Closable multipliers of Herz-Schur type

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Every weak* closable operator is closable.

Closability for operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be dual Banach spaces, with specified preduals $\mathcal{X}_*$ and $\mathcal{Y}_*$, respectively, and $D(\Phi) \subseteq \mathcal{X}$ be a weak* dense subspace.

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Note that, since the *-weak closure of the graph of T contains its norm-closure, each weak* closable operator is closable.

Closability for operators

The domain of the *adjoint operator* T^* is

$$D(T^*) = \{g \in \mathcal{Y}^* : \exists f \in \mathcal{X}^* \text{ s. t. } \langle T(x), g\rangle = \langle x, f\rangle \text{ for all } x \in D(T)\}$$

$T^* : D(T^*) \to \mathcal{X}^*$ is defined by letting $T^*(g) = f$, where f is the functional associated with g in the definition of $D(T^*)$.

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Proposition

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $D(T) \subseteq \mathcal{X}$, $T : D(T) \to \mathcal{Y}$ be a densely defined linear operator and set $\mathcal{D} = D(T^*)$. Consider the following conditions:

(i) T is weak** closable;

(ii) $\overline{\mathcal{D}}^{\|\cdot\|} = \mathcal{Y}^*$;

(iii) $\overline{\mathcal{D}}^{w^*} = \mathcal{Y}^*$;

(iv) T is closable.

Then (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv).

Closability for operators

Weak* closability can be characterised analogously:

Proposition

Let $D(T) \subseteq \mathcal{X}$ be a weak* dense subspace and $T : D(T) \to \mathcal{Y}$ be a linear operator. The following are equivalent:

(i) the operator T is weak* closable;

(ii) the space $D_*(T) = \{g \in \mathcal{Y}_* : x \to \langle T(x), g\rangle$ is w* -cont. on $D(T)\}$ is dense in $\mathcal{Y}_*$.

Closable multipliers on group C*-algebras – the setting

Let $\psi : G \to \mathbb{C}$ be a measurable function.

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it is easy to see that the operator $f \to \psi f$, $f \in D(\psi)$, viewed as a densely defined operator on $L^1(G)$, is closable.

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Since $\lambda(L^1(G))$ is dense in $C_r^*(G)$ and $\|\lambda(f)\| \leq \|f\|_1$, $f \in L^1(G)$, the space $\lambda(D(\psi))$ is dense in $C_r^*(G)$ in the operator norm.

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Thus, the operator $S_\psi : \lambda(D(\psi)) \to C_r^*(G)$ given by

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Thus, the operator $S_\psi : \lambda(D(\psi)) \to C_r^*(G)$ given by

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Recall that $B_\lambda(G) \subseteq B(G)$ is (isometric to) the dual of $C_r^*(G)$; the duality is given by

$$\langle \lambda(f), u \rangle = \int_G f(s)u(s)ds, \quad f \in L^1(G), u \in B_\lambda(G).$$

The domain of the adjoint of S_ψ

The domain of the dual S_ψ^* of the operator $S_\psi : \lambda(D(\psi)) \to C_r^*(G)$ is

$$J_\psi = \{g \in B_\lambda(G) : \psi g \in^m B_\lambda(G)\}$$

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Take a sequence $(K_N)_{N\in\mathbb{N}}$ of compacts such that $\cup_N K_N \sim G$ and $|\psi| \leq N$ on K_N. Then $L_1(K_N) \subseteq D(\psi)$.

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Thus, $\psi g \sim u$ and so $\psi g \in^m B_\lambda(G)$.

Conditions related to the closability of S_ψ

The aforementioned criterion of closability now implies that S_ψ is closable if and only if J_ψ is weak* dense in $B_\lambda(G)$.

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The null set $\mathrm{null}(J)$ of a subset $J \subseteq A(G)$ is

$$\mathrm{null}(J) = \{s \in G : u(s) = 0, \ \forall \ u \in J\}.$$

For a closed subset $E \subseteq G$ let $I(E)$ and $J(E)$ be the largest and the smallest closed ideal of $A(G)$ with null set E.

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Recall that a closed subset $E \subseteq G$ is called a *U-set* if $J(E)^\perp \cap C_r^*(G) = \{0\}$ and a *U_1-set* if $I(E)^\perp \cap C_r^*(G) = \{0\}$.

U-sets for arbitrary locally compact groups were first studied by Bożejko (1977).

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The set E is an *M-set* (resp. *M_1-set*) if it is not an U-set (resp. an U_1-set).

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Proposition

Let $E_\psi \overset{def}{=} \{t \in G : \psi$ does not almost belong to $A(G)$ at $t\}$. Then $\mathrm{null}(I_\psi) = E_\psi$.

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Let us say G possesses property (A) if there exists a net $(u_i) \subseteq A(G)$ such that, for every $g \in B_\lambda(G)$,

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In fact, if (u_i) is a net in $A(G)$ such that $\|u_i\|_{\mathrm{cbm}} \leq C$ for all i, and $u_i \to 1$ uniformly on compact sets then for $g \in B_\lambda(G)$ and $f \in C_c(G)$ we have

$$\langle \lambda(f), gu_i - g \rangle = \int_G f(t)g(t)(u_i(t) - 1)dt \to 0.$$

Since $\|gu_i - g\|_{B(G)} \leq (\|u_i\|_{\mathrm{cbm}} + 1)\|g\|_{B(G)}$, and $\lambda(C_c(G))$ is dense in $C_r^*(G)$, we are done.

Conditions related to the closability of S_ψ

Theorem

Suppose that G has property (A) and $\psi : G \to \mathbb{C}$ is a measurable function.

- If E_ψ is a U-set then S_ψ is closable;
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To see these statements, note that S_ψ is closable if and only if J_ψ is weak* dense in $B_\lambda(G)$, if and only if there is no non-zero $T \in C_r^*(G)$ such that

$$\langle T, u \rangle = 0, \quad \text{for all } u \in J_\psi.$$

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On the other hand, property (A) implies that the weak* closures of J_ψ and I_ψ in $B_\lambda(G)$ coincide.

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Thus, S_ψ is closable if and only if there is no non-zero $T \in C_r^*(G)$ such that

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The statements now follow from the fact that

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Corollary

Suppose G has property (A). If $\psi : G \to \mathbb{C}$ is a measurable function and $m(E_\psi) > 0$ then S_ψ is not closable.

Schur multipliers

Let (X, μ) and (Y, ν) be standard measure spaces.
For a function $\varphi \in L^\infty(X \times Y)$, let $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ be the corresponding multiplication operator

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The space $L^2(X \times Y)$ can be identified with the Hilbert-Schmidt class in $\mathcal{B}(L^2(X), L^2(Y))$ by

$$\xi \longrightarrow T_\xi, \qquad T_\xi f(y) = \int_X \xi(x, y) f(x) d\mu(x).$$

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A function $\varphi \in L^\infty(X \times Y)$ is called a *Schur multiplier* if there exists $C > 0$ such that

$$\|S_\varphi \xi\|_{\mathrm{op}} \leq C \|\xi\|_{\mathrm{op}}, \quad \xi \in L^2(X \times Y).$$

Local Schur multipliers

The function $\varphi : X \times Y \to \mathbb{C}$ is called a *local Schur multiplier* if there exists a family $\{\alpha_i \times \beta_i\}_{i=1}^{\infty}$ of measurable rectangles such that

$$\cup_{i=1}^{\infty} \alpha_i \times \beta_i \cong X \times Y$$

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Theorem

The measurable function $\varphi : X \times Y \to \mathbb{C}$ is a local Schur multiplier iff $\exists\ a_k, b_k$ such that

$$\sum_{k=1}^{\infty} |a_k(x)|^2 < \infty \text{ and } \sum_{k=1}^{\infty} |b_k(y)|^2 < \infty \text{ a.e.}$$

and

$$\varphi(x, y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{a.e. on } X \times Y.$$

Closable multipliers of Schur type

For a measurable function $\varphi : X \times Y \to \mathbb{C}$, let

$$D(\varphi) = \{\xi \in L^2(X \times Y) : \varphi\xi \in L^2(X \times Y)\}.$$

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Call the element of $\mathfrak{S}_{\mathrm{cl}}(X, Y)$ *closable multipliers*.

Weak** closable multipliers

Denoting by $\mathfrak{S}_{\mathrm{cl}^{**}}(X, Y)$ the set of all $\varphi : X \times Y \to \mathbb{C}$ for which the operator S_φ is weak** clocable, we have:

Theorem

$\varphi \in \mathfrak{S}_{\mathrm{cl}^{**}}(X, Y)$ if and ony if $\varphi = \frac{\varphi_1}{\varphi_2}$ such that φ_1 and φ_2 are local Schur multipliers with $\varphi_2(x, y) \neq 0$ for (marginally all) $(x, y) \in X \times Y$.

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Note that

$$\mathfrak{S}(X,Y) \subseteq \{\text{local Schur multipliers}\} \subseteq \mathfrak{S}_{\text{cl}^{**}}(X,Y) \subseteq \mathfrak{S}_{\text{cl}}(X,Y).$$

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All inclusions but the middle one are known to be proper.

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(v) An operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is *supported on* κ if

$$(\alpha \times \beta) \cap \kappa \simeq \emptyset \;\Rightarrow\; P(\beta) T P(\alpha) = 0,$$

where $P(\alpha)$ is the projection from $L^2(X)$ onto $L^2(\alpha)$.

Masa-bimodules

If $f \in L^\infty(X)$, let $M_f \in \mathcal{B}(L^2(X))$ be the operator of multiplication by f. Set

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Theorem (Arveson)

Given an ω-closed subset $\kappa \subseteq X \times Y$, there exists a maximal weak* closed masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ and a minimal weak* closed masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ with support κ.

Conditions related to closability of multipliers

Given a measurable $\varphi : X \times Y \to \mathbb{C}$, let

$$D^*(\varphi) = \{h \in L^2(X)\hat{\otimes}L^2(Y) : \varphi h \in L^2(X)\hat{\otimes}L^2(Y)\}.$$

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Let $\kappa_\varphi \subseteq X \times Y$ be the zero set of $D^*(\varphi)$:

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Theorem

Let $\varphi : X \times Y \to \mathbb{C}$ be measurable.

(i) If $\mathfrak{M}_{\max}(\kappa_\varphi)$ does not contain a compact operator then φ is a closable multiplier;

(ii) If $\mathfrak{M}_{\min}(\kappa_\varphi)$ contains a compact operator then φ is not a closable multiplier.

Sets of operator multiplicity

Let (X, μ) and (Y, ν) be standard measure spaces, $H_1 = L^2(X)$, $H_2 = L^2(Y)$, $\mathcal{K}$ the space of all compact operators from H_1 to H_2.

Definition

An ω-closed set $\kappa \subseteq X \times Y$ is called

(i) an *operator M-set* if $\mathcal{K} \cap \mathfrak{M}_{\max}(\kappa) \neq \{0\}$;

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The space $\mathcal{K}$ is a suitable substitute of $C_r^*(G)$ because

$$\mathcal{K}(L^2(G)) = \overline{\{M_a T M_b : a \in C_0(G), T \in C_r^*(G)\}}^{\|\cdot\|}.$$

Structure in $\mathfrak{S}_{\mathrm{cl}}(X, Y)$

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- $\mathfrak{S}_{\mathrm{cl}}(X, Y)$ is also an algebra but for a different reason:

Proposition

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are weak* dense $L^\infty(G)$-invariant subspaces of $T(X, Y)$ then the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is weak* dense in $T(X, Y)$, too.

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Proof for "+" : $D^*(\varphi_1) \cap D^*(\varphi_2) \subseteq D^*(\varphi_1 + \varphi_2)$.

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It is well-known that triangular truncation on $\ell^2(\mathbb{N})$ is unbounded.

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χ_Δ is closable but not weak** closable.

An example: triangular truncation

Proof.

$D^*(\chi_\Delta)$ contains the characteristic functions of measurable rectangles $\alpha \times \beta$ disjoint from the diagonal $\Lambda = \{(x, x) : x \in [0, 1]\}$.

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However, Λ does not support a compact operator and hence, by the above criterion, χ_Δ is closable.

χ_Δ cannot be weak** closable since it is not equivalent to an ω-continuous function.

Indeed, if $\chi_\Delta \sim \varphi$, and φ is ω-continuous, then φ has to be 1 on the interiour Δ^o of Δ and 0 on Δ^c. Since Λ is in the ω-closure of both Δ^o and Δ^c, φ must be both 1 and 0 m.a.e. on Λ, a contradiction. □

Passage from HA to OT

Theorem

Let $E \subseteq G$ be a closed set.

(i) E is an M-set if and only if E^* is an operator M-set;

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Theorem

Let G be a second countable locally compact group satisfying property (A), $\psi : G \to \mathbb{C}$ be a measurable function and $\varphi = N(\psi)$. The following are equivalent:

- the operator S_ψ is closable;
- the operator S_φ is closable.

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Corollary

The set $\mathrm{Clos}(G)$ of all closable multipliers on $C_r^*(G)$ is an algebra with respect to pointwise addition and multiplication.

About the proof

Theorem (A symbolic calculus)

For $\varphi \in T(G)$ let $E_\varphi : \mathcal{B}(L^2(G)) \to \mathrm{VN}(G)$ be the bounded linear trasformation with the property

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G), T \in \mathcal{B}(L^2(G)).$$

Then E_φ maps $\mathcal{K}$ into $C_r^*(G)$.

The map $\varphi \to E_\varphi$ is a contractive $\mathfrak{S}(G,G)$-module map from $T(G)$ into $CB^{w^*}(\mathcal{B}(L^2(G)), \mathrm{VN}(G))$.

If $T \neq 0$ then $E_{a\otimes b}(T) \neq 0$ for some $a, b \in L^2(G)$.

$N(\psi)$ closable $\Rightarrow$ ψ closable

Proof.

Let $\varphi = N(\psi)$ be a closable multiplier. If ψ is not closable, there exists a non-zero $T \in C_r^*(G)$ that annihilates I_ψ. Let $A = M_f$ be such that $f \in C_0(G)$ and $AT \neq 0$. It suffices to show that AT annihilates $D(S_\varphi^*)$.

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Let $h \in D(S_\varphi^*)$. A direct verification shows that

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(check first in the case $T = \lambda(f)$).

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Since $\varphi h \in T(G)$, $\psi P(h) = P(\varphi h) \in A(G)$ and hence $P(h) \in I_\psi$. Thus, $\langle T, P(h) \rangle = 0$ and hence $\langle T, h \rangle = 0$. $\square$

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Suppose that S_ψ is a closable operator but $S_{N(\psi)}$ is not.

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Thus

$$\langle E_{a\otimes b}(T), u\rangle = \langle T, (a \otimes b)N(u)\rangle = 0.$$

But $E_{a\otimes b}(T) \in C_r^*(G)$ and we are done by the closability criterion. □

An example: a non-closable multiplier on $C_r^*(\mathbb{T})$

Let $c = (c_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, $p > 2$, and $d = (d_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ be such that

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- $\mathrm{VN}_0(G) = \mathrm{span}\{\lambda_s : s \in G\}, \quad S'_\psi(\lambda_s) = \psi(s)\lambda_s.$

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Indeed, let $f \in B(\mathbb{R})$ such that $1/f \notin B(\mathbb{R})$, and let $\psi = 1/f$. Then $\psi \in A(\mathbb{R})^{\mathrm{loc}}$ but J_ψ is contained in the ideal of $B(\mathbb{R})$ generated by f and hence is not dense in $B(\mathbb{R})$.

- There exists $\psi \notin A(G)^{\mathrm{loc}}$ for which S_ψ is closable.

Indeed, this will be the case whenever E_ψ is a non-empty U-set. Continuous functions with this property are *e.g.* those odd ψ which are smooth on $(-\pi, \pi) \setminus \{0\}$, $\psi(0) = \psi(\pi) = 0$, $\psi'(\pi) = 0$, and $\int_0^1 \psi(t)/t dt$ diverges. For such ψ, we have $E_\psi = \{0\}$.

THANK YOU VERY MUCH