Injectivity and (quantum group) amenability

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1. **Amuse-gueule**
2. **Locally Compact Quantum Groups**
3. **Duality via $\mathcal{T}(L_2(\mathbb{G}))$**
4. **Amenability $=$ $\mathcal{T}(L_2(\mathbb{G}))$-Covariant Injectivity**

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We will answer this more generally for LC quantum groups.

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- $M = \mathcal{L}(\mathcal{G}) = A(\mathcal{G})^*$
  $\Gamma =$ adjoint of pointwise product $\bullet$

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Given an nsf weight φ on M, N_φ, equipped with $(x, y) := \varphi(y^*x)$, is a pre-Hilbert space; we denote by $L_2(M, \varphi)$ its completion.

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  $\varphi = \psi$ given by the Plancherel weight on $\mathcal{L}(\mathcal{G})$

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$\rightsquigarrow$ $L_1(\mathbb{G})$ Banach algebra: $f * g = \Gamma_*(f \otimes g)$

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We obtain dual quantum group $\hat{\mathbb{G}} = (\hat{M}, \hat{\Gamma}, \hat{\varphi}, \hat{\psi})$ with

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We can identify $L_2(M, \varphi) \cong L_2(\hat{M}, \hat{\varphi})$, and we write $L_2(\mathbb{G})$.

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...building on earlier work by Baaj–Skandalis, Effros–Ruan, Enock–Schwartz, Kac–Vainerman, Takesaki, ...

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  - Non-commutative C^*-algebra for $q \in (0, 1)$

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Theorem (Junge–N–Ruan)

$$\theta : \mathfrak{M}_{cb} L_1(\mathbb{G}) \cong \mathcal{NCB}_{L_\infty(\hat{\mathbb{G}})}^{L_\infty(\mathbb{G})}(\mathcal{B}(L_2(\mathbb{G})))$$

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Anti-Commutation Relation on Tensor Level!

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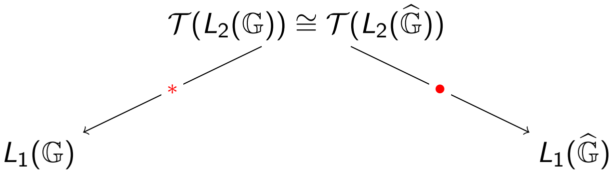

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$$\boxed{(\rho * \tau) \cdot \psi \;=\; (\rho \cdot \psi) * \tau}$$

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$\mathrm{LUC}(\mathbb{G})$

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Proposition (Hu–N–Ruan)

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Proposition (Hu–N–Ruan)

$\mathbb{G}$ *co-amenable. Then:*

$$\mathrm{LUC}(\mathbb{G}) = \mathrm{LUC}(\mathbb{G}) * L_1(\mathbb{G}) = L_\infty(\mathbb{G}) * L_1(\mathbb{G}) = \mathcal{B}(L_2(\mathbb{G})) * \mathcal{T}(L_2(\mathbb{G}))$$

$\mathrm{LUC}(\mathbb{G})$ as a C^*-algebra

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Definition

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Kac algebras, compact and discrete quantum groups are regular

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Theorem (Hu–N–Ruan)

If $\mathbb{G}$ is semi-regular, then $\mathrm{LUC}(\mathbb{G})$ is unital C^*-algebra.

Regularity and Discreteness

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Theorem (Hu–N–Ruan)

TFAE:

- $\mathbb{G}$ *is regular*
- *convolution in* $\mathcal{T}(L_2(\mathbb{G}))$ *is* w^**-cont. on the left*

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TFAE:

- *convolution in* $\mathcal{T}(L_2(\mathbb{G}))$ *is* w^**-cont. on the right*
- *convolution in* $\mathcal{T}(L_2(\mathbb{G}))$ *is separately* w^**-cont.*
- $\mathbb{G}$ *is discrete*

Arens (Ir-)Regularity

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Theorem (Hu–N–Ruan)

$\mathbb{G}$ bi-amenable with $L_1(\mathbb{G})$ separable. TFAE:

- *$\mathcal{T}(L_2(\mathbb{G}))$ is Arens Regular*
- *$\mathcal{T}(L_2(\mathbb{G}))$ is Strongly Arens Irregular (SAI)*
- *$\mathbb{G}$ is finite*

1. Amuse-gueule
2. Locally Compact Quantum Groups
3. Duality via $\mathcal{T}(L_2(\mathbb{G}))$
4. **Amenability = $\mathcal{T}(L_2(\mathbb{G}))$-Covariant Injectivity**

Amenability and Co-amenability, I

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$\mathbb{G}$ amenable if $\exists$ state $F \in L_\infty(\mathbb{G})^*$ s.t. $\langle F, h * g\rangle = \langle g, 1\rangle \langle F, h\rangle$ $\forall\ h \in L_\infty(\mathbb{G})$, $g \in L_1(\mathbb{G})$

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- $\mathbb{G} = \mathcal{L}(\mathcal{G})$ co-amenable $\Leftrightarrow$ $\mathcal{G}$ amenable (Leptin)

Amenability and Co-amenability, II

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Theorem (Bédos–Tuset '03; Tomatsu '06)

$\hat{\mathbb{G}}$ *co-amenable* $\Rightarrow$ $\mathbb{G}$ *amenable.*

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Proposition (Kalantar–N)

Let $\mathbb{G}$ be co-amenable. TFAE:

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- *$\lambda : L_1(\mathbb{G}) \to L_\infty(\hat{\mathbb{G}})$ isometry on $L_1(\mathbb{G})^+$*

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We get equivalence if we take into account action by $\mathcal{T}(L_2(\mathbb{G}))$!

Products on $\mathcal{T}(L_2(\mathbb{G}))$

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V induces extended co-multiplication

$$\Gamma^r : \mathcal{B}(L_2(\mathbb{G})) \ni x \mapsto V(x \otimes 1)V^* \in \mathcal{B}(L_2(\mathbb{G}))\overline{\otimes}\mathcal{B}(L_2(\mathbb{G}))$$

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Both $\triangleright$ and $\triangleleft$ yield $\mathcal{T}(L_2(\mathbb{G}))$-actions on $\mathcal{B}(L_2(\mathbb{G}))$

Amenability and Injectivity, II

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Theorem (Crann–N)

$\mathbb{G}$ *LC quantum group.*

$\mathbb{G}$ *amenable* $\Leftrightarrow \exists$ *conditional expectation* $E : \mathcal{B}(L_2(\mathbb{G})) \to L_\infty(\hat{\mathbb{G}})$ *commuting with right* $\mathcal{T}(L_2(\mathbb{G}))_\triangleright$*-action*

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In other words:

LC quantum group $\mathbb{G}$ amenable $\Leftrightarrow L_\infty(\hat{\mathbb{G}})$ covariantly injective

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Corollary (Crann–N)

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Here, $m \in \mathrm{LUC}(\mathbb{G})^*$ acts via

$$\langle \Theta(m)(T), \rho \rangle = \langle m, \underbrace{T \triangleright \rho}_{\in \mathrm{LUC}(\mathbb{G})} \rangle \ \forall\ T \in \mathcal{B}(L_2(\mathbb{G})),\ \rho \in \mathcal{T}(L_2(\mathbb{G}))$$

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$\rightsquigarrow$ cond. exp. $E_m : \mathcal{B}(L_2(\mathcal{G})) \to \mathcal{L}(\mathcal{G})$ (Paterson '92)

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- $L_\infty(\hat{\mathbb{G}})$ *covariantly injective via normal conditional expectation*

Discreteness vs. RNP

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Recall: $L_1(\mathcal{G})$ has RNP $\Leftrightarrow$ $\mathcal{G}$ discrete

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Let $d_2 \leq \alpha \leq \pi (= d_1)$ where d_p is the Chevet–Saphar tensor norm. TFAE:

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Proposition (Kalantar–N–Ruan)

The map $\mathcal{L}(\mathcal{G}) \otimes_{d_2} A(\mathcal{G}) \to \ell_2$, $T \mapsto \tau(\lambda_{x^{-1}} T(\lambda_x))$ *is surjective.*

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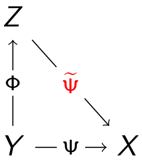
  morphism $\Psi : Y \to X$
  $\exists$ morphism $\widetilde{\Psi} : Z \to X$ such that $\widetilde{\Psi} \circ \Phi = \Psi$



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Regularity and Invariance

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Theorem (Hu–N–Ruan)

TFAE:

- $\mathbb{G}$ *regular*
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Theorem (Hu–N–Ruan)

$$\mathfrak{M}_{cb}L_1(\mathbb{G}) \;\cong\; \mathcal{NCB}_{\mathcal{T}}^{\mathcal{K}^*}(\mathcal{B}(L_2(\mathbb{G})))$$

$\mathbb{G}$-continuous operators, I

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$\mathbb{G}$-continuous operators, II

Theorem (Hu–N–Ruan)

Let $\mathbb{G}$ be co-amenable with $L_1(\mathbb{G})$ separable. TFAE:

- $X(L_2(\mathbb{G})) = \mathcal{B}(L_2(\mathbb{G}))$
- $L_\infty \subseteq X(L_2(\mathbb{G}))$
- $\mathbb{G}$ *discrete*

Theorem (Hu–N–Ruan)

Let $\mathbb{G}$ be semi-regular. Then $X(L_2(\mathbb{G}))$ is a unital C^*-subalgebra of $\mathcal{B}(L_2(\mathbb{G}))$.

Woronowicz's $SU_q(2)$

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C^*-algebra generated by a and b with

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Co-multiplication:

$$\begin{aligned} \Gamma(a) &= a \otimes a - q\ b^* \otimes b \\ \Gamma(b) &= b \otimes a + a^* \otimes b \end{aligned}$$