb{G}} \subseteq \mathfrak{M}_{cb} L_1(\mathbb{G})$ *in* $\mathrm{LUC}(\mathbb{G})^*$ *gives rise to*

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Then $\mathbb{G}^{\mathrm{LUC}}$ *is a compact right topological semigroup.*

Semigroup compactifications from quantum groups

Assume $\mathbb{G}$ co-amenable.

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Note: $\mathbb{G} = L_\infty(\mathcal{G})$ for a LC group $\mathcal{G}$ $\Rightarrow$ $\mathbb{G}^{\mathrm{LUC}} = \mathcal{G}^{\mathrm{LUC}}$

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$$\mathrm{m} \circ (\widehat{\mathrm{m}} \otimes \mathrm{id}) = \widehat{\mathrm{m}} \circ (\mathrm{m} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma)$$

Here, $\sigma(\varphi \otimes \tau) = \tau \otimes \varphi$ is the **flip**.

Duality = Anti-Commutation Relation on tensor level

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$$(\varphi * \tau) \bullet \psi = (\varphi \bullet \psi) * \tau$$

(Kalantar–N)

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Theorem (Kalantar–N)

$\mathbb{C}$ *is projective in mod–*$\mathcal{T}(L_2(\mathbb{G}))$

$\Leftrightarrow$ $\mathbb{G}$ *is compact*

Woronowicz's $SU_q(2)$

C^*-algebra generated by a and b with

$$\begin{aligned} b^* b &= bb^* \\ a^* a + b^* b &= 1 \\ ab &= q\ ba \\ ab^* &= q\ b^* a \\ aa^* + q^2\ bb^* &= 1 \end{aligned}$$

Co-multiplication:

$$\begin{aligned} \Gamma(a) &= a \otimes a - q\ b^* \otimes b \\ \Gamma(b) &= b \otimes a + a^* \otimes b \end{aligned}$$

Compact matrix pseudogroups

Definition (Woronowicz '87)

Given A unital C^*-algebra, $u \in M_n(A)_{\text{inv}}$.

$\mathbb{G} = (A, u)$ compact matrix pseudogroup if

- *-subalgebra $\mathcal{A}$ generated by u_{ij} is dense in A
- $\exists$ co-multiplication Γ on A
- $\exists\ \kappa : \mathcal{A} \to \mathcal{A}$ anti-multiplicative, $\kappa(\kappa(a^*)^*) = a\ (a \in \mathcal{A})$,

$$(\mathrm{id} \otimes \kappa)u = u^{-1}$$

$\rightsquigarrow$ For $SU_q(2)$: $\quad u = \begin{pmatrix} a & -q\ b^* \\ b & a^* \end{pmatrix}$