Classes of Herz-Schur multipliers

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- Approximation properties

Positive Herz-Schur multipliers

Definition

- Let X be a set. A function $k : X \times X \to \mathbb{C}$ is called *positive definite* if $(k(x_i, x_j))_{i,j=1}^n$ is a positive matrix, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$.

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- Let G be a group. A function $u : G \to \mathbb{C}$ is called *positive definite* if $N(u)$ is positive definite.

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- Let G be a group. A function $u : G \to \mathbb{C}$ is called *positive definite* if $N(u)$ is positive definite.

Schur product lemma

Let $A = (a_{i,j}) \in M_n$. The following are equivalent:

- for every positive matrix $B \in M_n$, the Schur product $A * B$ of A and B is a positive matrix;
- the matrix A is positive.

Positive Herz-Schur multipliers

Mercer's Theorem

If X is a locally compact Hausdorff space equipped with a regular Borel measure of full support and if $h \in L^2(X \times X) \cap C(X \times X)$, the operator T_h is positive if and only if h is positve definite.

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Proposition

Let X be a σ-compact metric space, equipped with a Radon measure μ with full support. Let $k : X \times X \to \mathbb{C}$ be a continuous Schur multiplier. The following are equivalent:

(i) S_k is positive;

(ii) k is positive definite.

Positive Herz-Schur multipliers

Proof.

(i)$\Rightarrow$(ii) By the assumption and Mercer's Theorem, kh is positive definite whenever $h \in L^2(X \times X) \cap C(X \times X)$ is positive definite. The statement now follows from the Schur product lemma.

Positive Herz-Schur multipliers

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(ii)$\Rightarrow$(i) follows from Mercer's Theorem, the Schur product lemma and the fact that

$$\overline{\{T_k : k \in L^2(X \times X) \cap C(X \times X), T_k \geq 0\}}^{\|\cdot\|} = \mathcal{K}(L^2(X))^+.$$

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(The latter can be seen as follows: suppose that $h \in T(X, X)$ and $\langle T_k, T_h \rangle \geq 0$ for each $k \in L^2(X \times X) \cap C(X \times X)$ with $T_k \geq 0$. By taking $k = a \otimes \bar{a}$, where $a \in C_c(X)$, we see that $(T_h a, a) \geq 0$ for all such a, and this implies that $T_h \geq 0$. It follows that $\langle T, T_h \rangle \geq 0$ for all $T \in \mathcal{K}(L^2(X))^+$, and the claim now follows by Hahn-Banach separation.) □

Positive Herz-Schur multipliers

Lemma

The unit ball of the subspace

$$\mathcal{A} = \left\{ \sum_{i=1}^{k} A_i T_i : A_i \in \mathcal{D}_G, T_i \in \mathrm{VN}(G) \right\}$$

is strongly dense in the unit ball of $\mathcal{B}(L^2(G))$.

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Proof.

Use the Stone-von Neumann Theorem, according to which the representation of the crossed product $G \times_\alpha C_0(G)$ arising from the covariant pair of representations (λ, π), where λ is the left regular representation of G and $\pi : C_0(G) \to \mathcal{B}(L^2(G))$ is given by $\pi(a) = M_a$, is faithful and its image coincides with the C*-algebra $\mathcal{K}(L^2(G))$ of all compact operators on $L^2(G)$. □

Positive Herz-Schur multipliers

Theorem (de Canniere-Haagerup, 1985)

Let G be a locally compact group and $u : G \to \mathbb{C}$ be a continuous function. The following are equivalent:

(i) $u \in M^{\rm cb}A(G)$ and S_u is completely positive;

(ii) u is positive definite.

If these conditions are fulfilled then $\|u\|_{\rm cbm} = u(e)$.

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Thus, its restriction S_u to $\mathrm{VN}(G)$ is completely positive. □

Positive Herz-Schur multipliers

Proof.

(i)⇒(ii) Let $T = SS^*$ for some contraction $S \in \mathcal{B}(L^2(G))$. Approximate S in the strong operator topology by contractions of the form $\sum_{i=1}^{k} A_i T_i$, where $A_i \in \mathcal{D}_G$ and $T_i \in \mathrm{VN}(G)$, $i = 1, \ldots, k$.

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Since $(T_i T_j^*)_{i,j} \geq 0$, we have $(S_u(T_i T_j^*))_{i,j} \geq 0$. Letting $A = (A_1, \ldots, A_k)$, we have

$$S_{N(u)} \left(\sum_{i,j=1}^k A_i T_i T_j^* A_j^* \right) = A(S_u(T_i T_j^*))_{i,j} A^* \geq 0.$$

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By weak* continuity, $S_{N(u)}$ is positive and so u is positive definite. Finally, note that $S_u(I) = S_u(\lambda_e) = u(e)I$. □

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Idempotent Herz-Schur multipliers

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Theorem (A.-M. Stan, 2009)

Let G be a locally compact group and $E \subseteq G$. The following are equivalent:

(i) $\chi_E \in M^{\mathrm{cb}}A(G)$ and $\|\chi_E\|_{\mathrm{cbm}} = 1$;

(ii) E belongs to the coset ring of G.

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If X and Y are sets, call a subset $E \subseteq X \times Y$ an *operator L-set* if $\ell^\infty(E) \subseteq \mathfrak{S}(X, Y)$.

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$\Lambda \subseteq G$ is an L-set if and only if $\Lambda^* = \{(s, t) : ts^{-1} \in \Lambda\}$ is an opetator L-set.

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$$|\{y \in Y : (x,y) \in E_1\}| \leq C, \ \forall x \in X,$$

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The set E of generators of $\mathbb{F}_\infty$ is an L-set.

Radial functions on $\mathbb{F}_r$

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Reduced word in $\mathbb{F}_r$: $t = t_1 \ldots t_k$, where $t_i \in \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots\}$ and $t_i^{-1} \neq t_{i+1}$ for all i.

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A function $\varphi : \mathbb{F}_r \to \mathbb{C}$ is called *radial* if it only depends on $|t|$; that is, if there exists a function $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) such that $\varphi(s) = \dot{\varphi}(|s|)$, $s \in \mathbb{F}_r$.

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Given φ radial, we always let $\dot{\varphi}$ be the corresponding "underlying" function on $\mathbb{N}_0$.

Sufficient conditions for a multiplier of $A(\mathbb{F}_r)$

Theorem (Haagerup (1979), Haagerup-Steenstrup-Szwarc (2010))

Let $\varphi : \mathbb{F}_r \to \mathbb{C}$.

(i) If $\sup_{s\in\mathbb{F}_r} |\varphi(s)|(1+|s|^2) < \infty$ then $\varphi \in MA(\mathbb{F}_r)$ and

$$\|\varphi\|_{\mathrm{m}} \leq \sup_{s\in\mathbb{F}_r} |\varphi(s)|(1+|s|^2).$$

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(ii) Suppose that φ is radial and $\sum_{n=0}^{\infty}(n+1)^2|\dot{\varphi}(n)|^2 < \infty$. Then $\varphi \in MA(\mathbb{F}_r)$ and

$$\|\varphi\|_{\mathrm{m}} \leq \left(\sum_{n=0}^{\infty}(n+1)^2|\dot{\varphi}(n)|^2\right)^{\frac{1}{2}}.$$

Radial multipliers and homogeneous trees

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The *Cayley graph* $\mathcal{C}_G$ of G is the graph whose vertices are the elements of G, and $\{s, t\} \subseteq G$ is an edge of $\mathcal{C}_G$ if $ts^{-1} \in \mathcal{E}$.

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A *tree* is a connected graph without cycles. The degree of a vertex is the number of edges containing the vertex, and a graph is called *locally finite* if the degrees of all vertices are finite.

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It is called *homogeneous* if all vertices have the same degree (called in this case the degee of the graph).

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Theorem (Figà-Talamanca, Nebbia, 1982)

Let G be a discrete finitely generated group. The Cayley graph $\mathcal{C}_G$ of G is a locally finite homogeneous tree if and only if G is of the form $G = (*_{i=1}^{M}\mathbb{Z}_2) * \mathbb{F}_N$; in this case, the degree q of $\mathcal{C}_G$ is equal to $2M + N - 1$.

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If $\mathcal{C}$ is a homogeneous tree with vertex set X, let $d(x, y)$ be the distance between two vertices x, y; that is, the length of the (unique) path connecting x and y; we set $d(x, x) = 0$.

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Theorem (Figà-Talamanca, Nebbia, 1982)

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If $\mathcal{C}$ is a homogeneous tree with vertex set X, let $d(x,y)$ be the distance between two vertices x, y; that is, the length of the (unique) path connecting x and y; we set $d(x,x) = 0$.

A function $\varphi : X \to \mathbb{C}$ is *radial* if there exists $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ with $\varphi(x) = \dot{\varphi}(d(x,o))$, $x \in X$ (here o is a fixed vertex).

Radial multipliers and homogeneous trees

Proposition

Let $\varphi : G \to \mathbb{C}$ be a radial function and $\tilde{\varphi} : G \times G \to \mathbb{C}$ be given by $\tilde{\varphi}(s,t) = \dot{\varphi}(d(s,t))$, $s, t \in G$.

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Then $\varphi \in M^{\mathrm{cb}}A(G)$ if and only if $\tilde{\varphi}$ is a Schur multiplier; in this case, $\|\varphi\|_{\mathrm{cbm}} = \|\tilde{\varphi}\|_{\mathfrak{S}}$.

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Proof.

Since the distance d is left-invariant, we have

$$\tilde{\varphi}(s,t) = \dot{\varphi}(d(s,t)) = \dot{\varphi}(d(t^{-1}s, e)) = \varphi(t^{-1}s).$$

The claim now follows from the embedding theorem. □

Radial multipliers and homogeneous trees

Theorem (Haagerup-Steenstrup-Szwarc, 2010)

Let X be a homogeneous tree of degree $q+1$ $(2 \leq q \leq \infty)$. Let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ be a function, $\varphi : X \to \mathbb{C}$ be the corresponding radial function and $\tilde{\varphi}(x,y) = \dot{\varphi}(d(x,y))$, $x, y \in X$. Set

$$h_{i,j} = \dot{\varphi}(i+j) - \dot{\varphi}(i+j+2), \quad i,j \in \mathbb{N}_0.$$

(i) $\tilde{\varphi}$ is a Schur multiplier if and only if $H = (h_{i,j})_{i,j\in\mathbb{N}_0}$ is trace class.
(ii) $(q = \infty)$. If (i) hold, the limits

$$\lim_{n\to\infty} \dot{\varphi}(2n) \text{ and } \lim_{n\to\infty} \dot{\varphi}(2n+1)$$

exist. If $c_\pm = \frac{1}{2}(\lim_{n\to\infty} \dot{\varphi}(2n) \pm \lim_{n\to\infty} \dot{\varphi}(2n+1))$, then $\|\tilde{\varphi}\|_{\mathfrak{S}} = |c_+| + |c_-| + \|H\|_1$.

Multipliers that are not completely bounded

Theorem (Haagerup-Steenstrup-Szwarc, 2010)

Let $G = (*_{i=1}^{M}\mathbb{Z}_2) * \mathbb{F}_N$. There exists a radial function φ which lies in $MA(G)$ but not in $M^{\mathrm{cb}}A(G)$.

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Let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ be given by $\dot{\varphi}(n) = 0$ if $n \neq 2^k$, $k \in \mathbb{N}$, and $\dot{\varphi}(2^k) = \frac{1}{k2^k}$, $k \in \mathbb{N}$.

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Then

$$\sum_{n=0}^{\infty}(n+1)^2|\dot{\varphi}(n)|^2 < \infty,$$

and so $\varphi \in MA(G)$. A direct verification shows that the corresponding matrix H is not of trace class. □

Radial multipliers on free products

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The number m id called the *block length* of g. Call a function $\varphi : G \to \mathbb{C}$ radial if it depends only on $\|g\|$.

Theorem (Wysoczański, 1995)

Let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ and $\varphi : G \to \mathbb{C}$ be the corresponding radial function with respect to the block length. The following are equivalent:

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(ii) the matrix $(\dot{\varphi}(i+j) - \dot{\varphi}(i+j+1))_{i,j}$ defines a trace class operator on $\ell^2(\mathbb{N}_0)$.

Positive multipliers of the Fourier algebra of a free group

Theorem (Haagerup, 1979)

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A function $k : X \times X \to \mathbb{R}$ is called *conditionally negative definite* if $k(x,x) = 0$, $k(x,y) = k(y,x)$ for all x, y, and

$$\sum_{i,j=1}^{m} k(x_i, x_j)\alpha_i\alpha_j \leq 0,$$

for all $x_1, \ldots, x_m \in X$ and all $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ with $\sum_{i=1}^{m} \alpha_i = 0$.

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Shoenberg's Theorem

If $k(x,y)$ is a conditionally negative definite kernel, then, for $\lambda > 0$, $e^{-\lambda k(x,y)}$ is a positive definite function.

Positive multipliers of the Fourier algebra of a free group

Proof.

Let $k(s,t) = |s^{-1}t|$, $s, t \in \mathbb{F}_n$. It suffices to show that k is conditionally negative definite.

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Fix generators $a_1, \dots, a_n$ of $\mathbb{F}_n$, let $\Lambda = \{(s,t) \in \mathbb{F}_n \times \mathbb{F}_n : s^{-1}t = a_i, \text{ for some } i\}$ and $H = \ell^2(\Lambda)$.

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Let $\{e_{(s,t)} : (s,t) \in \Lambda\}$ be the standard basis of H. If $s^{-1}t = a_i^{-1}$ for some i, then set $e_{(s,t)} = -e_{(t,s)}$.

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For $s = s_1 s_2 \dots s_k$, where s_i is either a generator or its inverse, let

$$b(s) = e_{(e,s_1)} + e_{(s_1,s_1s_2)} + \cdots + e_{(s_1s_2\dots s_{k-1},s)}.$$

□

Positive multipliers of the Fourier algebra of a free group

Theorem

Let $\theta \in \mathbb{R}$. Then the function $\varphi_\theta : t \to \theta^{|t|}$ on $\mathbb{F}_\infty$ is positive definite if and only if $-1 \leq \theta \leq 1$.

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(ii) There exists a probability measure μ on $[-1, 1]$ such that

$$\varphi(x) = \int \theta^{|x|} d\mu(\theta), \quad x \in \mathbb{F}_\infty.$$

The radial algebra

Fix $r \in \mathbb{N}$ and let $E_n = \{x \in \mathbb{F}_r : |x| = n\}$.

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For $n > 0$, let μ_n be the function taking the same constant value on the elements of E_n and zero on $\mathbb{F}_r \setminus E_n$, such that $\sum_x \mu_n(x) = 1$ (note that the constant value equals $\frac{1}{2r(2r-1)^{n-1}}$).

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Denote by $\mathcal{A}$ the subalgebra of the group algebra $\mathbb{C}[\mathbb{F}_r]$ generated by μ_n, $n \geq 0$ – this is the algebra of all radial functions on $\mathbb{F}_r$, equipped with the operation of convolution. Clearly, $\mathcal{A}$ is the linear span of $\{\mu_n : n \geq 0\}$.

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Lemma

Let $q = 2r - 1$. Then

$$\mu_1 * \mu_n = \frac{1}{q+1}\mu_{n-1} + \frac{q}{q+1}\mu_{n+1}, \quad n \geq 1.$$

The radial algebra

Proof.

We have

$$\mu_1 * \mu_n(x) = \sum_{y \in \mathbb{F}_r} \mu_1(y)\mu_n(y^{-1}x) = \frac{1}{q+1} \sum_{|y|=1} \mu_n(y^{-1}x). \quad (1)$$

Let $\{a_1, \ldots, a_{q+1}\}$ be the set of words of length one. If $|x| = n+1$, then among the words $a_j x$, $j = 1, \ldots, q$, there is only one of length n, namely, the word $a_j x$ for which $x = a_j^{-1} x'$ (for some $x' \in \mathbb{F}_r$). Thus, in this case

$$\mu_1 * \mu_n(x) = \frac{1}{q+1} \frac{1}{(q+1)q^{n-1}} = \frac{q}{q+1} \mu_{n+1}(x).$$

□

The radial algebra

Proof.

If $|x| = n - 1$, then among the words $a_j x$, $j = 1, \ldots, r$, there are q of length n, and thus

$$\mu_1 * \mu_n(x) = \frac{1}{q+1} \frac{q}{(q+1)q^{n-1}} = \frac{1}{q+1} \mu_{n-1}(x).$$

Finally, if x has length different from $n+1$ or $n-1$ then all words $a_j x$, $j = 1, \ldots, q$ have length different from n and hence the right hand side of (1) is zero. The claim follows. □

The polynomials P_n

Define a sequence (P_n) of polynomials by setting $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{n+1}(x) = \frac{q+1}{q} x P_n(x) - \frac{1}{q} P_{n-1}(x), \quad n \geq 1. \tag{2}$$

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By the definition of this sequence, we have that

$$\mu_n = P_n(\mu_1), \quad n \geq 0. \tag{3}$$

(Here, the product is taken with respect to the convolution.)

The Laplace operator

The *Laplace operator* is the linear map L acting on $\mathbb{C}[\mathbb{F}_r]$ and given by

$$L\varphi = \mu_1 * \varphi, \quad \varphi \in \mathbb{C}[\mathbb{F}_r] .$$

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If $\varphi \in \mathbb{C}[\mathbb{F}_r]$ and $x \in \mathbb{F}_r$, then

$$L\varphi(x) = \frac{1}{q+1} \sum_y \varphi(y),$$

where the sum is taken over all neighbours y of x in the Cayley graph of $\mathbb{F}_r$.

Spherical functions

Definition

Call a function $\varphi \in \mathbb{C}[\mathbb{F}_r]$ *spherical* if φ is radial, $\varphi(e) = 1$ and $L\varphi = s\varphi$ for some $s \in \mathbb{C}$.

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Suppose that $\varphi \in \mathbb{C}[\mathbb{F}_r]$ is spherical and let $\dot{\varphi}$ be as usual the underlying function defined on $\mathbb{N}_0$. We have

$$\dot{\varphi}(0) = 1,\ \dot{\varphi}(1) = s,\ \dot{\varphi}(n+1) = \frac{q+1}{q} s\dot{\varphi}(n) - \frac{1}{q}\dot{\varphi}(n-1).$$

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$$\dot{\varphi}(n) = P_n(s), \quad n \geq 0.$$

It also follows that for each $s \in \mathbb{C}$ there exists a unique spherical function corresponding to the eigenvalue s; we denote this function by φ_s.

Spherical functions

On the group algebra $\mathbb{C}[\mathbb{F}_r]$, consider the bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \sum_{x \in \mathbb{F}_r} f(x) g(x), \quad f, g \in \mathbb{C}[\mathbb{F}_r] \, .$$

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Thus,

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(ii) the functional $f \to \langle f, \varphi \rangle$ on $\mathcal{A}$ is multiplicative.

Proof.

(i)⇒(ii) Let $s \in \mathbb{C}$. We have $\langle P_n(\mu_1), \varphi_s \rangle = P_n(s), \quad n \geq 0$. The set $\{P_n : n \geq 0\}$ spans the set of all polynomials, and hence by linearity

$$\langle P(\mu_1), \varphi_s \rangle = P(s), \qquad P \text{ a polynomial.}$$

On the other hand, the map $P \to P(\mu_1)$, is a homomoprhism from the algebra of all polynomials onto $\mathcal{A}$. Statement (ii) now follows. □

Spherical functions

Proof.

(ii)⇒(i) We have

$$\langle \mu_n, \varphi \rangle = \langle \mu_0 * \mu_n, \varphi \rangle = \langle \mu_0, \varphi \rangle \langle \mu_n, \varphi \rangle$$

and hence $\dot{\varphi}(0) = \langle \mu_0, \varphi \rangle = 1$.
Let $s = \dot{\varphi}(1) = \langle \mu_1, \varphi \rangle$. Then

$$\langle \mu_1 * \mu_n, \varphi \rangle = \langle \mu_1, \varphi \rangle \langle \mu_n, \varphi \rangle = s\dot{\varphi}(n),$$

$$\begin{aligned} \langle \mu_1 * \mu_n, \varphi \rangle &= \left\langle \frac{1}{q+1}\mu_{n-1}, \varphi \right\rangle + \left\langle \frac{q}{q+1}\mu_{n+1}, \varphi \right\rangle \\ &= \frac{1}{q+1}\dot{\varphi}(n-1) + \frac{q}{q+1}\dot{\varphi}(n+1). \end{aligned}$$

Thus, $\varphi = \varphi_s$. □

The expectation onto $\mathcal{A}$

Let $\mathcal{E} : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ be the map given by

$$\mathcal{E}(f)(x) = \frac{1}{(q+1)q^{n-1}} \sum_{|y|=n} f(y), \quad |x| = n;$$

thus,

$$\mathcal{E}(f)(x) = \langle f, \mu_n \rangle, \quad |x| = n.$$

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(a) $\mathcal{E}(f) = f$ if $f \in \mathcal{A}$;

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Let $\mathcal{E} : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ be the map given by

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Moreover, if $\mathcal{E}' : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ satisfies (a) and (b) then $\mathcal{E}' = \mathcal{E}$.

(ii) Let $\mathcal{R}$ be the von Neumann subalgebra of $\mathrm{VN}(\mathbb{F}_r)$ generated by $\mathcal{A}$. Then the map $\mathcal{E}$ extends to a normal conditional expectation from $\mathrm{VN}(\mathbb{F}_r)$ onto $\mathcal{R}$.

The expectation onto $\mathcal{A}$

Proof.

(i) Properties (a) and (b) are straightforward. Suppose $\mathcal{E}' : \mathbb{C}[\mathbb{F}_r] \to \mathcal{A}$ satisfies (a) and (b). If $f \in \mathbb{C}[\mathbb{F}_r]$ and $x \in \mathbb{F}_r$ then

$$\mathcal{E}'(f)(x) = \langle \mathcal{E}'(f), \delta_x \rangle = \langle \mathcal{E}'(f), \mathcal{E}(\delta_x) \rangle = \langle f, \mathcal{E}(\delta_x) \rangle = \mathcal{E}(f)(x).$$

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(ii) By general von Neumann algebra theory, there exists a normal conditional expectation from $\mathrm{VN}(\mathbb{F}_r)$ onto $\mathcal{R}$. Its restriction on $\mathbb{C}[\mathbb{F}_r]$ must satisfy (a) and (b), and by (i) it must coincide with $\mathcal{E}$. □

Positive definiteness of spherical functions

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The function φ_s is positive definite if and only if $-1 \leq s \leq 1$.

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Suppose that $-1 \leq s \leq 1$. Then φ_s is real-valued. It was shown by Figà-Talamanca and Picardello that in this case φ_s is also bounded. Let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A}$ in $\ell^1(\mathbb{F}_r)$. Since φ_s is radial, $\varphi_s(x) = \varphi_s(x^{-1})$ for all $x \in \mathbb{F}_r$.

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We claim that the functional $f \to \langle f, \varphi_s \rangle$ on $\overline{\mathcal{A}}$ is positive. Indeed, if $f \in \overline{\mathcal{A}}$ then □

Positive definiteness of spherical functions

Proof.

$$\begin{aligned}
\langle f * f^*, \varphi_s \rangle &= \langle f, \varphi_s \rangle \langle f^*, \varphi_s \rangle \\
&= \left(\sum_{x \in \mathbb{F}_r} f(x) \varphi_s(x) \right) \left(\sum_{x \in \mathbb{F}_r} \overline{f(x^{-1})} \varphi_s(x) \right) \\
&= \left(\sum_{x \in \mathbb{F}_r} f(x) \varphi_s(x) \right) \left(\sum_{x \in \mathbb{F}_r} \overline{f(x^{-1}) \varphi_s(x^{-1})} \right) \\
&= \langle f, \varphi_s \rangle \overline{\langle f, \varphi_s \rangle} \geq 0.
\end{aligned}$$

□

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Proof.

Now let $f \in \ell^1(\mathbb{F}_r)$ be positive. Then $\mathcal{E}(f)$ is positive and, by the previous slide,

$$\langle f, \varphi_s \rangle = \langle \mathcal{E}(f), \varphi_s \rangle \geq 0.$$

It follows that the functional on $\ell^1(\mathbb{F}_r)$, $f \to \langle f, \varphi_s \rangle$, is positive, and hence φ_s is positive definite.

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Conversely, suppose that φ_s is positive definite. Then $\overline{\varphi_s(x)} = \varphi_s(x^{-1})$ and since $|x| = |x^{-1}|$, the function φ_s is real-valued. Since φ_s is also bounded, $-1 \leq s \leq 1$. $\square$

Radial positive definite functions on $\mathbb{F}_r$

Theorem (Haagerup-Knudby, 2013)

Let $\varphi : \mathbb{F}_r \to \mathbb{C}$ be a radial function with $\varphi(e) = 1$. The following are equivalent:

(i) φ is positive definite;

(ii) there exists a probability measure μ on $[-1, 1]$ such that

$$\varphi(x) = \int_{-1}^{1} \varphi_s(x) d\mu(s), \qquad x \in \mathbb{F}_r.$$

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(ii)$\Rightarrow$(i) follows from the previous proposition. □

Radial positive definite functions on $\mathbb{F}_r$

Proof.

(i)⇒(ii) Let Φ (resp. Φ_s, $-1 \leq s \leq 1$) be the state on $C^*(\mathbb{F}_r)$ which corresponds to φ (resp. φ_s, $-1 \leq s \leq 1$).

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Let $C^*(\mu_1)$ be the C*-subalgebra of $C^*(\mathbb{F}_r)$ generated by μ_1; since $\mathcal{A}$ is generated by μ_1 as an algebra, $C^*(\mu_1)$ coincides with the closure of $\mathcal{A}$ in $C^*(\mathbb{F}_r)$.

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We have that $\mu_1 = \mu_1^*$ and $\|\mu_1\| \le 1$ (indeed, in every represtentation of $\mathbb{F}_r$, the image of μ_1 is the average of r unitary operators and hence has norm at most 1), we have that the spectrum of μ_1 is contained in $[-1, 1]$.

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Conversely, since $\Phi_s(\mu_1) = \langle \mu_1, \varphi_s \rangle = s$, we have that the spectrum of μ_1 coincides with $[-1, 1]$. □

Radial positive definite functions on $\mathbb{F}_r$

Proof.

It follows that $C^*(\mu_1)$ is *-isomorphic to $C([-1,1])$. The restriction of Φ to $C^*(\mu_1)$ hence yields a state on $C([-1,1])$; by the Riesz Representation Theorem, there exists a probability measure μ on $[-1,1]$ such that

$$\Phi(f(\mu_1)) = \int_{-1}^{1} f(s)d\mu(s), \quad f \in C([-1,1]).$$

Now taking $f = P_n$, we obtain

$$\dot{\varphi}(n) = \Phi(\mu_n) = \Phi(P_n(\mu_1)) = \int_{-1}^{1} P_n(s)d\mu(s) = \int_{-1}^{1} \dot{\varphi}_s(n)d\mu(s).$$

□

Approximation properties for groups – weak amenability

We first recall that a locally compact group G is *amenable* if $A(G)$ possesses a bounded approximate identity. It is known that G is amenable if and only if there exists a net (u_i) of continuous compactly supported positive definite functions such that $u_i \to 1$ uniformly on compact sets.

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Definition

A locally compact group G is called *weakly amenable* if there exists a net $(u_i) \subseteq A(G)$ and a constant $C > 0$ such that $\|u_i\|_{\mathrm{cbm}} \leq C$ and $u_i \to 1$ uniformly on compact sets.

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If G is weakly amenable, the infimum of all constants C appearing in the last Definition is denoted by Λ_G.

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- u_i is compactly supported for each i;
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The weak amenability of $\mathbb{F}_n$ was generalised as follows:

Theorem (Bożejko-Picardello, 1993)

Let G_i, $i \in I$, be amenable locally compact groups, each of which contains a given open compact group H. Then the free product G of the family $(G_i)_{i \in I}$ over H is weakly amenable and $\Lambda_G = 1$.

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We point out a functoriality property of weak amenability: if G_1 and G_2 are discrete groups then $\Lambda_{G_1 \times G_2} = \Lambda_{G_1} \Lambda_{G_2}$.

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(iii) there exists a net $(u_i) \subseteq A(G)$ of functions with compact support such that $(u_i \otimes 1)$ is an approximate identity for $A(G \times SU(2))$.

More on amenability, weak amenability and (AP)

Theorem (Haagerup-Kraus, 1994)

Let G be a locally compact group.

(i) the group G is weakly amenable with $\Lambda_G \leq L$ if and only if the constant function 1 can be approximated in the weak* topology of $M^{\mathrm{cb}}A(G)$ by elements of the set $\{u \in A(G) : \|u\|_{\mathrm{cbm}} \leq L\}$.

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Theorem (Haagerup-Kraus, 1994)

Let G be a locally compact group and H be a closed normal subgroup of G. If H and G/H have (AP) then so does G.

A list of examples

- **(de Canniere-Haagerup)** $SO_o(1,n)$: the connected component of the identity of the group $SO(1,n)$ of all real $(n+1)\times(n+1)$ matrices with determinant 1, leaving the quadratic form $-t_0^1 + t_1^2 + \cdots + t_n^2$ invariant. Here $\Lambda_{SO_o(1,n)} = 1$.

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- (**de Canniere-Haagerup, Cowling-Haagerup, Hansen**) More generally, real simple Lie groups of real rank one are weakly amenable.

- (**Haagerup**) Real simple Lie groups of real rank at least two are not weakly amenable.

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- (**Ozawa**) Wreath products by non-amenable groups are not weakly amenable.
- (**Haagerup-de Laat**) Connected simple Lie groups with finite centre and real rank at least two do not have the (AP).
- (**Lafforgue-de la Salle**) $SL(3, \mathbb{Z})$ does not have (AP).

THANK YOU VERY MUCH