Introduction to operator spaces–Fields Institute, May 2014

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This course and these notes assume familiarity with some basic facts about C^*-algebras, and material from a graduate functional analysis course. They are a considerable expansion of most of Chapter 1 of my book [4] with Christian Le Merdy. The presentation here is thus greatly shaped by that book (indeed some is copied verbatim), and of course thanks go to Christian for permitting me to do this.

Each of the four chapters roughly corresponds to one lecture. Since these notes were aimed at the students in the class, I have not yet taken the trouble to compile an adequate bibliography, or to make sure that results are always attributed, etc. I thank Alex Bearden for finding many typos which have been corrected in the current draft.

Chapter 1

Lectures 1-2 (Operator spaces)

1.1 Introduction

Banach spaces or normed linear spaces are 'just' the linear subspaces of commutative C^*-algebras, while operator spaces are the linear subspaces of general C^*-algebras.

The importance of operator space theory may perhaps be best stated as follows: it is a generalization of Banach spaces which is particularly appropriate for studying 'noncommutative' or 'quantized' spaces, and 'linear' problems arising in noncommutative situations. They are particularly appropriate for studying spaces or algebras of operators on Hilbert space. Thus the field of operator spaces provides a bridge from the world of Banach and function spaces, to the world of spaces of operators on a Hilbert space, and of 'noncommutative mathematics'.

Crudely put, when generalizing classical arguments in functional analysis, one should often expect C^*-algebra theory to replace topology, von Neumann algebra to replace arguments using measure and integrals, and operator space theory to replace Banach space techniques.

After pioneering work by Arveson, Haagerup, and Wittstock, operator spaces were developed by Effros and Ruan, who were soon joined by B and Paulsen, Pisier, Junge, and many others.

1.2 Basic facts, examples, and constructions

1.2.1 (Matrix notation) Fix $m, n \in \mathbb{N}$. If X is a vector space, then so is $M_{m,n}(X)$, the set of $m \times n$ matrices with entries in X. This may also be thought of as the algebraic tensor product $M_{m,n} \otimes X$, where $M_{m,n} = M_{m,n}(\mathbb{C})$. We write I_n for the identity matrix of $M_n = M_{n,n}$. We write $M_n(X) = M_{n,n}(X)$, $C_n(X) = M_{n,1}(X)$ and $R_n(X) = M_{1,n}(X)$.

If x is a matrix, then x_{ij} or $x_{i,j}$ denotes the i-j entry of x, and we write x as

$[x_{ij}]$ or $[x_{i,j}]_{i,j}$. We write $(E_{ij})_{ij}$ for the usual (matrix unit) basis of $M_{m,n}$ (we allow m, n infinite here too). We write $A \mapsto A^t$ for the transpose on $M_{m,n}$, or more generally on $M_{m,n}(X)$. We will sometimes meet large matrices with row and column indexing that is sometimes cumbersome. For example, a matrix $[a_{(i,k,p),(j,l,q)}]$ is indexed on rows by (i, k, p) and on columns by (j, l, q), and may also be written as $[a_{(i,k,p),(j,l,q)}]_{(i,k,p),(j,l,q)}$ if additional clarity is needed. To illustrate this notation, the reader may want to write down the matrix $[\delta_{i,\ell}\delta_{kj}]_{(i,k),(j,\ell)}$. Here $\delta_{i,j}$ is Kronecker's delta.

1.2.2 (Norms of matrices with operator entries) Clearly $M_n(B(H))$ is a C^*-algebra for any Hilbert space H, with the norm that it gets via the $*$-isomorphism $M_n(B(H)) \cong B(H^{(n)})$. Reviewing, recall that $H^{(n)}$ is the Hilbert space direct sum of m copies of H, and the norm of a vector $\zeta = (\zeta_k)$ there is $(\sum_{k=1}^n \|\zeta_k\|^2)^{\frac{1}{2}}$. This $*$-isomorphism is the map taking a matrix $[T_{ij}] \in M_n(B(H))$ to the operator from $H^{(n)}$ to $H^{(n)}$:

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \zeta_n \end{bmatrix} = \begin{bmatrix} \sum_k T_{1k}\zeta_k \\ \sum_k T_{2k}\zeta_k \\ \cdot \\ \cdot \\ \sum_k T_{nk}\zeta_k \end{bmatrix}.$$

Thus $M_n(B(H))$ is a C^*-algebra, and $[T_{ij}] \in M_n(B(H))$ *has a natural norm*:

$$\|[T_{ij}]\|_n = \sup\{\|[T_{ij}]\vec{\zeta}\| : \vec{\zeta} \in H^{(n)}, \|\vec{\zeta}\| \leq 1\}.$$

Using the principle that $\|\xi\| = \sup\{|\langle \xi, \eta \rangle| : \eta \in \mathrm{Ball}(K)\}$ in any Hilbert space K, we deduce that

$$\|[T_{ij}]\|_n = \sup\{|\sum_{i,j} < T_{ij}\zeta_j, \eta_i > | : \vec{\zeta} = (\zeta_i), \vec{\eta} = (\eta_i) \in \mathrm{Ball}(H^{(n)})\}.$$

We can also view $M_n(B(H))$ as the *spatial tensor product* $M_n \otimes B(H)$ (we will review the spatial tensor product later).

Similar identities hold for rectangular matrices. Indeed if $m, n \in \mathbb{N}$, and K, H are Hilbert spaces, then we always assign $M_{m,n}(B(K, H))$ the norm (written $\|\cdot\|_{m,n}$) ensuring that

$$M_{m,n}(B(K, H)) \cong B(K^{(n)}, H^{(m)}) \quad \text{isometrically} \tag{1.1}$$

via the natural algebraic isomorphism.

1.2.3 (Completely bounded maps) Suppose that X and Y are vector spaces and that $u \colon X \to Y$ is a linear map. For a positive integer n, we write u_n for the associated map $[x_{ij}] \mapsto [u(x_{ij})]$ from $M_n(X)$ to $M_n(Y)$. This is often called the (nth) *amplification* of u, and may also be thought of as the map $I_{M_n} \otimes u$ on $M_n \otimes X$. Similarly one may define $u_{m,n} \colon M_{m,n}(X) \to M_{m,n}(Y)$. If each matrix space $M_n(X)$ and $M_n(Y)$ has a given norm $\|\cdot\|_n$, and if u_n is an isometry for all $n \in \mathbb{N}$, then we say that u is *completely isometric*, or is a *complete isometry.* Similarly, u is *completely contractive* (resp. *is a complete quotient map*) if each u_n is a contraction (resp. takes the open ball of $M_n(X)$ onto the open ball of $M_n(Y)$). A map u is *completely bounded* if

$$\|u\|_{\mathrm{cb}} \stackrel{def}{=} \sup\big\{\|[u(x_{ij})]\|_n \,:\, \|[x_{ij}]\|_n \leq 1, \text{ all } n \in \mathbb{N}\big\} \;< \infty.$$

As in the Banach space case, it is easy to prove that $\|u+v\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}} + \|v\|_{\mathrm{cb}}$, and $\|\lambda u\|_{\mathrm{cb}} = |\lambda|\|u\|_{\mathrm{cb}}$ for a scalar λ, and so on. Compositions of completely bounded maps are completely bounded, and one has the expected relation $\|u \circ v\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}$. If $u\colon X \to Y$ is a completely bounded linear bijection, and if its inverse is completely bounded too, then we say that u is a *complete isomorphism.* In this case, we say that X and Y are *completely isomorphic* and we write $X \approx Y$. If, further, u and u^{-1} are completely contractive, then just as in the Banach space case they are complete isometries.

1.2.4 (Operator spaces) A *concrete operator space* is a (usually closed) linear subspace X of $B(K, H)$, for Hilbert spaces H, K (indeed the case $H = K$ usually suffices, via the canonical inclusion $B(K, H) \subset B(H \oplus K)$). However we will want to keep track too of the norm $\|\cdot\|_{m,n}$ that $M_{m,n}(X)$ inherits from $M_{m,n}(B(K, H))$, for all $m, n \in \mathbb{N}$. We write $\|\cdot\|_n$ for $\|\cdot\|_{n,n}$; indeed when there is no danger of confusion, we simply write $\|[x_{ij}]\|$ for $\|[x_{ij}]\|_n$.

An *abstract operator space* is a pair $(X, \{\|\cdot\|_n\}_{n\geq 1})$, consisting of a vector space X, and a norm on $M_n(X)$ for all $n \in \mathbb{N}$, such that there exists a linear complete isometry $u\colon X \to B(K, H)$. In this case we call the sequence $\{\|\cdot\|_n\}_n$ an *operator space structure* on the vector space X. An *operator space structure* on a normed space $(X, \|\cdot\|)$ will usually mean a sequence of matrix norms as above, but with $\|\cdot\| = \|\cdot\|_1$.

Clearly subspaces of operator spaces are again operator spaces. We often identify two operator spaces X and Y if they are completely isometrically isomorphic. In this case we often write '$X \cong Y$ completely isometrically', or say '$X \cong Y$ *as operator spaces*'. Sometimes we simply write $X = Y$.

1.2.5 (C^*-algebras) If A is a C^*-algebra, a closed $*$-subalgebra of $B(H)$, then $M_n(A)$ may be viewed as a a closed $*$-subalgebra of $M_n(B(H)) \cong B(H^{(n)})$. Thus $M_n(A)$ is a C^*-algebra. A basic fact about C^*-algebras is that a one-to-one $*$-homomorphism between C^*-algebras is isometric. Thus there can be at most one norm on a $*$-algebra for which that $*$-algebra is a C^*-algebra. Thus the $*$-algebra $M_n(A)$ has a unique norm with respect to which it is a C^*-algebra. With respect to these matrix norms, A is an operator space. Indeed A is a concrete operator space in $B(H)$. We call this the *canonical operator space structure* on a C^*-algebra. If the C^*-algebra A is commutative, with $A = C_0(\Omega)$ for a locally compact space Ω, and then these matrix norms are determined via the canonical isomorphism $M_n(C_0(\Omega)) = C_0(\Omega; M_n)$. Explicitly, if $[f_{ij}] \in M_n(C_0(\Omega))$, then:

$$\|[f_{ij}]\|_n \;=\; \sup_{t\in\Omega} \big\|[f_{ij}(t)]\big\|. \tag{1.2}$$

To see this, note that by the above one only needs to verify that (1.2) does indeed define a C^*-norm on $M_n(C_0(\Omega))$. Clearly the right hand side of (1.2) is a finite number, since $\|[f_{ij}(t)]\| \leq n^2 \max_{i,j} |f_{ij}(t)|$, and each of the functions f_{ij} is bounded on Ω. Also, it is easy to check that (1.2) does define a norm. To see that (1.2) is a Banach algebra, note that

$$\begin{aligned} \|[f_{i,j}][g_{i,j}]\| &= \sup\{\|[f_{i,j}(t)][g_{i,j}(t)]\| : t \in \Omega\} \\ &\leq \sup\{\|[f_{i,j}(t)]\| : t \in \Omega\} \sup\{\|[g_{i,j}(t)]\| : t \in \Omega\} \\ &= \|[f_{i,j}]\|\|[g_{i,j}]\|. \end{aligned}$$

So $M_n(C_0(\Omega))$ is a Banach algebra. We check the C^*-identity:

$$\begin{aligned} \|[\overline{f_{j,i}}][f_{i,j}]\|_n &= \sup\{\|[\overline{f_{j,i}(t)}][f_{i,j}(t)]\| : t \in \Omega\} \\ &= \sup\{\|[f_{i,j}(t)]^*[f_{i,j}(t)]\| : t \in \Omega\} \\ &= \sup\{\|[f_{i,j}(t)]\|^2 : t \in \Omega\} \\ &= \sup\{\|[f_{i,j}(t)]\|_{M_n}\}^2. \end{aligned}$$

Thus $M_n(C_0(\Omega))$ is a C^*-algebra.

Proposition 1.2.6. For a homomorphism $\pi\colon A \to B$ between C^*-algebras, the following are equivalent: (i) π is contractive, (ii) π is completely contractive, and (iii) π is a $*$-homomorphism. If these hold, then $\pi(A)$ is closed, and π is a complete quotient map onto $\pi(A)$; moreover π is one-to-one if and only if it is completely isometric.

Proof. (ii) $\Rightarrow$ (i) Obvious.

(iii) $\Rightarrow$ (ii) Note that π_n is a $*$-homomorphism, and so contractive by C^*-algebra theory. So π is completely contractive.

Clearly if π is completely isometric it is one-to-one. Conversely, if π is one-to-one then π_n is a one-to-one $*$-homomorphism and so isometric by C^*-algebra theory. Thus π is completely isometric.

By C^*-algebra theory., any $*$-homomorphism π is a 1-quotient map (that is it maps the unit ball onto the unit ball) onto its (closed) range. Similarly, π_n is a 1-quotient map, so that π is a complete quotient map.

(i) $\Rightarrow$ (iii) This is a well fact about C^*-algebras that we shall not prove here. $\square$

1.2.7 (Maps into a commutative C^*-algebra) If $[a_{ij}] \in M_n$ then

$$\|[a_{ij}]\| = \sup \Big\{ \Big|\sum_{ij} a_{ij} z_j \overline{w_i}\Big| : z = [z_j],\ w = [w_i] \in \mathrm{Ball}(\ell^2_n)\Big\}.$$

Moreover, if $a_{ij} \in B(H)$, for a Hilbert space H, then

$$\|[a_{ij}]\| \geq \sup \Big\{ \|\sum_{ij} a_{ij} z_j \overline{w_i}\| : z = [z_j],\ w = [w_i] \in \mathrm{Ball}(\ell^2_n)\Big\}.$$

Indeed, if one uses the fact that $\|T\| = \sup\ \{|\langle T\zeta, \eta\rangle| : \zeta, \eta \in \mathrm{Ball}(H)\}$, for any $T \in B(H)$, then one sees that the right side of the last centered formula is

$$\sup\{\left|\left\langle [a_{i,j}] \begin{bmatrix} z_1\zeta \\ z_2\zeta \\ \vdots \\ z_n\zeta \end{bmatrix}, \begin{bmatrix} w_1\eta \\ w_2\eta \\ \vdots \\ w_n\eta \end{bmatrix} \right\rangle\right| : \zeta, \eta \in \mathrm{Ball}(H), \overrightarrow{z}, \overrightarrow{w} \in \mathrm{Ball}(l^2_n)\},$$

which is dominated by $\|[a_{i,j}]\|$.

Using these formulae, it is easy to see that any continuous linear functional $\varphi\colon X \to \mathbb{C}$ on an operator space X is completely bounded, with $\|\varphi\| = \|\varphi\|_{\mathrm{cb}}$. We have

$$\begin{aligned} \|[\varphi(x_{i,j})]\|_n &= \sup\left\{\left|\sum_{i,j} \varphi(x_{i,j}) z_j \overline{w_i}\right| : \overrightarrow{z}, \overrightarrow{w} \in \mathrm{Ball}(l^2_n)\right\} \\ &= \sup\left\{\left|\varphi\left(\sum_{i,j} x_{i,j} z_j \overline{w_i}\right)\right| : \overrightarrow{z}, \overrightarrow{w} \in \mathrm{Ball}(l^2_n)\right\} \\ &\leq \|\varphi\|\ \sup\left\{\left\|\sum_{i,j} x_{i,j} z_j \overline{w_i}\right\| : \overrightarrow{z}, \overrightarrow{w} \in \mathrm{Ball}(l^2_n)\right\} \\ &\leq \|\varphi\|\ \|[x_{i,j}]\|. \end{aligned}$$

Hence $\|\varphi_n\| \leq \|\varphi\|$, and so $\|\varphi\|_{cb} = \|\varphi\|$.

Next we claim that $\|u\| = \|u\|_{\mathrm{cb}}$ for any bounded linear map u from an operator space into a commutative C^*-algebra. We can assume that the commutative C^*-algebra is $C_0(\Omega)$, for a locally compact Ω. For fixed $w \in \Omega$ let $\phi_w \in X^*$ be defined by $\phi_w(x) = u(x)(w)$. Note that $|\phi_w(x)| = |u(x)(w)| \leq \|u(x)\| \leq \|u\|\|x\|$, if $x \in E$. Thus $\|\phi_w\| \leq \|u\|$. We then have

$$\|[u(x_{i,j})(w)]\| = \|[\phi_w(x_{i,j})]\| \leq \|\phi_w\|\|[x_{i,j}]\| \leq \|u\|\|[x_{i,j}]\|$$

Thus by equation (1.2), it follows that $\|[u(x_{i,j})]\| \leq \|u\|\|[x_{i,j}]\|$, and so $\|u\|_n \leq \|u\|$. Since this is true for all $n \in \mathbb{N}$ we have $\|u\|_{cb} = \|u\|$.

1.2.8 (Properties of matrix norms) If K, H are Hilbert spaces, and if X is a subspace of $B(K, H)$, then there are certain well-known properties satisfied by the matrix norms $\|\cdot\|_{m,n}$ described in 1.2.2. Most important for us are the following two.

(R1) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$, for all $n \in \mathbb{N}$ and all $\alpha, \beta \in M_n$, and $x \in M_n(X)$ (where multiplication of an element of $M_n(X)$ by an element of M_n is defined in the obvious way).

(R2) For all $x \in M_m(X)$ and $y \in M_n(X)$, we have

$$\left\|\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

We often write $x \oplus y$ for the matrix in (R2).

To see (R1), define

$$\tilde{\alpha} = \begin{bmatrix} \alpha_{11} I_H & \alpha_{12} I_H \ \ldots \ \alpha_{1n} I_H \\ \vdots & \\ \alpha_{n1} I_H & \alpha_{n2} I_H \ \ldots \ \alpha_{nn} I_H \end{bmatrix} = \alpha \otimes I_H \in M_n \otimes B(H) = M_n(B(H)).$$

Similarly define $\tilde{\beta} = \beta \otimes I_H \in M_n(B(H))$. Note that $\|\tilde{\alpha}\| = \|\alpha\|$ since the $*$-homomorphism $M_n \longrightarrow M_n(B(H))$ taking $\alpha \mapsto \tilde{\alpha}$ is one-to-one and hence is a (complete) isometry. Also $\alpha x \beta = \tilde{\alpha} x \tilde{\beta}$ for all $x \in M_n(B(H))$. Thus

$$\|\alpha x \beta\| = \|\tilde{\alpha} x \tilde{\beta}\| \leq \|\tilde{\alpha}\| \|x\| \|\tilde{\beta}\| = \|\alpha\| \|x\| \|\beta\|$$

since $M_n(B(H))$ is a Banach algebra. This proves (R1).

To prove (R2), note that if $a = [I_n : 0], b = [I_n : 0]^t$, then $x = a(x \oplus y)b$ (using the notation after the statement of (R2)). If we let $\tilde{a} = [I_n \otimes I_H : 0] \in M_{n,n+m}(B(H))$ (that is, $\tilde{a}$ is an $n \times (n+m)$ matrix consisting of all zero entries except for an I_H in the i-i entry for $i = 1, \cdots, n$), and if $\tilde{b} = \tilde{a}^t$, then as in the proof of (R1), $x = \tilde{a}(x \oplus y)\tilde{b}$. Hence

$$\|x\|_n = \|\tilde{a}(x \oplus y)\tilde{b}\|_n \leq \|\tilde{a}\| \|x \oplus y\|_{m+n} \|\tilde{b}\|.$$

Note that $\|\tilde{a}\| = \|\tilde{a}\tilde{a}^*\|^{\frac{1}{2}} = \|I\|^{\frac{1}{2}}_{M_n(B(H))} = 1$, and similarly $\|\tilde{b}\| = \|\tilde{b}^* \tilde{b}\|^{\frac{1}{2}} = 1$. Thus $\|x\|_n \leq \|x \oplus y\|_{m+n}$. Similarly, $\|y\|_n \leq \|x \oplus y\|_{m+n}$, so that $\max\{\|x\|_n, \|y\|_m\} \leq \|x \oplus y\|_{m+n}$.

For the other direction, let $\xi \in H^n, \eta \in H^m$, then

$$\|(x \oplus y) \begin{bmatrix} \xi \\ \eta \end{bmatrix}\|^2 = \| \begin{bmatrix} x\xi \\ y\eta \end{bmatrix} \|^2 = \|x\xi\|^2 + \|y\eta\|^2 \leq \|x\|^2 \|\xi\|^2 + \|y\|^2 \|\eta\|^2.$$

Clearly this is dominated by $\max\{\|x\|, \|y\|\}^2 (\|\xi\|^2 + \|\eta\|^2)$. Thus we deduce that $\|x \oplus y\| \leq \max\{\|x\|, \|y\|\}$. This proves (R2).

- It follows from (R1) that switching rows (or columns) of a matrix of operators does not change its norm. Such switching is equivalent to multiplying by a 'permutation' matrix U, namely a matrix which is all zeroes except for one 1 in each row and each column. Such a matrix has norm 1, being unitary, and so

$$\|Ux\| \leq \|x\| \leq \|U^* U x\| \leq \|Ux\|.$$

- Adding (or dropping) rows of zeros or columns of zeros does not change the norm of a matrix of operators. To see this, note that by the last paragraph we can suppose all the zero rows (resp. columns) are at the bottom (resp. right) of the matrix. But then it is elementary to see that the norm is unchanged if we remove those zero rows or columns. For example

$$\left\| \begin{bmatrix} x \\ 0 \end{bmatrix} \right\|_{m,n} = \sup\{\|x\zeta\| : \zeta \in \text{Ball}(H^{(n)})\} = \|x\|.$$

- By the principle in the last paragraph, we really only need to specify the norms for square matrices, that is, the case $m = n$ above, since $M_{m,n}(X)$ may be viewed as a subspace of $M_k(X)$ where $k = \max\{m, n\}$.
- If X is an operator space then the canonical algebraic isomorphisms

$$M_n(M_m(X)) \cong M_m(M_n(X)) \cong M_{mn}(X) \tag{1.3}$$

  are isometric. To see this, we can assume that X is a C^*-algebra, and then notice that these three canonical algebraic isomorphisms are $*$-isomorphisms, hence completely isometric.
- If X is an operator space then so is $M_n(X)$, the latter with the operator space structure for which the canonical isomorphism $M_m(M_n(X)) \cong M_{mn}(X)$ becomes an isometry. One way to see this[1] is to note that if X is a subspace of a C^*-algebra A, then $M_n(X) \subset M_n(A)$, and the latter is a C^*-algebra and hence is an operator space whose matrix norms are the ones making $M_m(M_n(A))$ a C^*-algebra. Then $M_n(X)$, with the inherited matrix norms, is an operator space. However, $M_m(M_n(A))$ is $*$-isomorphic to $M_{mn}(A)$, and hence the norm making $M_m(M_n(A))$ a C^*-algebra is exactly the one coming from the C^*-algebra $M_{mn}(A)$ via the canonical $*$-isomorphism $M_m(M_n(A)) \cong M_{mn}(A)$. Restricting the latter isomorphism to $M_m(M_n(X))$ gives the desired assertion.
- If $T \in B(K, H)$, and if $z_{ij} \in \mathbb{C}$ then $\|[z_{ij}T]\| = \|T\|\|[z_{ij}]\|$. We leave this as an exercise.
- It is easy to see from the above that $\max_{i,j} \ \|x_{ij}\| \leq \|[x_{ij}]\|_n \leq n \max_{i,j} \ \|x_{ij}\|$. It follows from this that a sequence (x_k) of matrices in $M_n(X)$ converges iff the i-j entry of x_k converges as $k \to \infty$ to the same entry of x, for all i, j.

1.2.9 (An operator that is not completely bounded) The canonical example of a map that is not completely bounded is the 'transpose' map $\pi(x) = x^t$ on $\mathbb{K} = K(\ell^2)$ thought of as infinite matrices (via the prescription $x \to [\langle x\vec{e_j}, \vec{e_i}\rangle]$). Note that π is an isometric linear $*$-antiisomorphism. Indeed if $\vec{z} = (z_i), \vec{w} = (w_i) \in \text{Ball}(\ell^2)$, then

$$\Big|\sum_{i,j=1}^{\infty} x_{ji} z_i \overline{w_j}\Big| = \Big|\sum_{i,j=1}^{\infty} x_{ij} \overline{w_i} z_j\Big| \leq \|[x_{ij}]\|.$$

This says that π is a contraction, and by symmetry (since $\pi^{-1} = \pi$) it is an isometry. If $\{E_{ij}\}$ is the usual basis for M_n and if $\rho : M_n \to \mathbb{K}$ is the 'top left corner embedding', i.e. $\rho(x) = x \oplus 0$, then π is a one-to-one $*$-homomorphism, and hence is a complete isometry. If $x_n = [\rho(E_{ji})] \in M_n(\mathbb{K})$ then

$$\|x_n\|_n = \|[\rho(E_{ji})]\| = \|[E_{ji}]\| = 1,$$

as can be seen by switching rows and columns of the matrix $[E_{ji}]$ to make it an 'identity matrix'. On the other hand, $\pi_n(x_n) = [\rho(E_{ij})]$, which has the same norm as $[E_{ij}]$. Erasing zero rows and columns, the latter becomes an $n \times n$ matrix with all entries 1. By the C^*-identity the latter has the same norm as $x^*x = n$, where

[1] Another way to see this is as in 1.2.23 (7).

x is a column of n entries each equal to 1. Thus we have $\|\pi_n(x_n)\|_n = n$, and so $\|\pi_n\| \geq n$. Hence π is not completely bounded.

Conditions (R1) and (R2) in 1.2.8 are often called *Ruan's axioms. Ruan's theorem* asserts that (R1) and (R2) characterize operator space structures on a vector space. This result is fundamental to our subject in many ways. At the most pedestrian level, it is used frequently to check that certain abstract constructions with operator spaces remain operator spaces. At a more sophisticated level, it is the foundational and unifying principle of operator space theory. We now proceed to Effros and Ruan's proof of Ruan's theorem. We omit the proof of the first lemma, which is an application of the geometric Hahn-Banach theorem, and which may be found on [p. 30,ERbook].

Lemma 1.2.10. If X is a vector space, and if $\|\cdot\|_n$ is a norm on $M_n(X)$, for each $n \in \mathbb{N}$, satisfying (R1) and (R2), and if $F \in \text{Ball}(M_n(X)^*)$, then there are states φ, ψ on M_n with

$$|F(\alpha x \beta)| \leq \varphi(\alpha\alpha^*)^{\frac{1}{2}} \, \|x\| \, \psi(\beta^*\beta)^{\frac{1}{2}}, \qquad \alpha, \beta \in M_n, x \in M_n(X).$$

Lemma 1.2.11. If $(X, \{\|\cdot\|_n\})$ are as in the last lemma, if $F \in \text{Ball}(M_n(X)^*)$, and if H is the Hilbert space ℓ^2_n, then there exist vectors $\zeta, \eta \in \text{Ball}(H^{(n)})$, and a completely contractive $u : X \to B(H) \cong M_n$ such that $F = \langle u_n(\cdot)\zeta, \eta\rangle$.

Proof. By the last lemma there are states φ, ψ on M_n with $|F(\alpha^* x\beta)| \leq \varphi(\alpha^*\alpha)^{\frac{1}{2}}\|x\|\psi(\beta^*\beta)^{\frac{1}{2}}$ for $\alpha, \beta \in M_n$. States on M_n are well understood. Indeed we can write

$$\varphi(x) = \sum_{k=1}^{n} \langle x\zeta_k, \zeta_k\rangle = \langle (x \otimes I_n)\zeta, \zeta\rangle, \qquad x \in M_n,$$

where $\zeta = (\zeta_k) \in \text{Ball}(H^{(n)})$, where $H = \ell^2_n$. It follows that for any $\alpha \in M_n$ we have

$$\varphi(\alpha^*\alpha) = \langle (\alpha^*\alpha \otimes I_n)\zeta, \zeta\rangle = \langle (\alpha \otimes I_n)\zeta, (\alpha \otimes I_n)\zeta\rangle = \|(\alpha \otimes I_n)\zeta\|^2.$$

Similarly, $\psi(\beta^*\beta)^{\frac{1}{2}} = \|(\beta \otimes I_n)\eta\|$ for some $\eta \in \text{Ball}(H^{(n)})$. The inequality in the first line of the proof then reads

$$|F(\alpha^* x\beta)| \leq \|x\| \, \|(\alpha \otimes I_n)\zeta\| \, \|(\beta \otimes I_n)\eta\|, \qquad \alpha, \beta \in M_n.$$

Let $E = (M_n \otimes I_n)\eta$ and $K = (M_n \otimes I_n)\zeta$, subspaces of $\mathbb{C}^{n^2}$. Fix $x \in M_n(X)$ for a moment and define $g : E \times K \to \mathbb{C}$ by $g((\beta \otimes I_n)\eta, (\alpha \otimes I_n)\zeta) = F(\alpha^* x\beta)$, for $\alpha, \beta \in M_n$. Thus

$$|g((\beta \otimes I_n)\eta, (\alpha \otimes I_n)\zeta)| \leq \|x\| \, \|(\alpha \otimes I_n)\zeta\| \, \|(\beta \otimes I_n)\eta\|, \qquad \alpha, \beta \in M_n.$$

It is easy to see from this that g is a well-defined and bounded sesquilinear form on $E \times K$. By Hilbert space theory there exists an operator in $B(E, K)$, which we shall write as $T(x)$, with $\|T(x)\| \leq \|x\|$, and

$$\langle T(x)(\beta \otimes I_n)\eta, (\alpha \otimes I_n)\zeta\rangle = F(\alpha^* x\beta), \qquad x \in M_n(X), \alpha, \beta \in M_n.$$

It is easy to see that T is linear. Let P be the projection from $\mathbb{C}^{n^2}$ onto E. Since E is invariant under $M_n \otimes I_n$, it follows from basic operator theory that $P \in (M_n \otimes I_n)'$. Let $R = T(\cdot)P \in B(\mathbb{C}^{n^2}) \cong M_{n^2}$. Then $R \in B(M_n(X), M_{n^2})$ is a linear contraction, since $\|R(x)\| \leq \|T(x)\|\|P\| \leq \|x\|$. We have

$$\langle R(x)\eta, \zeta\rangle = \langle T(x)(I_n \otimes I_n)\eta, (I_n \otimes I_n)\zeta\rangle = F(I_n x I_n) = F(x), \qquad x \in M_n(X).$$

Notice next that if $\alpha, \beta, \gamma \in M_n$ then

$$\langle T(x\gamma)(\beta \otimes I_n)\eta, (\alpha \otimes I_n)\zeta\rangle = F(\alpha^* x\gamma\beta) = \langle T(x)(\gamma\beta \otimes I_n)\eta, (\alpha \otimes I_n)\zeta\rangle.$$

That is,

$$\langle T(x\gamma)h, k\rangle = \langle T(x)(\gamma \otimes I_n)h, k\rangle, \qquad h \in E, k \in K,$$

which means that $T(x\gamma) = T(x)(\gamma \otimes I_n)$. Hence

$$R(x\gamma) = T(x\gamma)P = T(x)(\gamma \otimes I_n)P = R(x)(\gamma \otimes I_n), \qquad \gamma \in M_n.$$

A similar argument shows that $R(\gamma x) = (\gamma \otimes I_n)R(x)$ for $\gamma \in M_n$. It is a simple linear algebra exercise that if $S : M_n(Y) \to M_n(Z)$ is a linear map, where Y, Z are vector spaces, then $S = u_n$ for a linear $u : Y \to Z$ iff $S(\alpha x\beta) = \alpha S(x)\beta$ for all $x \in M_n(Y)$ and $\alpha, \beta \in M_n$. Hence $R = u_n$ for some $u : X \to M_n$ with $\|u_n\| = \|R\| \leq 1$. By Exercise 5 at the end of this section, this forces $\|u_m\| \leq 1$ for all $m \geq n$, so that u is completely contractive.

Thus $\langle u_n(x)\eta, \zeta\rangle = \langle R(x)\eta, \zeta\rangle = F(x)$ for all $x \in M_n(X)$. □

Corollary 1.2.12. If $(X, \{\|\cdot\|_n\})$ are as in the last lemma, and if $x \in M_n(X)$, then there exists a completely contractive $u : X \to M_n$ such that $\|u_n(x)\| = \|x\|_n$.

Proof. By the Hahn–Banach theorem there exists $F \in \mathrm{Ball}(M_n(X)^*)$ with $|F(x)| = \|x\|_n$. By the last lemma, there exist vectors $\zeta, \eta \in \mathrm{Ball}(\mathbb{C}^{n^2})$, and a completely contractive $u : X \to M_n$ such that $F(x) = \langle u_n(x)\zeta, \eta\rangle$. Thus

$$\|x\|_n = |F(x)| \leq \|u_n(x)\|\|\zeta\|\|\eta\| \leq \|u_n(x)\|.$$

However clearly $\|u_n(x)\| \leq \|x\|_n$. □

Theorem 1.2.13. (Ruan) Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$ satisfying conditions (R1) and (R2) above. Then X is linearly completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H.

Proof. Suppose that $(X, \{\|\cdot\|_n\})$ satisfies (R1) and (R2). Let I be the collection of all completely contractive $\varphi : X \to M_n$, for all $n \in \mathbb{N}$. We write $n_\varphi = n$ if $\varphi : X \to M_n$. Let $M = \oplus^\infty_{\varphi \in I} M_{n_\varphi}$. This is a von Neumann algebra, and therefore certainly an operator space. Define $u : X \to M$ by $u(x) = (\varphi(x))_{\varphi \in I}$. This is a complete contraction, as is very easy to check, and so u_n is a contraction for each $n \in \mathbb{N}$. Choose $x \in M_n(X)$, and by Corollary 1.2.12 choose completely contractive

$\varphi : X \to M_n$ such that $\|\varphi_n(x)\| = \|x\|_n$. If $x = [x_{ij}]$ then since the projection $P : M \to M_{n_\varphi}$ onto the 'φ-entry' is a $*$-homomorphism, and hence completely contractive, we have

$$\|u_n(x)\| = \|[u(x_{ij})]\| \geq \|[P(u(x_{ij}))]\| = \|[\varphi(x_{ij})]\| = \|\varphi_n(x)\| = \|x\|_n.$$

Thus u_n is an isometry, and hence u is a complete isometry. □

1.2.14 We next discuss some consequences and applications of Ruan's theorem. It follows immediately from this result that the 'abstract operator spaces' are precisely the vector spaces X with matrix norms satisfying (R1) and (R2). More precisely, a sequence of norms $\{\|\cdot\|_n\}$, with $\|\cdot\|_n$ a norm on $M_n(X)$, is an *operator space structure* (oss) on X in the sense of 1.2.4, iff they satisfy (R1) and (R2). The one direction of this follows immediately from Theorem 1.2.13. The other follows immediately from the fact noted earlier that every concrete, and hence every abstract, operator space satisfies (R1) and (R2).

1.2.15 (Quotient operator spaces) If $Y \subset X$ is a closed linear subspace of an operator space, then Ruan's theorem allows one to check that X/Y is an operator space with matrix norm on $M_n(X/Y)$ coming from the identification $M_n(X/Y) \cong M_n(X)/M_n(Y)$, the latter space equipped with its quotient Banach space norm. Explicitly, these matrix norms are given by the formula $\|[x_{ij}\dot{+}Y]\|_n = \inf\{\|[x_{ij} + y_{ij}]\|_n : y_{ij} \in Y\}$. Here $x_{ij} \in X$. Note that with this definition, the canonical quotient map $q : X \to X/Y$ is a complete quotient map.

To check the (R1) condition, note that more generally if $\alpha \in M_{n,m}, \beta \in M_{m,n}, x = [x_{ij}] \in M_m(X)$, then $q_n(\alpha x \beta) = \alpha q_n(x)\beta$ (an exercise in linear algebra), and so

$$\|\alpha q_n(x)\beta\| = \|q_n(\alpha x\beta)\| \leq \|\alpha x\beta\|_n \leq \|\alpha\|\|x\|_m\|\beta\|.$$

Given $\epsilon > 0$ we may choose x above so that $\|x\| < \|q_n(x)\| + \epsilon$, and then

$$\|\alpha q_n(x)\beta\| \leq \|\alpha\|\|x\|_m\|\beta\| < \|\alpha\|(\|q_n(x)\|_m + \epsilon)\|\beta\|.$$

Letting $\epsilon \to 0$ gives $\|\alpha q_n(x)\beta\| \leq \|\alpha\|\|q_n(x)\|_m\|\beta\|$ as desired.

To prove (R2) let x be as in the last paragraph, and choose $y \in M_k(X)$ with $\|y\|_k < \|q_k(y)\| + \epsilon$. Then $q_{n+k}(x \oplus y) = q_n(x) \oplus q_k(y)$, and so

$$\|q_n(x) \oplus q_k(y)\|_{n+k} = \|q_{n+k}(x \oplus y)\|_{n+k} \leq \|x \oplus y\|_{n+k} = \max\{\|x\|, \|y\|\}$$

which is dominated by $\max\{\|q_n(x)\|, \|q_k(y)\|\} + \epsilon$. Letting $\epsilon \to 0$ gives $\|q_n(x) \oplus q_k(y)\|_{n+k} \leq \max\{\|q_n(x)\|, \|q_k(y)\|\}$, which is 'one half' of the (R2) condition. The other half follows from our slightly more general version of (R1) in the last paragraph since, for example, if $\alpha = [I_n \ O]$ then

$$\|q_n(x)\|_n = \|\alpha(q_n(x) \oplus q_k(y))\alpha^*\| \leq \|q_n(x) \oplus q_k(y)\|.$$

1.2.16 (Factor theorem) If $u \colon X \to Z$ is completely bounded, and if Y is a closed subspace of X contained in $\mathrm{Ker}(u)$, then the canonical map $\tilde{u} \colon X/Y \to Z$ induced by u is also completely bounded, with $\|\tilde{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. If $Y = \mathrm{Ker}(u)$, then u is a complete quotient map if and only if $\tilde{u}$ is a completely isometric isomorphism. Indeed this follows exactly as in the usual Banach space case (exercise).

1.2.17 (The ∞-direct sum) Let $\{X_\lambda : \lambda \in I\}$ be a family of operator spaces, and we write $\oplus_\lambda X_\lambda$ (or $\oplus_\lambda^\infty X_\lambda$ if more clarity is needed), for their ∞-direct sum as Banach spaces. If $I = \{1, \ldots, n\}$ then we usually write this sum as $X_1 \oplus^\infty \cdots \oplus^\infty X_n$. Thus a tuple (x_λ) is in $\oplus_\lambda^\infty X_\lambda$ if and only if $x_\lambda \in X_\lambda$ for all λ, and $\sup_\lambda \|x_\lambda\| < \infty$. Let us write P_λ for the projection of $\oplus_\lambda^\infty X_\lambda$ onto X_λ. We assign $\oplus_\lambda^\infty X_\lambda$ an operator space structure by defining $\|x\|_n = \sup_\lambda \|x_\lambda\|_{M_n(X_\lambda)}$ if $x \in M_n(\oplus_\lambda X_\lambda)$ and $x_\lambda = (P_\lambda)_n(x)$. Another way to say this, is that we are assigning $M_n(\oplus_\lambda X_\lambda)$ the norm making the canonical linear algebraic isomorphism $M_n(\oplus_\lambda X_\lambda) \cong \oplus_\lambda M_n(X_\lambda)$ an isometry. It is easy to check by Ruan's theorem that this is an operator space structure on $\oplus_\lambda^\infty X_\lambda$. Or one can see this directly as follows. If $X_\lambda \subset A_\lambda$, where A_λ is a C^*-algebra, then $\oplus_\lambda^\infty X_\lambda$ is isometrically embedded in the C^*-algebra direct sum $\oplus_\lambda^\infty A_\lambda$. The canonical operator space structure on the C^*-algebra $\oplus_\lambda^\infty A_\lambda$ is given by the formula $\|x\|_n = \sup_\lambda \|x_\lambda\|_{M_n(A_\lambda)}$, where $x_\lambda = (P_\lambda)_n(x)$. This may be seen, as in the end of 1.2.5, by proving that the latter formula is a C^*-algebra norm on $M_n(\oplus_\lambda^\infty A_\lambda)$, which in turn follows easily for example from the $*$-isomorphism $M_n(\oplus_\lambda^\infty A_\lambda) \cong \oplus_\lambda^\infty M_n(A_\lambda)$. Then $\oplus_\lambda^\infty X_\lambda$ inherits this operator space structure from $\oplus_\lambda^\infty A_\lambda$.

Another way to say the above, is that if $X_\lambda \subset A_\lambda \subset B(H_\lambda)$ then $\oplus_\lambda X_\lambda$ may be regarded as the subspace of $B(\oplus_\lambda^2 H_\lambda)$ consisting of the operators which take $(\zeta_\lambda) \in \oplus_\lambda^2 H_\lambda$ to $(x_\lambda \zeta_\lambda)$. It is a simple exercise to see that the norm of the latter operator is $\sup_\lambda \|x_\lambda\|$, with a similar formula for matrix norms.

By definition, the canonical inclusion and projection maps between $\oplus_\lambda X_\lambda$ and its 'λth summand' are complete isometries and complete quotient maps respectively. As we said above, if X_λ are C^*-algebras then this direct sum is the usual C^*-algebra direct sum. If the X_λ are W^*-algebras then this direct sum is just the W^*-algebra direct sum.

The ∞-direct sum has the following universal property. If Z is an operator space and $u_\lambda : Z \to X_\lambda$ are completely contractive linear maps, then there is a canonical complete contraction $Z \to \oplus_\lambda X_\lambda$ taking $z \in Z$ to the tuple $(u_\lambda(z))$. We leave this as an easy exercise.

If $X_\lambda = X$ for all $\lambda \in I$, then we usually write $\ell_I^\infty(X)$ for $\oplus_\lambda X_\lambda$.

One may define a 'c_0-direct sum' of operator spaces to simply be the closure in $\oplus_\lambda^\infty X_\lambda$ of the set of tuples which are zero except in finitely many entries.

1.2.18 (Mapping spaces) If X, Y are operator spaces, then the space $CB(X, Y)$ of completely bounded linear maps from X to Y, is also an operator space, with matrix norms determined via the canonical isomorphism between $M_n(CB(X, Y))$ and $CB(X, M_n(Y))$. That is, if $[u_{ij}] \in M_n(CB(X, Y))$, then define

$$\|[u_{ij}]\|_n = \sup\big\{\|[u_{ij}(x_{kl})]\|_{nm} : [x_{kl}] \in \text{Ball}(M_m(X)),\, m \in \mathbb{N}\big\}. \tag{1.4}$$

Here the matrix $[u_{ij}(x_{kl})]$ is indexed on rows by i and k and on columns by j and l. Then

$$M_n(CB(X, Y)) \cong CB(X, M_n(Y)) \quad \text{isometrically.} \tag{1.5}$$

One may see that (1.4) defines an operator space structure on $CB(X, Y)$ by appealing to Ruan's theorem 1.2.13. Alternatively, one may see it as follows. Consider the set $I = \cup_n \mathrm{Ball}(M_n(X))$, and for $x \in \mathrm{Ball}(M_m(X)) \subset I$ set $n_x = m$. Consider the operator space direct sum (see 1.2.17) $\oplus^\infty_{x\in I} M_{n_x}(Y)$, which is an operator space. Then the map from $CB(X,Y)$ to $\oplus^\infty_{x\in I} M_{n_x}(Y)$ taking u to the tuple $(u_{n_x}(x))_x \in \oplus^\infty_x M_{n_x}(Y)$ is (almost tautologically) a complete isometry. For example, note that

$$\|(u_{n_x}(x))_{x\in I}\| = \sup\{\|u_{n_x}(x)\| : x \in I\} = \sup\{\|u_n\| : n \in \mathbb{N}\} = \|u\|_{\mathrm{cb}}.$$

Thus $CB(X,Y)$ is an operator space.

1.2.19 (The dual of an operator space) The special case when $Y = \mathbb{C}$ in 1.2.18 is particularly important. In this case, for any operator space X, we obtain by 1.2.18 an operator space structure on $X^* = CB(X,\mathbb{C})$. The latter space equals $B(X,\mathbb{C})$ isometrically by 1.2.7. We call X^*, viewed as an operator space in this way, the *operator space dual* of X. This duality will be studied further in later sections. By (1.5) we have

$$M_n(X^*) \cong CB(X, M_n) \quad \text{isometrically.} \tag{1.6}$$

(Note that the map implementing this isomorphism is also exactly the canonical map θ from $M_n \otimes X^*$ to $B(X, M_n)$, where $\theta(a\otimes\varphi)(x) = \varphi(x)a$, for $a \in M_n, \varphi \in X^*, x \in X$.)

1.2.20 (Minimal operator spaces) Let E be a Banach space, and consider the canonical isometric inclusion of E in the commutative C^*-algebra $C(\mathrm{Ball}(E^*))$. Here E^* is equipped with the w^*-topology. This inclusion induces, via 1.2.5, an operator space structure on E, which is denoted by $\mathrm{Min}(E)$. We call $\mathrm{Min}(E)$ a *minimal operator space*. By (1.2), the resulting matrix norms on E are given by

$$\|[x_{ij}]\|_n = \sup\bigl\{\|[\varphi(x_{ij})]\| \,:\, \varphi \in \mathrm{Ball}(E^*)\bigr\} \tag{1.7}$$

for $[x_{ij}] \in M_n(E)$. Thus every Banach space may be canonically considered to be an operator space. Since $\mathrm{Min}(E) \subset C(\mathrm{Ball}(E^*))$, we see from 1.2.7 that for any bounded linear u from an operator space Y into E, we have

$$\|u\colon Y \longrightarrow \mathrm{Min}(E)\|_{\mathrm{cb}} \;=\; \|u\colon Y \longrightarrow E\|. \tag{1.8}$$

From this last fact one easily sees that $\mathrm{Min}(E)$ is the smallest operator space structure on E. For if $\{|||\cdot|||_n\}$ was an operator space structure on E, with $|||\cdot|||_1 = \|\cdot\|$, write X for the abstract operator space which is E with these matrix norms. Then $I_E : \mathrm{Min}(E) \to X$ is a linear isometry, and so by (1.8) we have $\|I_E^{-1}\|_{\mathrm{cb}} = \|I_E^{-1}\| = 1$. But this says precisely that $|||\cdot|||_n$ dominates the norm in (1.7).

Also, if Ω is any compact space and if $i\colon E \to C(\Omega)$ is an isometry, then the matrix norms inherited by E from the operator space structure of $C(\Omega)$, coincide again with those in (1.7). That is, the norms in (1.7) equal $\|[i(x_{ij})]\|_n$. This may be seen by applying 1.2.7 to i and i^{-1} (the latter defined on the range of i). By 1.2.7 we have i completely contractive. But since $\mathrm{Min}(E) \subset C(\mathrm{Ball}(E^*)))$ we have by 1.2.7 again that i^{-1} is completely contractive. So i is a complete isometry, which is the desired identity.

This means that 'minimal operator spaces' are exactly the operator spaces completely isometrically isomorphic to a subspace of a $C(K)$-space. Note too that the category of Banach spaces and bounded linear maps is 'the same' as the category of minimal operator spaces and completely bounded linear maps.

1.2.21 (Maximal operator spaces) If E is a Banach space then $\mathrm{Max}(E)$ is the largest operator space structure we can put on E. We define the matrix norms on $\mathrm{Max}(E)$ by the following formula

$$\|[x_{ij}]\|_n = \sup\big\{\|[u(x_{ij})]\| \,:\, u \in \mathrm{Ball}(B(E,Y)),\ \text{all operator spaces } Y\big\}. \tag{1.9}$$

This may be seen to be an operator space structure on E by using Ruan's theorem. However again a direct sum argument is more elementary: Define a map $i : x \mapsto (u(x))_u$ from E into the operator space $Z = \oplus^\infty_u Y_u$, where the latter sum is indexed by every $u : E \to Y$ as in (1.9), and writing such Y as Y_u. We may assume that the cardinality of Y is dominated by that of E so that there are no set theoretic issues. Since there exists at least one such u which is an isometry (see e.g. 1.2.20), it is evident that i is an isometry. Thus $\|\cdot\|_1$ is the usual norm on E. Then the matrix norms inherited by E from the operator space structure of Z, gives E an operator space structure. However the latter coincides again with the one in (1.9). That is, the norms in (1.9) equal $\|[i(x_{ij})]\|_n$.

It is clear from this formula that $\mathrm{Max}(E)$ has the property that for any operator space Y, and for any bounded linear $u\colon E \to Y$, we have

$$\|u\colon \mathrm{Max}(E) \longrightarrow Y\|_{\mathrm{cb}} \;=\; \|u\colon E \longrightarrow Y\|. \tag{1.10}$$

Indeed to prove this we may assume that u is a contraction, and then from (1.9) we see that $\|[u(x_{ij})]\|$ is dominated by the norm in (1.9). That is, u_n is a contraction, so that u is completely contractive, as a map from E with the matrix norms from (1.9). This proves (1.10).

It is also clear that $\mathrm{Max}(E)$ is the largest operator space structure we can put on E. For if $\{|||\cdot|||_n\}$ was another operator space structure on E, with $|||\cdot|||_1 = \|\cdot\|$, write X for the abstract operator space which is E with these matrix norms. Then $I_E : \mathrm{Max}(E) \to X$ is a linear isometry, and so by (1.10) we have $\|I_E\|_{\mathrm{cb}} = \|I_E\| = 1$. But this says precisely that $|||\cdot|||_n$ is dominated by the norm in (1.9).

1.2.22 (Hilbert column and row spaces) If H is a Hilbert space then there are two canonical operator space structures on H most commonly considered. The first is the *Hilbert column space* H^c. Informally one should think of H^c as a 'column in $B(H)$'. Thus if $H = \ell^2_n$ then $H^c = M_{n,1}$, thought of as the matrices in M_n which are 'zero except on the first column'. We write this operator space also as C_n, and the 'row' version as R_n. Note that for such a matrix x the norm $\|x\| = \|x^*x\|^{\frac{1}{2}}$ is precisely the ℓ^2_n norm of the entries in x. So $C_n \cong \ell^2_n$ isometrically. However C_n is not completely isometric to R_n, and they fail to even be completely isomorphic if n is infinite (see the discussion after Proposition 1.2.25).

For a general Hilbert space H there are several simple ways of describing H^c more precisely. For example, one may identify H^c with the concrete operator space

$B(\mathbb{C}, H)$. If $\zeta \in H$ write $T_\zeta : \mathbb{C} \to H$ for the operator taking 1 to ζ. It is easy to see that $T_\zeta^* T_\eta$ is the operator on $\mathbb{C}$ taking 1 to $\langle \eta, \zeta \rangle$. Thus if $[\zeta_{ij}] \in M_n(H)$ then by the C^*-identity

$$\|[T_{\zeta_{ij}}]\| = \|[\sum_{k=1}^{n} T_{\zeta_{ki}}^* T_{\zeta_{kj}}]\|^{\frac{1}{2}} = \|[\sum_{k=1}^{n} \langle \zeta_{kj}, \zeta_{ki} \rangle]\|^{\frac{1}{2}}.$$

Another equivalent description of Hilbert column space is as follows: If η is a fixed unit vector in H, then the set $H \otimes \eta$ of rank one operators $\zeta \otimes \eta$ is a closed subspace of $B(H)$ which is isometric to H via the map $\zeta \mapsto \zeta \otimes \eta$. (By convention, $\zeta \otimes \eta$ maps $\xi \in H$ to $\langle \xi, \eta \rangle \zeta$.) Thus we may transfer the operator space structure on $H \otimes \eta$ inherited from $B(H)$ over to H. The resulting operator space structure is independent of η and coincides with H^c. To see this, we will use the C^*-identity in $M_n(B(H))$ applied to $\|[\zeta_{ij} \otimes \eta]\|$. Note that

$$[\zeta_{ij} \otimes \eta]^* [\zeta_{ij} \otimes \eta] = [\eta \otimes \zeta_{ji}][\zeta_{ij} \otimes \eta] = [\sum_{k=1}^{n} (\eta \otimes \zeta_{ki})(\zeta_{kj} \otimes \eta)].$$

However, $(\eta \otimes \zeta_{ki})(\zeta_{kj} \otimes \eta) = \langle \zeta_{kj}, \zeta_{ki} \rangle \, R$, where $R = \eta \otimes \eta$. It was left as an exercise in 1.2.8 that $\|[z_{ij} R]\| = \|R\| \|[z_{ij}]\|$ for scalars z_{ij}, and so we conclude using the C^*-identity that $\|[\zeta_{ij} \otimes \eta]\| = \|[\sum_{k=1}^{n} \langle \zeta_{kj}, \zeta_{ki} \rangle]\|^{\frac{1}{2}}$. That is,

$$\|[\zeta_{ij}]\|_{M_n(H^c)} = \Big\| \Big[\sum_{k=1}^{n} \langle \zeta_{kj}, \zeta_{ki} \rangle \Big] \Big\|^{\frac{1}{2}}, \qquad [\zeta_{ij}] \in M_n(H). \tag{1.11}$$

This shows that this is the same operator space structure on H as the previous one. If $H = \ell_n^2$ and we take $\eta = (1, 0, \cdots, 0)$ then $\{\zeta \otimes \eta\}$ is precisely the matrices in M_n which are 'zero except on the first column'.

If $T \in B(H, K)$ then $\|T\| = \|T\|_{\mathrm{cb}}$, where the latter is the norm taken in $CB(H^c, K^c)$. Indeed let $[\zeta_{ij}] \in M_n(H^c)$, and let $\alpha \in B(\ell_n^2, \ell_n^2(H))$ correspond to this matrix via the identity $M_n(H^c) = M_n(B(\mathbb{C}, H)) = B(\ell_n^2, \ell_n^2(H))$. Similarly, let $\beta \in B(\ell_n^2, \ell_n^2(K))$ corresponding to $[T\zeta_{ij}]$. Then $\beta = (I_{\ell_n^2} \otimes T) \circ \alpha$, and hence $\|\beta\| \leq \|I_{\ell_n^2} \otimes T\| \|\alpha\| \leq \|T\| \|\alpha\|$. This shows that $\|T_n\| \leq \|T\|$, and so $\|T\|_{\mathrm{cb}} \leq \|T\|$.

More generally, we have

$$B(H, K) = CB(H^c, K^c) \quad \text{completely isometrically} \tag{1.12}$$

We will give a quick proof of this identity at the end of this section.

A subspace K of a Hilbert column space H^c is again a Hilbert column space, as may be seen by considering (1.11). Similarly the quotient H^c/K^c is a Hilbert column space completely isometric to $(H \ominus K)^c$. This may be seen by considering the canonical projection P from H^c onto $(H \ominus K)^c$. Note P is applying completely contractive by the fact at the start of the second last paragraph, and is therefore clearly a complete quotient map. Now apply 1.2.16 to see that $H^c/\mathrm{Ker}(P) = H^c/K^c \cong (H \ominus K)^c$ completely isometrically.

We define *Hilbert row space* similarly. Recalling that $H^* \cong \bar{H}$ is a Hilbert space too, we identify H^r with the concrete operator space $B(\bar{H}, \mathbb{C})$. Analogues of the above results for H^c hold, except that there is a slight twist in the corresponding version of (1.12). Namely, although $B(H,K) = CB(H^r, K^r)$ isometrically, this is not true *completely isometrically.* Instead, as we shall see, there is a canonical completely isometric isomorphism $B(H,K) \cong CB(\bar{K}^r, \bar{H}^r)$. We write C and R for ℓ^2 with its column and row operator space structures respectively.

We have

$$(H^c)^* \;\cong\; \bar{H}^r \quad \text{and} \quad (H^r)^* \;\cong\; \bar{H}^c \tag{1.13}$$

completely isometrically using the operator space dual structure in 1.2.19. The first relation is obtained by setting $K = \mathbb{C}$ in (1.12). The second relation follows e.g. from the first if we replace H there by $K = \bar{H}$, and take the operator space dual, using the fact that Hilbert spaces are reflexive, and also the first result in the next Section 1.3, which states that $X \subset X^{**}$ completely isometrically. Thus $K^c \cong (K^c)^{**} \cong (H^r)^*$ completely isometrically.

Just as in one of the exercises for Chapter 1, the map $T \mapsto T^*$ is a complete isometry from $CB(X,Y)$ into $CB(Y^*, X^*)$ and this map is onto if X is reflexive. Thus if H, K are Hilbert spaces then we have

$$B(H,K) \cong CB(H^c, K^c) \cong CB((K^c)^*, (H^c)^*) \cong CB(\bar{K}^r, \bar{H}^r),$$

using (1.13).

1.2.23 (Matrix spaces) If X is an operator space, and I, J are sets, then we write $\mathbb{M}_{I,J}(X)$ for the set of $I \times J$ matrices whose finite submatrices have uniformly bounded norm. We explain: By an '$I \times J$ matrix' we mean a matrix $x = [x_{i,j}]_{i \in I, j \in J}$, where $x_{i,j} \in X$. For such a matrix x, and for a subset $\Delta = C \times D \subset I \times J$, we write x_Δ for the 'submatrix' $[x_{i,j}]_{i \in C, j \in D}$. Sometimes we also write x_Δ for the same matrix viewed as an element of $\mathbb{M}_{I,J}(X)$, and with all other entries zero. We say the submatrix is finite if Δ is finite. We define $\|x\|$ to be the supremum of the norms of its finite submatrices, and $\mathbb{M}_{I,J}(X)$ consists of those matrices x with $\|x\| < \infty$. Similarly there is an obvious way to define a norm on $M_n(\mathbb{M}_{I,J}(X))$ by equating this space with $\mathbb{M}_{I,J}(M_n(X))$, and one has $M_n(\mathbb{M}_{I,J}(X)) \cong \mathbb{M}_{n.I,n.J}(X)$, for $n \in \mathbb{N}$.

We write $\mathbb{M}_I(X) = \mathbb{M}_{I,I}(X), C_I^w(X) = \mathbb{M}_{I,1}(X)$, and $R_I^w(X) = \mathbb{M}_{1,I}(X)$. If $I = \aleph_0$ we simply denote these spaces by $\mathbb{M}(X)$, $C^w(X)$ and $R^w(X)$ respectively. Also, $\mathbb{M}_{I,J}^{\text{fin}}(X)$ will denote the vector subspace of $\mathbb{M}_{I,J}(X)$ consisting of 'finitely supported matrices', that is, those matrices with only a finite number of nonzero entries. We write $\mathbb{K}_{I,J}(X)$ for the norm closure in $\mathbb{M}_{I,J}(X)$ of $\mathbb{M}_{I,J}^{\text{fin}}(X)$. We set $\mathbb{K}_I(X) = \mathbb{K}_{I,I}(X), C_I(X) = \mathbb{K}_{I,1}(X)$, and $R_I(X) = \mathbb{K}_{1,I}(X)$. Again we merely write $\mathbb{K}(X)$, $R(X)$ and $C(X)$ for these spaces if $I = \aleph_0$. If $X = \mathbb{C}$ then we write $C_I(\mathbb{C}) = C_I$. Similarly, $R_I = R_I(\mathbb{C})$. We write $\mathbb{K}_{I,J}$ for $\mathbb{K}_{I,J}(\mathbb{C})$, and $\mathbb{M}_{I,J}$ for $\mathbb{M}_{I,J}(\mathbb{C})$.

We leave the following assertions about matrix spaces as exercises, for the most part. Throughout, I, J, I_0, J_0 are sets and X, Y are operator spaces.

(1) If $X \subset Y$ (completely isometrically), then $\mathbb{M}_{I,J}(X) \subset \mathbb{M}_{I,J}(Y)$ completely isometrically. Thus if $X \subset B(H,K)$ then $\mathbb{M}_{I,J}(X) \subset \mathbb{M}_{I,J}(B(H,K))$. This

is important, since this reduces most facts about $\mathbb{M}_{I,J}(X)$ to facts about $\mathbb{M}_{I,J}(B(H,K))$, which we shall see in (5) is a simple space to deal with.

(2) If $u\colon X \to Y$ is completely bounded, then so is the obvious *amplification* $u_{I,J}\colon \mathbb{M}_{I,J}(X) \to \mathbb{M}_{I,J}(Y)$, and $\|u_{I,J}\|_{\rm cb} = \|u\|_{\rm cb}$. Clearly $u_{I,J}$ also restricts to a completely bounded map from $\mathbb{K}_{I,J}(X)$ to $\mathbb{K}_{I,J}(Y)$. If u is a complete isometry, then so is $u_{I,J}$ (see (1)). Thus the $\mathbb{M}_{I,J}(\cdot)$ and $\mathbb{K}_{I,J}(\cdot)$ constructions are 'injective' in some sense.

(3) $\mathbb{M}_{I,J} \cong B(\ell^2_J, \ell^2_I)$ completely isometrically. Via this identification, $\mathbb{K}_{I,J} = S^\infty(\ell^2_J, \ell^2_I)$ completely isometrically. Thus for any Hilbert spaces K, H we have that $B(K,H) \cong \mathbb{M}_{I_0,J_0}$ completely isometrically, for some sets I_0, J_0.

(4) We have $\mathbb{M}_{I,J}(\mathbb{M}_{I_0,J_0}) \cong \mathbb{M}_{I\times I_0, J\times J_0} \cong \mathbb{M}_{I_0,J_0}(\mathbb{M}_{I,J})$ completely isometrically.

(5) Putting (3) and (4) together, it follows easily that for any sets I, J, we have $\mathbb{M}_{I,J}(B(K,H)) \cong B(K^{(J)}, H^{(I)})$ completely isometrically.

(6) Fix $i \in I, j \in J$. The map which takes $x \in X$ to the matrix in $\mathbb{M}_{I,J}(X)$ which is all zero except for an x in the i-j-entry, is a complete isometry. The map $\mathbb{M}_{I,J}(X) \to X$ which takes a matrix to its i-j-entry, is a complete contraction. The map which takes $x \in X$ to the matrix in $\mathbb{M}_I(X)$ which is all zero except for an x in all the entries on the 'main diagonal', is a complete isometry.

(7) If X is an operator space then so is $\mathbb{M}_{I,J}(X)$. Indeed if $X \subset B(H)$, then by (1) and (5) we have $\mathbb{M}_{I,J}(X) \subset \mathbb{M}_{I,J}(B(H)) \cong B(H^{(J)}, H^{(I)})$ completely isometrically. If X is complete then so is $\mathbb{M}_{I,J}(X)$. To see this, we can suppose that X is a closed subspace of $B(H)$. Then $\mathbb{M}_{I,J}(X) \subset \mathbb{M}_{I,J}(B(H)) \cong B(H^{(J)}, H^{(I)})$, and the latter space is complete. Suppose that $a_n \in \mathbb{M}_{I,J}(X)$, with $a_n \to a \in \mathbb{M}_{I,J}(B(H))$. Then by (6) the i-j-entry of a_n converges to the i-j-entry of a, and so the latter is in X. Hence $a \in \mathbb{M}_{I,J}(X)$. So $\mathbb{M}_{I,J}(X)$ is norm closed in $\mathbb{M}_{I,J}(B(K,H))$.

(8) We have $\mathbb{M}_{I,J}(X) = C^w_I(R^w_J(X)) = R^w_J(C^w_I(X))$ completely isometrically. One way to see this is to first check this identity in the case $X = B(H)$ using (5) repeatedly, and then use this fact to do the general case.

(9) By a similar argument, $\mathbb{M}_{I,J}(\mathbb{M}_{I_0,J_0}(X)) \cong \mathbb{M}_{I\times I_0, J\times J_0}(X)$ for any operator space X, generalizing (4).

(10) $C^w_I(\mathbb{C}) = C_I = (\ell^2_I)^c$ (see 1.2.22 for this notation). Indeed, by (5) we have $C^w_I(\mathbb{C}) = B(\mathbb{C}, \ell^2_I) = (\ell^2_I)^c$, and this must equal C_I since 'finitely supported tuples' are dense in ℓ^2_I. Similarly, $R_I = R^w_I(\mathbb{C}) = (\ell^2_I)^r$.

(11) $\mathbb{K}_{I,J}(X)$ is the set of $x \in \mathbb{M}_{I,J}(X)$ such that the net (x_Δ) converges to x, where the net is indexed by the finite subsets $\Delta = C \times D$ of $I \times J$, ordered by inclusion.

(12) For any operator spaces X, Y we have $CB(X, \mathbb{M}_{I,J}(Y)) \cong \mathbb{M}_{I,J}(CB(X,Y))$ isometrically. We leave it as an exercise to write down the obvious isomorphism here, and to check that this is a (complete) isometry.

1.2.24 (Infinite sums) Suppose that X, Y are subspaces of a C^*-algebra $A \subset B(H)$. Let I be an infinite set. If $x \in R^w_I(X)$ and $y \in C_I(Y)$, then the 'product' $xy = \sum_i x_i y_i$, if x and y have ith entries x_i and y_i respectively, actually converges in norm to an element of A, and we have $\|xy\| \leq \|x\|\|y\|$. This is clear if I is finite, in this case we can view $x \in R_n(B(H)) = B(H^{(n)}, H)$ and similarly $y \in B(H, H^{(n)})$, and then clearly $\|xy\| \leq \|x\|\|y\|$. To see the general case, we use the following notation. If z is an element of $R_I^w(X)$ or $C_I(Y)$, and if $\Delta \subset I$, write z_Δ for z but with all entries outside Δ 'switched to zero'. Since $y \in C_I(Y)$, by 1.2.23 (11) given $\epsilon > 0$ there is a finite set $\Delta \subset I$, such that $\|y - y_\Delta\| = \|y_{\Delta^c}\| < \epsilon$. If Δ' is a finite subset of I not intersecting Δ then

$$\Big\| \sum_{i \in \Delta'} x_i y_i \Big\| = \|x_{\Delta'} y_{\Delta'}\| \leq \|x_{\Delta'}\|\|y_{\Delta'}\| \leq \|x\|\|y_{\Delta'}\| < \|x\|\epsilon.$$

Hence the sum converges in norm as claimed. For any finite $\Delta \subset I$, a computation identical to the first part of the second last centered equation shows that $\| \sum_{i \in \Delta} x_i y_i\| \leq \|x\|\|y\|$. Taking the limit over Δ, we have $\|xy\| \leq \|x\|\|y\|$.

Proposition 1.2.25. For any operator space X and set I, we have that $CB(C_I, X) \cong R_I^w(X)$ and $CB(R_I, X) \cong C_I^w(X)$ completely isometrically.

Proof. We sketch the proof of just the first relation. Define $L\colon R_I^w(X) \to CB(C_I, X)$ by $L(x)(z) = \sum_i x_i z_i$, for $x \in R_I^w(X), z \in C_I$. This map is well defined, by the argument for 1.2.24 for example. It is also easy to check, by looking at the partial sums of this series as in 1.2.24, that L is contractive. Alternatively, this can be seen by viewing $X \subset B(H)$, and $L(x)(z)$ as the product (composition) TS of the operator $S_z : H \to H^{(I)} : \zeta \mapsto [z_i \zeta]$, and the operator $T_x : H^{(I)} \to H : [\eta_i] \to \sum_i x_i \eta_i$. It is easy to argue that

$$\|[L(x)(z_{ij})]\| = \|[T_x S_{z_{ij}}]\| \leq \|T_x\|\|[S_{z_{ij}}]\| = \|x\|\|[z_{ij}]\|,$$

so that L is a contraction.

Conversely, for u in $CB(C_I, X)$, let x be a 1 by I matrix whose ith entry is $u(e_i)$, where (e_i) is the canonical basis. If $\Delta = \{i_1, i_2, \cdots, i_m\} \subset I$ then

$$\|x_\Delta\| = \|[u(e_{i_1})\ u(e_{i_2})\ \cdots\ u(e_{i_m})]\| \leq \|u\|_{\text{cb}}\|[e_{i_1}\ e_{i_2}\ \cdots\ e_{i_m}]\| = \|u\|_{\text{cb}},$$

since the last matrix after erasing rows and columns of zeros is an identity matrix. Thus $x \in R_I^w(X)$ and $\|x\|_{R_I^w(X)} \leq \|u\|_{\text{cb}}$. It is easy to see that $L(x)(z) = \sum_i u(e_i) z_i = u(\sum_i e_i z_i) = u(z)$ if $z \in C_I$. Thus $L(x) = u$, and so L is a surjective isometry. This together with (1.5) yields

$$M_m(CB(C_I, X)) \cong CB(C_I, M_m(X)) \cong R_I^w(M_m(X)) \cong M_m(R_I^w(X))$$

isometrically. From this one sees that L is a complete isometry. □

We will use the last lemma to verify two facts that were mentioned earlier. First, that C_I is not completely isomorphic to R_I if I is infinite. One way to see this is to note that $CB(C_I, R_I) \cong R_I^w(R_I) \cong R_{I \times I}$ by Proposition 1.2.25, and (10) and (4) of 1.2.23. This is saying that for an operator $T : C_I \to R_I$, the 'cb-norm' equals its Hilbert-Schmidt norm (that is, its norm in the Hilbert-Schmidt class $S^2(\ell_I^2)$. Similarly if $S : R_I \to C_I$. So if $C_I \cong R_I$ completely isomorphically, then there is an *invertible* operator between them which is in $S^2(\ell^2_I)$. Since $S^2(\ell^2_I)$ is known from operator theory to be an ideal, this implies that the identity map ($= TT^{-1}$) is in $S^2(\ell^2_I)$, which is absurd if I is infinite.

Second, we show that $B(H,K) \cong CB(H^c, K^c)$ completely isometrically. Indeed,

$$CB(C_I, C_J) \cong R^w_I(C_J) = R^w_I(C^w_J) \cong \mathbb{M}_{J,I} \cong B(\ell^2_I, \ell^2_J),$$

using Proposition 1.2.25, and 1.2.23 (10), (8), and (3).

Historical note: The results in Section 2.1 are almost all due to Arveson, Effros, and Ruan [1, 17]. Hamana studied matrix spaces (see 1.2.23) in some of his papers (see e.g. [20]), and they are studied in more detail by Effros, and Ruan in [13, 14]. Maximal operator spaces were first considered by Blecher and Paulsen [8]. Preliminary forms of some of the results in 1.2.22 were noted in the latter paper; and the fact that $B(H,K) \cong CB(H^c, K^c)$ isometrically is due to Wittstock [32]. In the generality listed here, the main source for the results towards the end of 1.2.22 is [16], although Blecher independently discovered a couple of these [3].

Exercises.

(1) If $T \in B(K,H)$, and if $[z_{ij}] \in M_n$ show that $\|[z_{ij}T]\| = \|T\|\|[z_{ij}]\|$. Deduce that $\mathbb{C}$ has a unique operator space structure (up to complete isometry).

(2) If $X \subset B(H,K)$, and if $S : K \to K'$ and $T : H' \to H$ are operators between Hilbert spaces, prove that the map $x \mapsto SxT$ is completely bounded with 'cb-norm' dominated by $\|S\|\|T\|$.

(3) If K is a closed subspace of a Hilbert space H, prove that the map $x \mapsto P_K x_{|K}$ is completely contractive from $B(H)$ to $B(K)$.

(4) Prove that if $S : M_n(Y) \to M_n(Z)$ is a linear map, where Y, Z are vector spaces, then $S = u_n$ for a linear $u : Y \to Z$ iff $S(\alpha x \beta) = \alpha S(x)\beta$ for all $x \in M_n(Y)$ and $\alpha, \beta \in M_n$.

(5) (R. R. Smith) If $u : X \to M_n$ satisfies $\|u_n\| \leq 1$, use linear algebra to show that $\|u_m\| \leq 1$ for all $m \geq n$, so that u is completely contractive. [Hint: if $m \geq n$ and $\zeta, \eta \in \mathbb{C}^{mn}$, then we can write $\zeta = \sum_{k=1}^n \zeta_k \otimes \vec{e_k}$, where $\zeta_k \in \mathbb{C}^m$. Since $\text{Span}\{\zeta_k : k = 1, \cdots, n\}$ has dimension $\leq n$, there is an isometry $\beta \in M_{m,n}$ and vectors $\widetilde{\zeta_k} \in \mathbb{C}^n$ with $\beta\widetilde{\zeta_k} = \zeta_k$. Similarly, there is an isometry $\alpha \in M_{m,n}$ with $\alpha\widetilde{\eta_k} = \eta_k$. Use this to find an upper bound for the number $|\langle u_m(x)\zeta, \eta\rangle|$.]

(6) Prove that (R1) and (R2) together are equivalent to requiring that: (R1)$'$ $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_m\|\beta\|$, for all $n, m \in \mathbb{N}$ and all $\alpha \in M_{n,m}, \beta \in M_{m,n}$, and $x \in M_m(X)$, and (R2)$'$ $\|x \oplus y\|_{n+m} \leq \max\{\|x\|_n, \|y\|_m\}$ for $x \in M_n(X), y \in M_m(X)$. Prove also that (R1)$'$ and (R2)$'$ actually imply that $\|\cdot\|_n$ is a norm.

(7) Suppose that $u_i : X \to B(K_i, H_i)$ are completely contractive, and that $K = \oplus_i K_i$ and $H = \oplus_i H_i$ (Hilbert space sum). Define $u : X \to B(K,H)$ by $u(x) = (u_i(x))$, where the latter denotes the operator $(\zeta_i) \mapsto (u_i(x)\zeta_i)$ on K. Show that $\|u_i\|_{\text{cb}} \leq \|u\|_{\text{cb}} \leq 1$.

(8) Prove that if in Ruan's theorem X is also separable, then one may take the Hilbert space there to be ℓ^2.

(9) If X, Y are (possibly incomplete) operator spaces, and if $\theta : X \to Y$ is a linear isomorphism such that the map $\varphi \mapsto \varphi \circ \theta$ is a well defined complete isometry from Y^* onto X^*, then θ is completely isometric.

(10) Prove the facts stated in 1.2.23.

1.3 Duality of operator spaces

An operator space Y is said to be a *dual operator space* if Y is completely isometrically isomorphic to the operator space dual (see 1.2.19) X^* of an operator space X. We also say that X is an *operator space predual* of Y, and sometimes we write X as Y_*. If X, Y are dual operator spaces then we write $w^*CB(X, Y)$ for the space of w^*-continuous completely bounded maps from X to Y.

Unless otherwise indicated, in what follows the symbol X^* denotes the dual space *together with its dual operator space structure* as defined in 1.2.19. Of course X^{**} is considered as the dual operator space of X^*.

Proposition 1.3.1. If X is an operator space then $X \subset X^{**}$ completely isometrically via the canonical map i_X.

Proof. We can suppose that X is a subspace of $B(H)$, for a Hilbert space H. Fix $n \in \mathbb{N}$ and $[x_{ij}] \in M_n(X)$. We first show that $\|[i_X(x_{ij})]\|_n \leq \|[x_{ij}]\|_n$. By definition, the norm $\|[i_X(x_{ij})]\|_n$ in $M_n((X^*)^*)$ equals

$$\begin{aligned} &\sup\big\{\|[i_X(x_{ij})(f_{kl})]\|_{nm} \;:\; [f_{kl}] \in \text{Ball}(M_m(X^*)),\, m \in \mathbb{N}\big\} \\ &= \sup\big\{\|[f_{kl}(x_{ij})]\|_{nm} \;:\; [f_{kl}] \in \text{Ball}(M_m(X^*)),\, m \in \mathbb{N}\big\} \\ &\leq \|[x_{ij}]\|_n, \end{aligned}$$

the last line by definition of $[f_{kl}] \in \text{Ball}(M_m(X^*))$.

Since $\|[i_X(x_{ij})]\|_n$ equals the supremum above, and since $M_m(X^*) \cong CB(X, M_m)$, to see that i_X is completely isometric, it suffices to prove the Claim: for a given $n \in \mathbb{N}$, $\epsilon > 0$, and $[x_{kl}] \in M_n(X)$, there exists an integer m and a completely contractive $u\colon X \to M_m$ such that $\|[u(x_{kl})]\| \geq \|[x_{kl}]\| - \epsilon$. In fact this Claim follows immediately (with $\epsilon = 0$ and $m = n$) from Corollary 1.2.12. □

Remark. Because of its independent interest, we will give another alternative proof of the Claim in the last proof. Let $[x_{ij}] \in M_n(X) \subset M_n(B(H)) \cong B(H^{(n)})$. Thus $[x_{ij}]$ may be viewed as an operator on $H^{(n)}$. The norm of any operator $T \in B(K)$, for any Hilbert space K, is given by the formula $\|T\| = \sup\{|\langle Ty, z\rangle| : y, z \in \text{Ball}(K)\}$. Thus in our case,

$$\|[x_{ij}]\|_n = \sup\{|\langle [x_{ij}]y, z\rangle| : y, z \in \text{Ball}(H^{(n)})\}.$$

So, if $\epsilon > 0$ is given, there exists $y, z \in \text{Ball}(H^{(n)})$ such that $|\langle [x_{ij}]y, z\rangle| > \|[x_{ij}]\|_n - \epsilon$. If $y = (\zeta_k)$ and $z = (\eta_k)$, with $\zeta_k, \eta_k \in H$, then $\langle [x_{ij}]y, z\rangle = \sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle$, and so

$$\Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big| \geq \|[x_{ij}]\| - \epsilon.$$

Let $K = \text{Span}\,\{\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n\}$ in H. This is finite dimensional, and so there is an isometric $*$-isomorphism $\pi : B(K) \to M_m$, where $m = \dim(K)$. Then π is completely contractive by Proposition 1.2.6. Let P_K be the projection from H onto K. Let $T\colon B(H) \to B(K)$ be the function $T(x) = P_K x_{|K}$. By an exercise at the end of the section, T is completely contractive. Let $u = \pi \circ T$, which will be completely contractive too. Now $\langle [T(x_{ij})]y, z\rangle = \sum_{i,j} \langle T(x_{ij})\zeta_j, \eta_i\rangle$, and so

$$\|[T(x_{ij})]\|_n \geq \Big|\sum_{i,j} \langle T(x_{ij})\zeta_j, \eta_i\rangle\Big| = \Big|\sum_{i,j} \langle P_K x_{ij}\zeta_j, \eta_i\rangle\Big| = \Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big|,$$

the last step since $\eta_i \in K$. Thus,

$$\|[u(x_{ij})]\|_n = \|[\pi(T(x_{ij}))]\|_n = \|[T(x_{ij})]\|_n \geq \Big|\sum_{i,j} \langle x_{ij}\zeta_j, \eta_i\rangle\Big| \geq \|[x_{ij}]\| - \epsilon,$$

using the fact at the end of the last paragraph. This proves the Claim.

1.3.2 From 1.3.1 we have for any $[x_{ij}] \in M_n(X)$ that

$$\|[x_{ij}]\|_n = \sup\{\|[\varphi_{kl}(x_{ij})]\| : m \in \mathbb{N},\ [\varphi_{kl}] \in \text{Ball}(M_m(X^*))\} \tag{1.14}$$

There is a canonical map $\theta : M_n(X) \to CB(X^*, M_n)$, namely $\theta([x_{ij}])(\varphi) = [\varphi(x_{ij})]$, and (1.14) says that θ is an isometry. Note that if $[x_{ij}] \in M_n(X)$, and if (φ_t) is a net in X^* converging weak* to $\varphi \in X^*$, then $\varphi_t(x_{ij}) \to \varphi(x_{ij})$, and so $[\varphi_t(x_{ij})] \to [\varphi(x_{ij})]$ in norm in M_n, and hence also weak*. Thus the range of θ is inside $w^*CB(X^*, M_n)$. On the other hand, if $u \in w^*CB(X^*, M_n)$, then u corresponds to a matrix $[\eta_{ij}] \in M_n(X^{**})$. If (φ_t) is a net in X^* converging weak* to $\varphi \in X^*$, then $u(\varphi_t) = [\eta_{ij}(\varphi_t)] \to u(\varphi) = [\eta_{ij}(\varphi)]$ in M_n, and so $\eta_{ij}(\varphi_t) \to \eta_{ij}(\varphi)$ for each i, j. Thus η_{ij} is weak* continuous and so $\eta_{ij} = i_X(x_{ij})$ for some $x_{ij} \in X$. Clearly $\theta([x_{ij}]) = u$. In other words, θ is an isometry from $M_n(X)$ onto $w^*CB(X^*, M_n)$:

$$M_n(X) \cong w^*CB(X^*, M_n) \subset CB(X^*, M_n). \tag{1.15}$$

Another consequence of 1.3.1, is that if X is an operator space which as a Banach space is reflexive, then $X \cong X^{**}$ completely isometrically.

1.3.3 (The adjoint map) The 'adjoint' or 'dual' u^* of a completely bounded map $u\colon X \to Y$ between operator spaces is completely bounded from Y^* to X^*, with $\|u^*\|_{\text{cb}} = \|u\|_{\text{cb}}$. Indeed if $[u_{ij}] \in M_n(CB(X,Y))$ then $u_{ij}^* : Y^* \to X^*$ and

$$\|[u_{ij}^*]\|_n = \sup\{\|[u_{ij}^*(\varphi_{kl})]\| : [\varphi_{kl}] \in \text{Ball}(M_m(Y^*)), m \in \mathbb{N}\}.$$

However

$$\begin{aligned}\|[u_{ij}^*(\varphi_{kl})]\| &= \sup\{\|[u_{ij}^*(\varphi_{kl})(x_{rs})]\| : [x_{rs}] \in \text{Ball}(M_p(X)), p \in \mathbb{N}\},\\ &= \sup\{\|[\varphi_{kl}(u_{ij}(x_{rs}))]\| : [x_{rs}] \in \text{Ball}(M_p(X)), p \in \mathbb{N}\},\end{aligned}$$

and it follows by combining the last two centered equations, and using (1.14), that

$$\begin{aligned}\|[u_{ij}^*]\|_n &= \sup\{\|[u_{ij}(x_{rs})]\| : [x_{rs}] \in \mathrm{Ball}(M_p(X)), p \in \mathbb{N}\} \\ &= \|[u_{ij}]\|_n.\end{aligned}$$

Thus $* : CB(X,Y) \to CB(Y^*, X^*)$ is a complete isometry.

Direct computations from the definitions also show that if u is a complete quotient map then u^* is a complete isometry (exercise). It is slightly harder to see that if u is completely isometric then u^* is a complete quotient map. This requires Wittstock's extension theorem, which we will prove later using elementary properties of the 'Haagerup tensor product'. The crux of Wittstock's result is that if $X \subset Y$ then any complete contraction $w : X \to M_n$ has a completely contractive extension $\hat{u} : Y \to M_n$. To see that u^* is a complete quotient map if $u : X \to Y$ is completely isometric, let $[\varphi_{ij}] \in \mathrm{Ball}(M_n(X^*))$. By (1.6) we may regard $[\varphi_{ij}]$ as a complete contraction $g : X \to M_n$. By Wittstock's extension theorem there exists a complete contraction $w : Y \to M_n$ with $w_{|u(X)} = g \circ u^{-1}$ on $u(X)$. By (1.6) we may regard w as a matrix $[\psi_{ij}] \in \mathrm{Ball}(M_n(Y^*))$. We claim that $[u^*(\psi_{ij})] = [\varphi_{ij}]$. Indeed, if $x \in X$ then

$$[u^*(\psi_{ij})(x)] = [\psi_{ij}(u(x))] = w(u(x)) = g(u^{-1}(u(x)) = g(x) = [\varphi_{ij}(x)].$$

Thus u^* is a complete quotient map. Conversely, if u^* is a complete quotient map then u^{**} is a complete isometry, so that u is a complete isometry (using 1.3.1). Thus u is a complete isometry if and only if u^{**} is a complete isometry.

1.3.4 (Duality of subspaces and quotients) The operator space versions of the usual Banach duality of subspaces and quotients follow easily from 1.3.3. If X is a subspace of Y, then we have $X^* \cong Y^*/X^\perp$ and $(Y/X)^* \cong X^\perp$. Indeed the dual of the inclusion map $i : X \hookrightarrow Y$ will be a complete quotient map $i^* : Y^* \to X^*$, which induces a complete isometry $X^* \cong Y^*/\mathrm{Ker}(i^*) = Y^*/X^\perp$. Similarly, the dual of the canonical quotient map $q : Y \to Y/X$ is the canonical complete isometry $q^* : (Y/X)^* \to Y^*$ which we know from the Banach space case has range $X^\perp$.

The predual versions go through too with the same proofs as in the Banach space case in a functional analysis course: if X is a w^*-closed subspace of a dual operator space Y, then $(Y_*/X_\perp)^* \cong (X_\perp)^\perp = X$ as dual operator spaces. Also, $(X_\perp)^* \cong Y/(X_\perp)^\perp = Y/X$ completely isometrically. These use the facts in the last paragraph, and the Banach space fact that $(X_\perp)^\perp = X$.

1.3.5 (Good and bad preduals) If X is an operator space which has a predual Banach space Z, then there is only one way to give Z an operator space structure with any hope that $Z^* \cong X$ completely isometrically. Namely, view $Z \subset X^*$ and give Z the operator space structure inherited from X^*. That is, define

$$\|[z_{ij}]\|_n = \sup\{\|[\langle x_{pq}, z_{ij}\rangle]\| : [x_{pq}] \in \mathrm{Ball}(M_m(X)), m \in \mathbb{N}\}, \tag{1.16}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between X and Z. Unfortunately, even then Z^* may fail to be completely isometric to X. We shall see an example of this later. Thus there may be 'good' and 'bad' Banach space preduals of an operator space X (the bad ones having no operator space structure whose operator space dual is X).

1.3.6 (The trace class operator space) If H is a Hilbert space then $B(H)$ is a dual Banach space, and like any von Neumann algebra its predual is unique (this is Sakai's theorem). By operator theory, the predual Banach space is the *trace class* $S^1(H)$. Fortunately, this (unique) predual Banach space $S^1(H)$ is 'good' in the sense of 1.3.5. More precisely, let us equip its predual $S^1(H)$ with the operator space structure it inherits from $B(H)^*$ via the canonical isometric inclusion $S^1(H) \hookrightarrow B(H)^*$. Then we claim that $B(H) \cong S^1(H)^*$ completely isometrically. Indeed the canonical map $\rho : B(H) \to S^1(H)^*$ is completely contractive by definition. Indeed if X, Z are as in 1.3.5, then by definition we equip Z with the the operator space structure making the canonical map $\theta : Z \to X^*$ a complete isometry. Then the map $X \to Z^*$ taking $x \in X$ to $\theta^*(\hat{x})$ is a complete contraction, and this is the canonical map from X into Z^*. To see that ρ is completely isometric, we use the second proof of 1.3.1, given in the Remark after that result. This shows that for any $n \in \mathbb{N}$, $\epsilon > 0$, and $[x_{kl}] \in M_n(B(H))$, we can find an integer m and a completely contractive $u\colon B(H) \to M_m$ such that $\|[u(x_{kl})]\| \geq \|[x_{kl}]\| - \epsilon$. We recall that u is the composition of maps π and T there. Since M_m and $B(K)$ are finite dimensional, π is w^*-continuous. If $x_t \to x$ weak* in $B(H)$, and if $\zeta, \eta \in K$ then

$$\langle T(x_t)\zeta, \eta\rangle = \langle x_t\zeta, \eta\rangle \to \langle x\zeta, \eta\rangle = \langle T(x)\zeta, \eta\rangle,$$

so that T is w^*-continuous. Hence u is w^*-continuous. By the argument in 1.3.2, u corresponds to a matrix $[z_{ij}] \in M_m(S^1(H))$, and by (1.16) the norm of this matrix equals the 'cb-norm' of u, which is ≤ 1. Finally, we have

$$\|[x_{kl}]\| - \epsilon \leq \|[u(x_{kl})]\| = \|[\langle x_{kl}, z_{ij}\rangle]\| \leq \|[\rho(x_{kl})]\|_n.$$

Since $\epsilon > 0$ was arbitrary, we have proved the desired reverse inequality. Thus ρ is a complete isometry.

Similarly, $B(K, H)$ is the dual operator space of the space $S^1(H, K)$ of trace class operators, the latter regarded as a subspace of $B(K, H)^*$. Henceforth, when we write $S^1(H, K)$ we will mean the operator space predual of $B(K, H)$ described above. Similarly, we will henceforth also view $S^1_n = M^*_n$ as an operator space.

Lemma 1.3.7. Any w^*-closed subspace X of $B(H)$ is a dual operator space. Indeed, if $Y = S^1(H)/X_\perp$ is equipped with its quotient operator space structure inherited from $S^1(H)$, then $X \cong Y^*$ completely isometrically.

Proof. This follows from 1.3.4 and 1.3.6. □

In particular this shows that any von Neumann algebra equipped with its 'natural' operator space structure (see 1.2.5) is a dual operator space. So they also have 'good' preduals.

The converse of 1.3.7 is true too, as we see next, so that 'dual operator spaces', and the w^*-closed subspaces of some $B(H)$, are essentially the same thing.

Lemma 1.3.8. Any dual operator space is completely isometrically isomorphic, via a homeomorphism for the w^*-topologies, to a w^*-closed subspace of $B(H)$, for some Hilbert space H.

Proof. Suppose that W is a dual operator space, with predual X. Let $Y = \mathbb{C}$, and recall from 1.2.18 the construction of a complete isometry

$$W \;=\; CB(X,Y) \;\longrightarrow\; \oplus^{\infty}_{x\in I}\, M_{n_x}(Y) \;=\; \oplus_{x\in I}\, M_{n_x},$$

namely the map J taking $w \in W$ to the tuple $([\langle w, x_{ij}\rangle])_x$ in $\oplus_x M_{n_x}$. Since the maps $w \mapsto \langle w, x_{ij}\rangle$ are w^*-continuous for any fixed $x \in I$, and since $\oplus^{\rm fin}_x S^1_{n_x}$ is dense in the Banach space predual $\oplus^1_x S^1_{n_x}$ of $\oplus_x M_{n_x}$, it is easy to see that J is w^*-continuous too. We recall a basic convergence principle from functional analysis: If D is a set whose span is dense in a normed space E, then a bounded net $\varphi_t \stackrel{w*}{\longrightarrow} \varphi$ in E^* if and only if $\varphi_t(x) \to \varphi(x)$ for all $x \in D$. Thus to show that J is w^*-continuous, it suffices to show that if $a \in I$ is fixed, and if $z \in S^1_n$ where $n = n_a$, and if $\epsilon_a : S^1_{n_a} \to \oplus^1_x S^1_{n_x}$ is the canonical inclusion map, and if $\varphi_t \to \varphi$ weak* in X^*, then $\langle J(\varphi_t), \epsilon_a(z)\rangle \to \langle J(\varphi), \epsilon_a(z)\rangle$. However

$$\langle J(\varphi_t), \epsilon_a(z)\rangle = \sum_{i,j}^{n} \varphi_t(a_{ij}) z_{ij} \to \sum_{i,j}^{n} \varphi(a_{ij}) z_{ij} = \langle J(\varphi), \epsilon_a(z)\rangle.$$

So J is w^*-continuous. We will use a consequence of the Krein-Smulian theorem, namely, that a linear w*-continuous isometry $u \colon E \to F$ between dual Banach spaces has w*-closed range, and u is a w*-w*-homeomorphism onto $\text{Ran}(u)$. Hence W is completely isometrically and w^*-homeomorphically isomorphic to a w^*-closed subspace of the W^*-algebra $\oplus_x M_{n_x}$. If the latter is regarded as a von Neumann subalgebra of $B(H)$ say, then W is completely isometrically and w^*-homeomorphically isomorphic to a w^*-closed subspace of $B(H)$. □

1.3.9 (W^*-continuous extensions) If X and Y are two operator spaces and if $u \colon X \to Y^*$ is completely bounded, then its (unique) w^*-continuous linear extension $\tilde{u} \colon X^{**} \to Y^*$ is completely bounded, with $\|\tilde{u}\|_{\rm cb} = \|u\|_{\rm cb}$. Indeed recall that this w^*-continuous extension is $\tilde{u} = i^*_Y \circ u^{**}$; and clearly

$$\|\tilde{u}\|_{\rm cb} = \|i^*_Y \circ u^{**}\|_{\rm cb} \le \|i^*_Y\|_{\rm cb}\, \|u^{**}\|_{\rm cb} = \|u\|_{\rm cb},$$

using the first paragraph in 1.3.3, whereas $\|\tilde{u}\|_{\rm cb} \ge \|u\|_{\rm cb}$ since $\tilde{u}$ extends u. Note that since $\tilde{u}$ is w^*-continuous, we have $\tilde{u}(X^{**}) \subset \overline{u(X)}^{w*}$. The above also shows that

$$CB(X, Y^*) = w^*CB(X^{**}, Y^*) \tag{1.17}$$

isometrically via the mapping $u \mapsto \tilde{u}$. Indeed note that if $g \in w^*CB(X^{**}, Y^*)$ then $g = \widetilde{g_{|X}}$, since both of these maps are w^*-continuous and they agree on the w^*-dense subset X.

By 1.3.6, the last paragraph applies in particular to $B(H)$ valued maps.

1.3.10 (The second dual) Let X be an operator space, and fix $n \in \mathbb{N}$. We wish to compare the spaces $M_n(X^{**})$ (equipped with its 'operator space dual' matrix norms as in 1.2.19), and $M_n(X)^{**}$. First note that they can be canonically identified as topological vector spaces, as may $M_n(X^*)$ and $M_n(X)^*$. Indeed note that $M_n(X)$

is a direct sum of n^2 copies of X, and so we can apply the principles in Exercise (3) at the end of this section. Applying these Banach space principles, we see that we have bicontinuous isomorphisms $M_n(X) \cong X \oplus^\infty \cdots \oplus^\infty X$ and $M_n(X^{**}) \cong X^{**} \oplus^\infty \cdots \oplus^\infty X^{**}$. Hence we have

$$M_n(X)^{**} \cong (X^* \oplus^1 \cdots \oplus^1 X^*)^* \cong X^{**} \oplus^\infty \cdots \oplus^\infty X^{**} \cong M_n(X^{**}).$$

If $\eta \in M_n(X)^{**}$, let $[\eta_{ij}]$ be the corresponding matrix in $M_n(X^{**})$, via the isomorphisms in the last centered equation. We will prove in 1.3.12 below that the map $\eta \to [\eta_{ij}]$ is an isometry. As a first easy step, let us check that it is a contraction. If $\eta \in \mathrm{Ball}(M_n(X)^{**})$, then by Goldstine's lemma in functional analysis, there is a net $(x^s)_s$ in $\mathrm{Ball}(M_n(X))$ such that $x^s \to \eta$ in the w^*-topology of $M_n(X)^{**}$. This means that $\varphi(x^s) \to \eta(\varphi)$ for any $\varphi \in M_n(X)^*$. Since $M_n(X)^* \cong M_n(X^*)$ and $M_n(X)^{**} \cong M_n(X^{**})$ bicontinuously, this is equivalent to

$$\sum_{i,j=1}^{n} \varphi_{i,j}(x^s_{i,j}) \to \sum_{i,j=1}^{n} \eta_{i,j}(\varphi_{i,j}), \qquad \varphi_{i,j} \in X^*,$$

which in turn is equivalent to $\varphi(x^s_{i,j}) \to \eta_{i,j}(\varphi)$ for all $i, j = 1, \cdots, n$ and $\varphi \in X^*$. Let $[\varphi_{pq}] \in \mathrm{Ball}(M_m(X^*))$, for some $m \geq 1$. We deduce that

$$\|[\langle \eta_{ij}, \varphi_{pq} \rangle]\| \;=\; \lim_s \|[\langle \varphi_{pq}, x^s_{ij} \rangle]\| \leq 1,$$

by (1.4) or (1.14). Thus $\|[\langle \eta_{ij}, \varphi_{pq} \rangle]\| \leq 1$. By (1.4) again, we deduce that $\|[\eta_{ij}]\|_{M_n(X^{**})} \leq 1$, which proves the result.

Note too that the map $\eta \to [\eta_{ij}]$ above restricts to the identity map on $M_n(X)$, by the last part of the aforementioned Exercise (3) at the end of the section.

1.3.11 (The second dual of a C^*-algebra) If A is a C^*-algebra, then there are at least three canonical norms one could put on $M_n(A^{**})$. Fortunately, they are all the same, as we now show. The first two are the ones discussed in 1.3.10. The third is the one from 1.2.5, arising from the fact that the second dual A^{**} of any C^*-algebra is a C^*-algebra, and hence has a canonical operator space structure. To see that these three are the same, we will need to state some notation. To avoid confusion, we state that whenever we write $M_n(A^{**})$ below, we are equipping this space with its 'operator space dual' matrix norms (see 1.2.19); thus $M_n(A^{**}) \cong CB(A^*, M_n)$ isometrically. Let $\pi_u \colon A \to B(H_u)$ denote the universal representation of A, and we write $A^{\dagger\dagger}$ for the von Neumann algebra $\pi_u(A)''$. The claim will follow if we can prove for any fixed $n \geq 1$ that

$$M_n(A)^{**} \cong M_n(A^{**}) \cong M_n(A^{\dagger\dagger}) \quad \text{isometrically} \tag{1.18}$$

via the canonical maps. The first of these maps is the contraction from $M_n(A)^{**}$ to $M_n(A^{**})$ discussed in 1.3.10. The second map in (1.18) is $(\widetilde{\pi_u})_n$, which is a contraction since according to 1.3.9, the mapping $\widetilde{\pi_u}$ is a complete contraction. To establish (1.18), we need only prove that the resulting contraction $\rho \colon M_n(A)^{**} \longrightarrow M_n(A^{\dagger\dagger})$ is isometric. It is clearly one-to-one. Of course $M_n(A)^{**}$ is also a C^*-algebra.

Claim: ρ is w^*-continuous. Regarding ρ as valued in $B(H_u^{(n)})$, we have $\langle \rho(\eta)\,\zeta, \xi\rangle = \sum_{i,j} \langle \widetilde{\pi_u}(\eta_{ij})\zeta_j, \xi_i\rangle$, for $\zeta = [\zeta_i], \xi = [\xi_i] \in H_u^{(n)}$, and $\eta = [\eta_{ij}]$ as in 1.3.10. If $\eta^s \to \eta$ weak* in $M_n(A)^{**}$ then the argument in 1.3.10 shows also that $\eta^s_{i,j} \to \eta_{i,j}$ weak* in A^{**}, and so $\langle \widetilde{\pi_u}(\eta^s_{ij})\zeta_j, \xi_i\rangle \to \langle \widetilde{\pi_u}(\eta_{ij})\zeta_j, \xi_i\rangle$. Hence $\langle \rho(\eta^s)\,\zeta, \xi\rangle \to \langle \rho(\eta)\,\zeta, \xi\rangle$, which implies that ρ is w^*-continuous. Thus ρ is the unique w^*-continuous extension of $(\pi_u)_n$ to $M_n(A)^{**}$, which is a $*$-homomorphism. Since it is one-to-one it is isometric.

The last result has many consequences. For example, we can use it to see that $S^\infty(H)^* = S^1(H)$ completely isometrically. Indeed, since $S^\infty(H)^{**} = B(H)$ completely isometrically, $S^\infty(H)^*$ must be the unique operator space predual $S^1(H)$ of $B(H)$ (see 1.3.6). Also we obtain:

Theorem 1.3.12. If X is an operator space then $M_n(X)^{**} \cong M_n(X^{**})$ isometrically for all $n \in \mathbb{N}$ (via an isomorphism extending the identity map on $M_n(X)$).

Proof. Choose a C^*-algebra A with $X \subset A$ completely isometrically. Then $X^{**} \subset A^{**}$ completely isometrically by 1.3.3, hence we have both $M_n(X)^{**} \subset M_n(A)^{**}$, and $M_n(X^{**}) \subset M_n(A^{**})$, isometrically. Under the identifications between $M_n(A)^{**}$ and $M_n(A^{**})$ and between $M_n(X)^{**}$ and $M_n(X^{**})$ discussed above, these two embeddings are easily seen to be the same. That is, the diagram below commutes:

$$\begin{array}{ccc} M_n(A)^{**} & \longrightarrow & M_n(A^{**}) \\ \uparrow & & \uparrow \\ M_n(X)^{**} & \longrightarrow & M_n(X^{**}) \end{array}$$

Hence the isometry $M_n(A^{**}) = M_n(A)^{**}$ provided by 1.3.11, implies that we also have $M_n(X^{**}) = M_n(X)^{**}$ isometrically. $\square$

1.3.13 (Duality of Min and Max) We will prove later in 3.1.10 that for any Banach space E, we have

$$\mathrm{Min}(E)^* = \mathrm{Max}(E^*) \qquad \text{and} \qquad \mathrm{Max}(E)^* = \mathrm{Min}(E^*).$$

1.3.14 (The 1-direct sum) For a family $\{X_\lambda : \lambda \in I\}$ of operator spaces, we give $\oplus^1_\lambda X_\lambda$ its canonical 'predual operator space structure' (see 1.3.5), as the predual of the operator space $\oplus^\infty_\lambda X^*_\lambda$. It is easy to argue directly from the definitions that the canonical inclusion and projection maps ϵ_λ and π_λ between $\oplus^1_\lambda X_\lambda$ and its 'λth summand' are complete isometries and complete quotient maps respectively. Or, to see that $\epsilon_\lambda : X_\lambda \to \oplus^1_\lambda X_\lambda$ is a complete isometry, consider the following sequence of canonical maps:

$$X_\lambda \longrightarrow \oplus^1_\lambda X_\lambda \subset (\oplus^\infty_\lambda X^*_\lambda)^*,$$

and let u be the composition of all these maps. On the other hand, the dual of the canonical projection map $\oplus^\infty_\lambda X^*_\lambda \to X^*_\lambda$ (which is a complete quotient map), is a complete isometry $j : X^{**}_\lambda \to (\oplus^\infty_\lambda X^*_\lambda)^*$. Moreover, the range of u falls within $j(X^{**}_\lambda)$, and $j^{-1} \circ u$ is the complete isometry from X_λ into its second dual. This implies that the first of these maps in the sequence, ϵ_λ, is a complete isometry.

Next we observe that $\oplus^1_\lambda X_\lambda$ is a 'good predual' of $\oplus^\infty_\lambda X^*_\lambda$, in the sense of 1.3.5. That is,

$$(\oplus^1_\lambda X_\lambda)^* \cong \oplus^\infty_\lambda X^*_\lambda \quad \text{as dual operator spaces.} \tag{1.19}$$

Indeed, it is easy to see (if necessary by an argument early in 1.3.6), that the canonical map $\theta : \oplus^\infty_\lambda X^*_\lambda \to (\oplus^1_\lambda X_\lambda)^*$ is a complete contraction. On the other hand, an element in $\mathrm{Ball}(M_n((\oplus^1_\lambda X_\lambda)^*))$ may be regarded as a complete contraction from $\oplus^1_\lambda X_\lambda$ into M_n. Composing this map with each ϵ_λ, we get a tuple in the ball of $\oplus^\infty_\lambda CB(X_\lambda, M_n)$. Since $CB(X_\lambda, M_n) \cong M_n(X^*_\lambda)$, we actually obtain an element in the ball of $M_n(\oplus^\infty_\lambda X^*_\lambda) \cong \oplus^\infty_\lambda M_n(X^*_\lambda)$. It is easy to see from all this that θ is a complete isometry.

Corollary 1.3.15. Any operator space X is a complete quotient of a 1-sum of spaces of the form $S^1_n = M^*_n$.

Proof. The map J in the proof of 1.3.8 is a weak* continuous complete isometry. Thus by Exercise (5) below, $J = q^*$ for a complete quotient map q from a 1-sum of spaces of the form $S^1_n = M^*_n$, onto X. ☐

1.3.16 We end this section with an example of an operator space which is a dual Banach space, but has no 'good' operator space predual in the sense of 1.3.5. Let $\mathbb{B} = B(H)$ with its canonical matrix norms, and let $\mathbb{K}$ be the compact operators on H. Then $Q = \mathbb{B}/\mathbb{K}$ is the well known *Calkin algebra*, which is a C^*-algebra and hence has a canonical operator space structure. The only fact we will need about the Calkin algebra is that it is not commutative. Let $X = B(H)$ but with matrix norms

$$|||[x_{ij}]|||_n = \max\{\|[x_{ij}]\|_{M_n(\mathbb{B})}, \|[q(x_{ji})]\|_{M_n(Q)}\}, \qquad [x_{ij}] \in M_n(X),$$

where $q : \mathbb{B} \to Q$ is the canonical quotient map. One can easily check that X is an operator space, for example by appealing to Ruan's theorem. As a Banach space X is just $\mathbb{B}$, since q is a contraction (so that $|||x|||_1 = \|x\|_{\mathbb{B}}$). Thus X has a unique Banach space predual $S^1(H)$, the trace class. We will show that this is a 'bad predual'.

Notice that $|||\cdot|||_n$ restricted to the copy of $M_n(\mathbb{K})$ is just the usual norm, since q annihilates $\mathbb{K}$. Thus if $Y = S^1(H)$ with its canonical 'predual matrix norms' from 1.3.5, that is, the matrix norms coming from its duality with $(X, \{|||\cdot|||_n\})$, then for $[y_{ij}] \in M_n(Y)$ we have

$$\begin{aligned} \|[y_{ij}]\|_{M_n(Y)} &= \sup\{\|[< y_{ij}, x_{kl} >]\| : [x_{kl}] \in \mathrm{Ball}(M_m(X)), m \in \mathbb{N}\} \\ &\geq \sup\{\|[< y_{ij}, x_{kl} >]\| : [x_{kl}] \in \mathrm{Ball}(M_m(\mathbb{K})), m \in \mathbb{N}\} \\ &= \|[y_{ij}]\|_{M_n(\mathbb{K}^*)} = \|[y_{i,j}]\|_{M_n(S^1(H))}, \end{aligned}$$

where the last norm is the usual operator space structure of $S^1(H)$ (see 1.3.6). Thus, if $[x_{ij}] \in M_n(X)$, then

$$\begin{aligned} \|[x_{ij}]\|_{M_n(Y^*)} &= \sup\{\|[< x_{ij}, y_{kl} >]\| : [y_{kl}] \in \mathrm{Ball}(M_m(Y)), m \in \mathbb{N}\} \\ &\leq \sup\{\|[< x_{ij}, y_{kl} >]\| : [y_{kl}] \in \mathrm{Ball}(M_m(S^1(H))), m \in \mathbb{N}\} \\ &= \|[x_{ij}]\|_{M_n(\mathbb{B})}, \end{aligned}$$

by the fact in 1.3.6 that $S^1(H)^* = \mathbb{B}$ completely isometrically. Hence if $Y^* = X$ completely isometrically, then $|||[x_{ij}]|||_n \leq \|[x_{ij}]\|_{M_n(\mathbb{B})}$. The reverse inequality follows from the definition of $|||[x_{ij}]|||_n$, and so $|||[x_{ij}]|||_n = \|[x_{ij}]\|_{M_n(\mathbb{B})}$. Therefore, $\|[q(x_{ji})]\|_{M_n(\mathbb{Q})} \leq \|[x_{ij}]\|_{M_n(\mathbb{B})}$ for all $[x_{ij}] \in M_n(X)$. If $[z_{ij}] \in M_n(\mathbb{K})$, then $\|[q(x_{ji})]\|_{M_n(\mathbb{Q})} = \|[q(x_{ji} + z_{ji})]\|_{M_n(\mathbb{Q})} \leq \|[x_{ij} + z_{ij}]\|_{M_n(\mathbb{B})}$. Taking the infimum over such $z_{ij} \in \mathbb{K}$, we get $\|[q(x_{ji})]\|_{M_n(\mathbb{Q})} \leq \|[q(x_{ij})]\|_{M_n(\mathbb{Q})}$. Symmetry implies that this inequality is in fact an equality. But we claim that the only unital C*-algebras A with $\|[a_{ji}]\|_n = \|[a_{ij}]\|_n$, for all $n \in \mathbb{N}$ and $[a_{ij}] \in M_n(A)$, are the commutative ones, and the Calkin algebra is not commutative! Indeed if A is any unital C*-algebra, and if A° is A with the reversed multiplication, then A° is a unital C*-algebra, and its canonical matrix norm are given by $\|[a_{ij}]\|_{M_n(A^\circ)} = \|[a_{ji}]\|_{M_n(A)}$ (we leave this as an exercise). Thus if the identity in the claim holds, then the identity map $A \to A^\circ$ is a complete isometry. By Corollary 2.1.7, it is a homomorphism, so that A is commutative.

Historical note: The results in Section 2.2 are due to Blecher (see [2], which was written close to the date of [8, 15], although it appeared much later), with the following main exceptions. The fact that $X \subset X^{**}$ completely isometrically was independently noticed in [8, 15]. Effros and Ruan had noticed 1.3.8 via a different route [14]. Item 1.3.16 is a simplification by Blecher and Magajna [5] of examples of Effros-Ozawa-Ruan, and Peters-Wittstock. Le Merdy was the first to find an example of a 'bad predual' in the sense of 1.3.5.

Exercises.

(1) As in the Exercise 1 at the end of Chapter 1, show that the map $T \mapsto T^*$ is a complete isometry from $CB(X, Y)$ into $CB(Y^*, X^*)$, and show that this map is onto if X and Y are reflexive.
(2) Show that if u is a complete quotient map then u^* is a complete isometry.
(3) Show that if F is any Banach space, and if $E = F \oplus \cdots \oplus F$ is a finite direct sum of n copies of F, equipped with any norm such that the n canonical inclusions of F into E are isometries, then $E \cong F \oplus^1 \cdots \oplus^1 F \cong F \oplus^\infty \cdots \oplus^\infty F$ bicontinuously. Also, if $Z = F^* \oplus \cdots \oplus F^*$, equipped with any norm such that the n canonical inclusions of F^* into Z are isometries, show that $Z \cong E^*$ bicontinuously, via the map $\theta(\varphi_1, \cdots, \varphi_n)(x_1, \cdots, x_n) = \sum_{k=1}^n \varphi_k(x_k)$, for $\varphi_k \in F^*, x_k \in F$. Show that a similar statement holds for $W = F^{**} \oplus \cdots \oplus F^{**}$ equipped with any norm such that the n canonical inclusions of F^{**} into W are isometries. Moreover, show that the resulting isomorphism $E^{**} \cong W$ 'restricts' on E to the 'identity map'.
(4) Show that the 1-sum has the following universal property: If Z is an operator space and if $u_\lambda \colon X_\lambda \to Z$ are completely contractive linear maps, then there is a canonical complete contraction $u \colon \oplus^1_\lambda X_\lambda \to Z$ such that $u \circ \epsilon_\lambda = u_\lambda$.
(5) Show that if $u : X^* \to Y^*$ is a weak* continuous isometry (resp. complete isometry), where X and Y are Banach spaces (resp. complete operator spaces), then $u = q^*$ for a 1-quotient map (resp. complete quotient map) $q : Y \to X$. [Hint: By Krein-Smulian, u is a weak* homeomorphism and its range N is

weak* closed. Thus we can assume $X^* = N, X = Y/N_\perp$, and u is the inclusion $N \to Y^*$, in which case we can take $q : Y \to Y/N_\perp$ to be the canonical quotient map.]

(6) If A is any unital C*-algebra, and if A° is A with the reversed multiplication, show that A is a unital C*-algebra, and its canonical matrix norms are given by $\|[a_{ij}]\|_{M_n(A^\circ))} = \|[a_{ji}]\|_{M_n(A))}$.

Chapter 2

Addendum to Lecture 2/3 (Operator spaces)

2.1 Operator systems

2.1.1 (Unital operator spaces) A *unital operator space* is a subspace $\mathcal{S}$ of a unital C^*-algebra A, which contains the identity of A. There are important in studying noncommutative function spaces (we recall in the study of classical abstract function spaces (spaces of functions on a topological space) one often assumes the space 'contains constants'. We will not say much about these spaces–just giving their abstract characterization (without proof) due to Blecher and Neal:

Theorem 2.1.2. If X is an operator space and $u \in X$ with $\|u\| = 1$ then (X, u) is a unital operator space iff

$$\max\{\|u_n + i^k x\| : k = 0, 1, 2, 3\} \geq \sqrt{1 + \|x\|}, \qquad n \in \mathbb{N}, x \in M_n(X).$$

This is also equivalent to

$$\|[\ u_n \quad x\ ]\| = \left\| \begin{bmatrix} u_n \\ x \end{bmatrix} \right\|, \qquad n \in \mathbb{N}, x \in M_n(X), \|x\| = 1.$$

Here u_n is the diagonal matrix $u \otimes I_n$ in $M_n(X)$ with u in each diagonal entry. Indeed in the first result one only needs x of 'small norm', where 'small' can differ for each n.

2.1.3 (Operator systems) An *operator system* is a subspace $\mathcal{S}$ of a unital C^*-algebra A, which contains the identity of A, and which is *selfadjoint*, that is, $x^* \in \mathcal{S}$ if and only if $x \in \mathcal{S}$. There is an abstract characterization of these due to Choi and Effros. A *subsystem* of an operator system $\mathcal{S}$ is a selfadjoint linear subspace of $\mathcal{S}$ containing the 'identity' 1 of $\mathcal{S}$. If $\mathcal{S}$ is an operator system, a subsystem of a C^*-algebra A, then $\mathcal{S}$ has a distinguished 'positive cone' $\mathcal{S}_+ = \{x \in \mathcal{S} : x \geq 0 \text{ in } A\}$. We also write $\mathcal{S}_{sa}$ for the real vector space of selfadjoint elements x (i.e. those satisfying $x = x^*$) in $\mathcal{S}$. Then $\mathcal{S}$ has an associated ordering $\leq$, namely we say that $x \leq y$ if x, y are selfadjoint and $y - x \in \mathcal{S}_+$. Note that if $x \in \mathcal{S}$ then $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2i}$ are selfadjoint, and so any $x \in \mathcal{S}$ is of the form $x = h + ik$ for $h, k \in \mathcal{S}_{sa}$. Also, if $h \in \mathcal{S}_{sa}$ then $\|h\|1 + h$ and $\|h\|1 - h$ are positive. Thus $\mathcal{S}_{sa} = \mathcal{S}_+ - \mathcal{S}_+$.

A linear map $u : \mathcal{S} \to \mathcal{S}'$ between operator systems is called $*$-*linear* if $u(x^*) = u(x)^*$ for all $x \in \mathcal{S}$. Some authors say that such a map is *selfadjoint*. We say that u is *positive* if $u(\mathcal{S}_+) \subset \mathcal{S}'_+$. By facts at the end of the last paragraph, any $x \in \mathcal{S}$ may be written as $x = x_1 - x_2 + i(x_3 - x_4)$, and from this it is easy to see that a positive map $u : \mathcal{S} \to \mathcal{S}'$ is $*$-linear. Indeed,

$$u(x^*) = u(x_1 - x_2 - i(x_3 - x_4)) = u(x_1) - u(x_2) - i(u(x_3) - u(x_4)),$$

whereas

$$u(x)^* = (u(x_1) - u(x_2) + i(u(x_3) - u(x_4)))^* = u(x_1) - u(x_2) - i(u(x_3) - u(x_4)).$$

The operator system $M_n(\mathcal{S})$, which is a subsystem of $M_n(A)$, has a 'positive cone' too, and thus it makes sense to talk about *completely positive maps* between operator systems. These are the maps u such that $u_n = I_{M_n} \otimes u \colon M_n(\mathcal{S}) \to M_n(\mathcal{S}')$ is positive for all $n \in \mathbb{N}$. Indeed the morphisms in the category of operator systems are often taken to be the unital completely positive maps. Any $*$-homomorphism π between C^*-algebras is clearly positive, and applying this fact to π_n shows that π is completely positive. Completely positive maps are discussed in very many places in the literature (see e.g. [10, 23]), and we shall be brief here.

Suppose that $\mathcal{S}$ is a subsystem of a unital C^*-algebra. By the Hahn–Banach theorem, the set of states of $\mathcal{S}$ (that is, the set of $\varphi \in \mathcal{S}^*$ with $\varphi(1) = \|\varphi\| = 1$) is just the set of restrictions of states on the containing C^*-algebra to $\mathcal{S}$. Using this fact, we have that $\mathcal{S}_{sa}$ (resp. $\mathcal{S}_+$) is exactly the set of elements $x \in \mathcal{S}$ such that $\varphi(x) \in \mathbb{R}$ (resp. $\varphi(x) \geq 0$) for all states φ of $\mathcal{S}$ (by the C^*-algebra case of these results). From this it is clear that if $u \colon \mathcal{S}_1 \to \mathcal{S}_2$ is a contractive unital linear map between operator systems, then u is a positive map (for if $x \in \mathcal{S}_{1+}$, and if φ is a state on $\mathcal{S}_2$ then $\varphi \circ u$ is a state of $\mathcal{S}_1$, so that $\varphi(u(x)) \geq 0$; and so $u(x) \geq 0$). Applying this principle to u_n, we see that a completely contractive unital linear map between operator systems is completely positive.

Clearly an isomorphism between operator systems which is unital and completely positive, and has completely positive inverse, preserves all the 'order'. Such a map is called a *complete order isomorphism*. The range of a completely positive unital map between operator systems is clearly also an operator system; we say that such a map is a *complete order injection* if it is a complete order isomorphism onto its range.

The following simple fact relates the norm to the matrix order, and is an elementary exercise using the definition of a positive operator. Namely, if x is an element of a unital C^*-algebra or operator system A, or if $x \in B(K, H)$, then

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \|x\| \leq 1. \tag{2.1}$$

Here '≥ 0' means 'positive in $M_2(A)$' (or 'positive in $B(H \oplus K)$').

2.1.4 It is easy to see from (2.1) that a completely positive unital map u between operator systems is completely contractive. (For example, to see that u is contractive, take $\|x\| \leq 1$, and apply u_2 to the associated positive matrix in (2.1). This is positive, so that using (2.1) again we see that $\|u(x)\| \leq 1$.) Putting this together with some facts from 2.1.3 we see that a unital map between operator systems is completely positive if and only if it is completely contractive; and in this case the map is $*$-linear. If, further, u is one-to-one, then by applying the above to u and u^{-1} one sees immediately that a unital map between operator systems is a complete order injection if and only if it is a complete isometry.

Theorem 2.1.5. (Stinespring) Let A be a unital C^*-algebra. A linear map $u\colon A \to B(H)$ is completely positive if and only if there is a Hilbert space K, a unital $*$-homomorphism $\pi\colon A \to B(K)$, and a bounded linear $V\colon H \to K$ such that $u(a) = V^*\pi(a)V$ for all $a \in A$. This can be accomplished with $\|u\|_{\mathrm{cb}} = \|V\|^2$. Also, this equals $\|u\|$. If u is unital then we may take V to be an isometry; in this case we may view $H \subset K$, and we have $u(\cdot) = P_H\pi(\cdot)_{|H}$.

Proof. The usual proof of this may be found in many places (e.g. [1, 10, 23]), and it is very similar to the proof of the 'GNS construction' from C^*-algebra theory. Thus we just give a sketch. Given a completely positive u, the idea to construct π, as in the GNS construction proof, is to find an inner product defined on a simple space containing H on which A has a natural algebraic representation. In this case, the space is $A \otimes H$, and we define the representation of A by $\pi(a)(b \otimes \zeta) = ab \otimes \zeta$ for $a, b \in A, \zeta \in H$. We define the inner product on $A \otimes H$ by

$$\langle a \otimes \eta, b \otimes \zeta\rangle = \langle T(b^*a)\eta, \zeta\rangle\,, \qquad a, b \in A, \eta, \zeta \in H.$$

The rest can be left as an exercise, following the model of the GNS construction. □

Proposition 2.1.6. (A Kadison–Schwarz inequality) If $u\colon A \to B$ is a unital completely positive (or equivalently unital completely contractive) linear map between unital C^*-algebras, then $u(a)^*u(a) \leq u(a^*a)$, for all $a \in A$.

Proof. By 2.1.5 we have $u = V^*\pi(\cdot)V$, with $\|V\| \leq 1$ and π a $*$-homomorphism. Thus $u(a)^*u(a) = V^*\pi(a)^*VV^*\pi(a)V \leq V^*\pi(a)^*\pi(a)V = u(a^*a)$. □

Corollary 2.1.7. Let $u\colon A \to B$ be a completely isometric unital surjection between unital C^*-algebras. Then u is a $*$-isomorphism.

Proof. By 2.1.6 applied to both u and u^{-1} we have $u(x)^*u(x) \leq u(x^*x)$, and $u^{-1}(u(x)^*u(x)) \geq u^{-1}(u(x))^*u^{-1}(u(x)) = x^*x$, for all $x \in A$. Applying u to the last inequality gives $u(x)^*u(x) \geq u(x^*x)$. Hence $u(x)^*u(x) = u(x^*x)$. Now use the polarization identity ($\Phi(x, y) = \sum_{k=0}^{3} i^k\Phi(x, x)$ for any sesquiliear map $\Phi(x, y)$), to conclude that $u(x)^*u(y) = u(x^*y)$ for $x, y \in A$. Setting $y = 1$ gives $u(x)^* = u(x^*)$, and so $u(x^*y) = u(x^*)u(y)$. So u is a $*$-isomorphism. □

Proposition 2.1.8. Let $u\colon A \to B$ be as in 2.1.6. Suppose that $c \in A$, and that c satisfies $u(c)^*u(c) = u(c^*c)$. Then $u(ac) = u(a)u(c)$ for all $a \in A$.

Proof. Suppose that $B \subset B(H)$. We write $u = V^*\pi(\cdot)V$ as in Stinespring's theorem, with $V^*V = I_H$. Let $P = VV^*$ be the projection onto $V(H)$. By hypothesis $V^*\pi(c)^*P\pi(c)V = V^*\pi(c)^*\pi(c)V$. For $\zeta \in H$, set $\eta = \pi(c)V\zeta$. Then $\|P\eta\|^2 = \langle V^*\pi(c)^*P\pi(c)V\zeta, \zeta\rangle = \|\eta\|^2$. Thus $P\eta = \eta$, and $VV^*\pi(c)V = \pi(c)V$. Therefore $u(a)u(c) = V^*\pi(a)VV^*\pi(c)V = V^*\pi(a)\pi(c)V = u(ac)$. □

2.1.9 (Completely positive bimodule maps) An immediate consequence of 2.1.8: Suppose that $u\colon A \to B$ is as in 2.1.6, and that there is a C^*-subalgebra C of A with $1_A \in C$, such that $\pi = u_{|C}$ is a $*$-homomorphism. Then

$$u(ac) = u(a)\pi(c) \quad \text{and} \quad u(ca) = \pi(c)u(a) \qquad (a \in A, c \in C).$$

We recall that a map Φ is *idempotent* if $\Phi \circ \Phi = \Phi$.

Theorem 2.1.10. (Choi and Effros) Suppose that A is a unital C^*-algebra, and that $\Phi\colon A \to A$ is a unital, completely positive (or equivalently by 2.1.4, completely contractive), idempotent map. Then we may conclude:

(1) $R = \mathrm{Ran}(\Phi)$ is a C^*-algebra with respect to the original norm, involution, and vector space structure, but new product $r_1 \circ_\Phi r_2 = \Phi(r_1 r_2)$.
(2) $\Phi(ar) = \Phi(\Phi(a)r)$ and $\Phi(ra) = \Phi(r\Phi(a))$, for $r \in R$ and $a \in A$.
(3) If B is the C^*-subalgebra of A generated by the *set* R, and if R is given the product $\circ_\Phi$, then $\Phi_{|B}$ is a $*$-homomorphism from B onto R.

Proof. (2) By linearity and the fact that a positive map is $*$-linear (see 2.1.3), we may assume that a, r are selfadjoint. Set

$$d = d^* = \begin{bmatrix} 0 & r \\ r^* & a \end{bmatrix}.$$

Then $\Phi_2(d^2) \geq (\Phi_2(d))^2$ by the Kadison–Schwarz inequality 2.1.6, so that

$$\begin{bmatrix} \Phi(r^2) & \Phi(ra) \\ \Phi(ar) & * \end{bmatrix} \geq \begin{bmatrix} r^2 & r\Phi(a) \\ \Phi(a)r & * \end{bmatrix}.$$

Here $*$ is used for a term we do not care about. Applying Φ_2 gives

$$\begin{bmatrix} \Phi(r^2) & \Phi(ra) \\ \Phi(ar) & * \end{bmatrix} \geq \begin{bmatrix} \Phi(r^2) & \Phi(r\Phi(a)) \\ \Phi(\Phi(a)r) & * \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & \Phi(ra) - \Phi(r\Phi(a)) \\ \Phi(ar) - \Phi(\Phi(a)r) & * \end{bmatrix} \geq 0,$$

which implies that $\Phi(ra) - \Phi(r\Phi(a)) = 0$ and $\Phi(ar) - \Phi(\Phi(a)r) = 0$.

(1) By (2) we have for $r_1, r_2, r_3 \in R$ that

$$(r_1 \circ_\Phi r_2) \circ_\Phi r_3 = \Phi(\Phi(r_1 r_2) r_3) = \Phi(r_1 r_2 r_3).$$

Similarly, $r_1 \circ_\Phi (r_2 \circ_\Phi r_3) = \Phi(r_1 r_2 r_3)$, which shows that the multiplication is associative. It is easy to check that R (with original norm, involution, and vector space structure, but new multiplication) satisfies the conditions necessary to be a C^*-algebra. For example:

$$(r_1 \circ_\Phi r_2)^* \;=\; \Phi(r_1 r_2)^* \;=\; \Phi(r_2^* r_1^*) \;=\; r_2^* \circ_\Phi r_1^*.$$

We check the C^*-identity using the Kadison–Schwarz inequality 2.1.6:

$$\|r^* \circ_\Phi r\| \;=\; \|\Phi(r^* r)\| \;\geq\; \|\Phi(r)^*\Phi(r)\| = \|r^* r\| = \|r\|^2,$$

and conversely,

$$\|r\|^2 \;=\; \|r^* r\| \;\geq\; \|\Phi(r^* r)\| \;=\; \|r^* \circ_\Phi r\|.$$

(3) This will follow if we can prove that $\Phi(r_1 r_2 \cdots r_n) = r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_n$, for $r_i \in R$. This follows in turn by induction on n. Supposing that it is true for $n = k$, we see that $r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_{k+1}$ equals

$$\Phi((r_1 \circ_\Phi r_2 \cdots \circ_\Phi r_k) r_{k+1}) = \Phi(\Phi(r_1 r_2 \cdots r_k) r_{k+1}) = \Phi(r_1 r_2 \cdots r_k r_{k+1}),$$

using (2) in the last equality. □

2.1.11 (The Paulsen system) If X is a subspace of $B(H)$, we define the *Paulsen system* to be the operator system

$$\mathcal{S}(X) \;=\; \begin{bmatrix} \mathbb{C}I_H & X \\ X^\star & \mathbb{C}I_H \end{bmatrix} \;=\; \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \;:\; x, y \in X,\; \lambda,\, \mu \in \mathbb{C} \right\}$$

in $M_2(B(H))$, where the entries λ and μ in the last matrix stand for λI_H and μI_H respectively. The following important lemma shows that as an operator system (i.e. up to complete order isomorphism) $\mathcal{S}(X)$ only depends on the operator space structure of X, and not on its representation on H.

Lemma 2.1.12. (Paulsen) Suppose that for $i = 1, 2$, we are given Hilbert spaces H_i, K_i, and linear subspaces $X_i \subset B(K_i, H_i)$. Suppose that $u\colon X_1 \to X_2$ is a linear map. Let $\mathcal{S}_i$ be the following operator system inside $B(H_i \oplus K_i)$:

$$\mathcal{S}_i = \begin{bmatrix} \mathbb{C}I_{H_i} & X_i \\ X_i^\star & \mathbb{C}I_{K_i} \end{bmatrix}.$$

If u is contractive (resp. completely contractive, completely isometric), then

$$\Theta : \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \;\mapsto\; \begin{bmatrix} \lambda & u(x) \\ u(y)^* & \mu \end{bmatrix}$$

is positive (resp. completely positive and completely contractive, a complete order injection) as a map from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Proof. Suppose that z is a positive element of $\mathcal{S}_1$. Thus

$$z \;=\; \begin{bmatrix} a & x \\ x^* & b \end{bmatrix}$$

where a and b are positive. Since $z \geq 0$ if and only if $z + \epsilon 1 \geq 0$ for all $\epsilon > 0$, we may assume that a and b are invertible. Then

$$\begin{bmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & b^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} a & x \\ x^* & b \end{bmatrix} \begin{bmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & b^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 1 & a^{-\frac{1}{2}} x b^{-\frac{1}{2}} \\ b^{-\frac{1}{2}} x^* a^{-\frac{1}{2}} & 1 \end{bmatrix} \geq 0.$$

Hence by (2.1), we have that $\|a^{-\frac{1}{2}} x b^{-\frac{1}{2}}\| \leq 1$. Applying u we obtain that $\|a^{-\frac{1}{2}} u(x) b^{-\frac{1}{2}}\| \leq 1$. Reversing the argument above now shows that $\Theta(z) \geq 0$. So Θ is positive, and a similar argument shows that it is completely positive if u is completely contractive. By 2.1.4 we have that Θ is completely contractive in that case. If in addition u is a complete isometry, then applying the above to u and u^{-1} we obtain the final assertion. □

Historical notes: This section is a slight variant of [4, Section 1.3]; historical attributions are given there.

Chapter 3

Lecture 3

3.1 Operator space tensor products

Recall that a map $T : E \times F \to Z$ between vector spaces is called *bilinear* if $T(x, \cdot)$ is linear for each fixed $x \in E$, and $T(\cdot, y)$ is linear for each fixed $y \in F$. If E and F are vector spaces, we recall that the (algebraic) tensor product is a pair $(E \otimes F, \otimes)$ consisting of a vector space $E \otimes F$, and a bilinear map $\otimes : E \times F \to E \otimes F$ (where we write $x \otimes y$ for $\otimes$ applied to the pair $(x, y) \in E \times F$), with the universal property in the following result:

Proposition 3.1.1. If E and F are vector spaces then there exists a vector space $E \otimes F$, and a bilinear map $\otimes : E \times F \to E \otimes F$ whose range spans $E \otimes F$, with the following property:

> For every vector space Z, and every bilinear $T : E \times F \to Z$, there exists a linear map $\tilde{T} : E \otimes F \to Z$ such that $\tilde{T}(x \otimes y) = T(x, y)$ for all $x \in E, y \in F$.

Moreover, this vector space is essentially unique; that is, if V is another vector space, and $\psi : E \times F \to V$ is a bilinear map, with the above property, then there is a vector space isomorphism $\theta : E \otimes F \to V$ such that $\theta(x \otimes y) = \psi(x, y)$ for all $x \in E, y \in F$.

Proof. Existence: There are several ways to show that there exists a space with this property. We assume that the reader has seen one such method in an algebra course.

Uniqueness: If (V, ψ) is another pair with the above property, then since $E \otimes F$ has the above property there exists a linear map $\tilde{\psi} : E \otimes F \to V$ such that $\tilde{\psi}(x \otimes y) = \psi(x, y)$ for all $x \in E, y \in F$. Similarly, since V has the above property, applying the property to the bilinear map $T = \otimes$, there exists a linear map $\tilde{T} : V \to E \otimes F$ such that $\tilde{T}(\psi(x, y)) = x \otimes y$ for all $x \in E, y \in F$. It follows that

$$\tilde{\psi}(\tilde{T}(\psi(x, y))) = \tilde{\psi}(x \otimes y) = \psi(x, y).$$

That is $\tilde{\psi} \circ \tilde{T}$ =Id on the range of ψ. Since the range of ψ spans V, and since $\tilde{\psi} \circ \tilde{T}$ is linear, we deduce that $\tilde{\psi} \circ \tilde{T}$ =Id on V. A similar argument shows that $\tilde{T} \circ \tilde{\psi} =$

Id on $E \otimes F$. So $\tilde{\psi}$ is an isomorphism from $E \otimes F \to V$, and we already saw that $\tilde{\psi}(x \otimes y) = \psi(x, y)$ for all $x \in E, y \in F$. □

Thus the algebraic tensor product 'linearizes bilinear maps'.

A special case of this of interest is if $u_i \colon E_i \to F_i$ are linear maps for $i = 1, 2$. The map $E_1 \times E_2 \to F_1 \otimes F_2$ defined by $(x_1, x_2) \mapsto u_1(x_1) \otimes u_2(x_2)$, is bilinear. Linearizing this bilinear map by 3.1.1, we obtain a linear map $E_1 \otimes E_2 \to F_1 \otimes F_2$. This map is written as $u_1 \otimes u_2$ and has the defining property that

$$(u_1 \otimes u_2)(x_1 \otimes x_2) = u_1(x_1) \otimes u_2(x_2), \qquad x_1 \in E_1, x_2 \in E_2.$$

3.1.2 (The injective tensor product) Suppose that E, F are normed vector spaces. If $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ are finite families in E and F respectively, then one may define for $z = \sum_{k=1}^n x_k \otimes y_k$ in the algebraic tensor product $E \otimes F$, the quantity

$$\Big\| \sum_k x_k \otimes y_k \Big\|_\vee \;=\; \sup\Big\{ \Big| \sum_k \varphi(x_k)\, \psi(y_k) \Big| : \varphi \in \mathrm{Ball}(E^*),\; \psi \in \mathrm{Ball}(F^*) \Big\}.$$

This is a norm on $E \otimes F$. To see this notice that it is fairly obviously a seminorm (exercise). To see that this is a norm, we rewrite z. Choose an Auerbach basis $(w_k)_{k=1}^m$ for the space $W = \mathrm{Span}(\{x_k : k = 1, \cdots, n\})$ (look this up on Wiki if you havent seen this before). Thus we have linear functionals $\varphi_j \in \mathrm{Ball}(W^*)$ with $\varphi_j(w_i) = \delta_{ij}$. By the Hahn-Banach theorem these extend to continuous $\widetilde{\varphi_j} \in E^*$. We can rewrite each x_k in terms of this basis, and this allows us to write $z = \sum_{k=1}^m w_k \otimes y'_k$ say. If $\|z\|_\vee = 0$, then for every $\varphi \in \mathrm{Ball}(E^*)$, $\psi \in \mathrm{Ball}(F^*)$ we have $\sum_k \varphi(w_k)\psi(y'_k) = 0$. Thus $\psi\Big(\sum_k \varphi(w_k) y'_k\Big) = 0$. By a corollary to the Hahn-Banach theorem, we have $\sum_k \varphi(w_k) y'_k = 0$. Setting $\varphi = \widetilde{\varphi_j}$ shows that $y'_j = 0$ for all j, and so $z = 0$.

The completion of $E \otimes F$ in this norm is called the *injective tensor product*, and is written as $E \check{\otimes} F$.

3.1.3 (Hilbert tensor product) We recall from operator theory that if H_1, H_2 are Hilbert spaces then there is at most one inner product on $H_1 \otimes H_2$ satisfying

$$\langle \zeta_1 \otimes \zeta_2, \eta_1 \otimes \eta_2 \rangle = \langle \zeta_1, \eta_1 \rangle \, \langle \zeta_2, \eta_2 \rangle, \qquad \zeta_1, \eta_1 \in H_1, \zeta_2, \eta_2 \in H_2. \tag{3.1}$$

. Then the completion in the associated norm is a Hilbert space. The latter is the *Hilbert space tensor product*, and is written as $H_1 \otimes^2 H_2$ or $H_1 \otimes H_2$. Note that we have a unitary equivalence

$$L^2(X) \otimes^2 L^2(Y) \cong L^2(X \times Y)$$

where we are using the product measure on $X \times Y$. For $\zeta \in H_1, \eta \in H_2$ we have

$$\|\zeta \otimes \eta\|^2 = \langle \zeta \otimes \eta, \zeta \otimes \eta \rangle = \langle \zeta, \zeta \rangle \, \langle \eta, \eta \rangle = \|\zeta\|^2 \|\eta\|^2,$$

so that $\|\zeta \otimes \eta\| = \|\zeta\| \|\eta\|$. It is easy now to prove that if $T_k : H_k \to K_k$ are contractions between Hilbert spaces, then there is an induced contraction $T_1 \otimes T_2 : H_1 \otimes H_2 \to K_1 \otimes K_2$.

3.1.4 (Minimal tensor product) Let X and Y be operator spaces, and let $X \otimes Y$ denote their algebraic tensor product. We recall from above that any $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ can be associated with a map $\tilde{u} : Y^* \to X$ defined by $\tilde{u}(\psi) = \sum_k x_k \ \psi(y_k)$, for $\psi \in Y^*$. If $u = \sum_{k=1}^n x_k \otimes \psi_k \in X \otimes Y^*$ then u can be associated with a map $\hat{u} : Y \to X$ defined by $\tilde{u}(y) = \sum_k x_k \ \psi_k(y)$, for $y \in Y$. Both $\tilde{u}$ and $\hat{u}$ are automatically completely bounded by 1.2.7, since they are linear combinations of scalar functionals multiplied by fixed operators. Thus the above correspondences between tensor products and finite rank mappings yield embeddings $X \otimes Y \hookrightarrow CB(Y^*, X)$ and $X \otimes Y^* \hookrightarrow CB(Y, X)$. The *minimal tensor product* $X \otimes_{\min} Y$ may then be defined to be (the completion of) $X \otimes Y$ in the matrix norms inherited from the operator space structure on $CB(Y^*, X)$ described in 1.2.18. That is,

$$X \otimes_{\min} Y \hookrightarrow CB(Y^*, X) \quad \text{completely isometrically.} \tag{3.2}$$

Explicitly, if $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$, then the norm of u in $X \otimes_{\min} Y$ equals

$$\sup \Big\| \Big[\sum_k x_k \ \psi_{ij}(y_k) \Big] \Big\|_{M_m(X)}, \tag{3.3}$$

the supremum taken over all $[\psi_{ij}]$ in the ball of $M_m(Y^*)$, and all $m \in \mathbb{N}$. Applying (1.14) to (3.3), we see that $\|u\|_{\min}$ equals the more symmetric form

$$\sup \Big\| \Big[\varphi_{rs} \Big(\sum_k x_k \ \psi_{ij}(y_k) \Big) \Big] \Big\|_{M_{ms}} = \sup \Big\| \Big[\sum_k \varphi_{rs}(x_k) \, \psi_{ij}(y_k) \Big] \Big\|_{M_{ms}}, \tag{3.4}$$

the supremum taken over $[\varphi_{rs}]$ and $[\psi_{ij}]$ in the ball of $M_s(X^*)$ and $M_m(Y^*)$ respectively, and all $m, s \in \mathbb{N}$. A similar formula holds in $M_n(X \otimes_{\min} Y)$:

$$\|[w_{rs}]\|_{M_n(X \otimes_{\min} Y)} \ = \ \sup\{ \|[(\varphi_{kl} \otimes \psi_{ij})(w_{rs})]\| \} \tag{3.5}$$

for $[w_{rs}] \in M_n(X \otimes Y)$, where the supremum is taken over all $[\varphi_{rs}]$ and $[\psi_{ij}]$ in the ball of $M_s(X^*)$ and $M_m(Y^*)$ respectively, and all $m, s \in \mathbb{N}$, and where $\varphi_{kl} \otimes \psi_{ij}$ denotes the obvious functional on $X \otimes Y$ formed from φ_{kl} and ψ_{ij}.

- We see from (3.5) that $\otimes_{\min}$ is *commutative*, that is

$$X \otimes_{\min} Y \ = \ Y \otimes_{\min} X$$

  as operator spaces. The underlying reason for this is because in the formulae above we have $\varphi_{rs}(x)\psi_{ij}(y) = \psi_{ij}(y)\varphi_{rs}(x)$.
- It is also easy to see from (3.5) that $\otimes_{\min}$ is *functorial*. That is, if X_i and Y_i are operator spaces for $i = 1, 2$, and if $u_i \colon X_i \to Y_i$ are completely bounded, then the map $x \otimes y \mapsto u_1(x) \otimes u_2(y)$ on $X_1 \otimes X_2$ has a unique continuous extension to a map $u_1 \otimes u_2 \colon X_1 \otimes_{\min} X_2 \to Y_1 \otimes_{\min} Y_2$, with $\|u_1 \otimes u_2\|_{\mathrm{cb}} \leq \|u_1\|_{\mathrm{cb}} \|u_2\|_{\mathrm{cb}}$. One way to see this is to note that if $[\varphi_{rs}]$ and $[\psi_{ij}]$ are in the ball of $M_s(Y_1^*)$ and $M_m(Y_2^*)$ respectively, for $m, s \in \mathbb{N}$, then $\frac{1}{\|u_1\|_{\mathrm{cb}}}[\varphi_{rs} \circ u_1]$ and $\frac{1}{\|u_2\|_{\mathrm{cb}}}[\psi_{ij} \circ u_2]$ are in the ball of $M_s(X_1^*)$ and $M_m(X_2^*)$ respectively. Hence by (3.5) we have

$$\frac{1}{\|u_1\|_{\mathrm{cb}} \|u_2\|_{\mathrm{cb}}} \ \|[(\varphi_{kl} \otimes \psi_{ij})((u_1 \otimes u_2) w_{rs})]\| \leq \|[w_{rs}]\|_{M_n(X_1 \otimes_{\min} X_2)},$$

for $[w_{rs}] \in M_n(X_1 \otimes X_2)$, and taking the supremum over $[\varphi_{rs}]$ and $[\psi_{ij}]$, by (3.5) again we have

$$\frac{1}{\|u_1\|_{\rm cb}\|u_2\|_{\rm cb}} \, \|[(u_1 \otimes u_2)w_{rs}]\|_{M_n(Y_1 \otimes_{\min} Y_2)} \leq \|[w_{rs}]\|_{M_n(X_1 \otimes_{\min} X_2)}.$$

Thus $\|u_1 \otimes u_2\|_{\rm cb} \leq \|u_1\|_{\rm cb}\|u_2\|_{\rm cb}$. (As an exercise, the reader could check that $\|u_1 \otimes u_2\|_{\rm cb} = \|u_1\|_{\rm cb}\|u_2\|_{\rm cb}$, but we shall not need this.)

- If, further, the u_i are completely isometric, then so is $u_1 \otimes u_2$. This latter fact is called the *injectivity* of the tensor product. To prove it, since $u_1 \otimes u_2 = (u_1 \otimes I) \circ (I \otimes u_2)$, we may by symmetry reduce the argument to the case that $Y_2 = X_2$, and $u_2 = I_{X_2}$. Then it is easy to see that we can suppose that $X_1 \subset Y_1$ and that u_1 is this inclusion map. In this case, consider the commutative diagram

$$\begin{array}{ccc} CB(X_2^*, X_1) & \longrightarrow & CB(X_2^*, Y_1) \\ \uparrow & & \uparrow \\ X_1 \otimes_{\min} X_2 & \stackrel{u_1 \otimes I}{\longrightarrow} & Y_1 \otimes_{\min} X_2 \end{array}$$

where the vertical arrows are complete isometries by definition of $\otimes_{\min}$, and the top arrow is a complete isometry (since a map into a subspace of an operator space clearly has the same norm as when it is viewed as a map into the bigger space). Hence the bottom arrow is a complete isometry too, which is what we need.

- For any operator spaces X, Y, we have

$$X \otimes_{\min} Y^* \hookrightarrow CB(Y, X) \quad \text{completely isometrically,} \tag{3.6}$$

via the map $\wedge : u \to \hat{u}$ mentioned at the start of 3.1.4. We first prove this in the case that $X = B(H)$. Consider the sequence of maps

$$B(H) \otimes_{\min} Y^* \stackrel{\wedge}{\longrightarrow} CB(Y, B(H)) \cong w^*CB(Y^{**}, B(H)) \subset CB(Y^{**}, B(H)),$$

where the '$\cong$' is from (1.17). The composition of these maps is the complete isometry $u \mapsto \tilde{u}$ implementing (3.2). Since the last few maps in the sequence are isometries so is the first one.

For a general operator space $X \subset B(H)$ we have a commutative diagram

$$\begin{array}{ccc} B(H) \otimes_{\min} Y^* & \stackrel{\wedge}{\longrightarrow} & CB(Y, B(H)) \\ \uparrow & & \uparrow \\ X \otimes_{\min} Y^* & \stackrel{\wedge}{\longrightarrow} & CB(Y, X) \end{array}$$

where the left vertical arrow is a complete isometry by the injectivity of $\otimes_{\min}$, and right one is obviously a complete isometry as we observed earlier. By the last paragraph, the top arrow is an isometry, and so the bottom arrow is an isometry too. We leave the proof that it is a complete isometry to the interested reader.

3.1.5 (Properties of $\check{\otimes}$) Simpler versions of all the computations above give the analogous properties for $\check{\otimes}$. For example, $X \check{\otimes} Y^* \hookrightarrow B(Y, X)$ isometrically, for Banach spaces X and Y.

3.1.6 (The spatial tensor product and $\otimes_{\min}$) Suppose that H_1, H_2 are Hilbert spaces, and consider the canonical map $\pi\colon B(H_1) \otimes B(H_2) \to B(H_1 \otimes^2 H_2)$. This is the map taking a rank one tensor $S \otimes T$ in $B(H_1) \otimes B(H_2)$ to the map $S \otimes T$ on $H_1 \otimes^2 H_2$ taking $\zeta \otimes \eta$ to $S(\zeta) \otimes T(\eta)$. We claim that π actually is a complete isometry when $B(H_1) \otimes B(H_2)$ is given its norm as a subspace of $B(H_1) \otimes_{\min} B(H_2)$. To see this, we choose a set I such that $H_1 = \ell^2_I$, so that we both have $\mathbb{M}_I \cong B(H_1)$ $*$-isomorphically, and also $H_1 \otimes^2 H_2 \cong H_2^{(I)}$ as Hilbert spaces. By (3.6) and 1.3.6, $B(H_1) \otimes_{\min} B(H_2) \hookrightarrow CB(S^1(H_2), \mathbb{M}_I)$. However, by 1.3.6 and 1.2.23 (5) and (12), we have

$$CB(S^1(H_2), \mathbb{M}_I) \cong \mathbb{M}_I(S^1(H_2)^*) \cong \mathbb{M}_I(B(H_2)) \cong B(H_2^{(I)}) \cong B(H_1 \otimes^2 H_2),$$

isometrically. A similar argument proves the complete isometry, and proves the claim.

Thus if X and Y are subspaces of $B(H_1)$ and $B(H_2)$ respectively, then by the injectivity of this tensor product, we have that $X \otimes_{\min} Y$ is completely isometrically isomorphic to the closure in $B(H_1 \otimes^2 H_2)$ of the span of the operators $x \otimes y$ on $H_1 \otimes^2 H_2$, for $x \in X, y \in Y$. Thus the minimal tensor product of $X \otimes Y$ may alternatively be defined to be this subspace of $B(H_1 \otimes^2 H_2)$.

The above implies that the minimal tensor product of C^*-algebras coincides with the tensor product of the same name used in C^*-algebra theory, or with the so-called *spatial tensor product*. We recall that if A and B are C^*-subalgebras of $B(H_1)$ and $B(H_2)$ respectively, then $A \otimes B$ may be identified (as above) with a subspace of $B(H_1 \otimes^2 H_2)$, which is easy to see is actually a $*$-subalgebra. The closure of this, which is a C^*-algebra, is called the *spatial tensor product* of A and B, and is written as $A \otimes_{\min} B$. If A and B are also commutative, then so is $A \otimes_{\min} B$, since it is the closure of a commutative $*$-subalgebra.

From the last paragraph it is clear that for any operator space X,

$$M_n \otimes_{\min} X \;\cong\; M_n(X) \tag{3.7}$$

completely isometrically, since both can be completely isometrically identified with the same subspace of $B(\ell^2_n \otimes H) \cong B(H^{(n)})$, if $X \subset B(H)$. Similarly, $M_{mn} \otimes_{\min} X \cong M_{mn}(X)$.

3.1.7 (Uncompleted tensor products) For what follows, it is convenient to state separately a simple property of tensor product norms. If E and F are incomplete spaces, and α is a tensor norm on $\bar{E} \otimes \bar{F}$, then it is usual to write $\bar{E} \otimes_\alpha \bar{F}$ for the completion of $\bar{E} \otimes \bar{F}$ with respect to α. We will always deal with so-called 'cross norms'; that is, $\alpha(x \otimes y) = \|x\|\|y\|$ for $x \in E, y \in F$. Let us write $E \otimes^\alpha F$ for the (possibly incomplete) subspace $E \otimes F$ of $\bar{E} \otimes \bar{F}$, equipped with the norm α. Claim: $\bar{E} \otimes_\alpha \bar{F}$ is the closure (and also the completion) of $E \otimes^\alpha F$. To see this, we need to show that any $u \in \bar{E} \otimes_\alpha \bar{F}$ may be approximated in the norm topology by elements in $E \otimes F$. However, such u may first be approximated in norm by a finite sum of elementary tensors $x \otimes y$, with $x \in \bar{E}$ and $y \in \bar{F}$. Then we can approximate $x \otimes y$ in norm by $x' \otimes y'$, with $x' \in E$ and $y' \in F$. Hence u is approximable by elements in $E \otimes^{\alpha} F$.

3.1.8 (Further properties of $\otimes_{\min}$) For any set I we have

$$\mathbb{K}_I \otimes_{\min} X \;\cong\; \mathbb{K}_I(X). \tag{3.8}$$

To see this, first note that if $X \subset B(H)$, then by the injectivity of $\otimes_{\min}$ we have $\mathbb{K}_I \otimes_{\min} X \;\subset \mathbb{M}_I \otimes_{\min} B(H)$. By 3.1.6, the latter space can be identified with a subspace of $B(\ell^2_I \otimes H) \cong B(H^{(I)}) \cong \mathbb{M}_I(B(H))$ (see 1.2.23 (5)). On the other hand, $\mathbb{K}_I(X)$ is the closure of $\mathbb{M}_I^{\text{fin}}(X)$ in $\mathbb{M}_I(B(H))$. We can express this in the commutative diagram

$$\begin{array}{ccc}
B(\ell^2_I \otimes^2 H) & \longrightarrow & \mathbb{M}_I(B(H)) \\
\uparrow & & \uparrow \\
\mathbb{M}_I \otimes_{\min} B(H) & & \mathbb{M}_I(X) \\
\uparrow & & \uparrow \\
\mathbb{M}_I^{\text{fin}} \otimes^{\min} X & \longrightarrow & \mathbb{M}_I^{\text{fin}}(X).
\end{array}$$

The complete isometry in the top row, restricts to a complete isometry in the bottom row. Taking completions, and using 3.1.7, gives (completely isometrically)

$$\mathbb{K}_I \;\otimes_{\min} X = \overline{\mathbb{M}_I^{\text{fin}} \otimes^{\min} X} \cong \overline{\mathbb{M}_I^{\text{fin}}(X)} = \mathbb{K}_I(X).$$

There is a 'rectangular variant' of (3.8): for any sets I, J we have

$$\mathbb{K}_{I,J} \otimes_{\min} X \;\cong\; \mathbb{K}_{I,J}(X). \tag{3.9}$$

To see this, suppose that I has a bigger cardinality than J (the contrary case is almost identical). Then we may regard $\mathbb{K}_{I,J} \subset \mathbb{K}_I$ and $\mathbb{K}_{I,J}(X) \subset \mathbb{K}_I(X)$. By the injectivity of $\otimes_{\min}$ we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{K}_I \otimes_{\min} X & \longrightarrow & \mathbb{K}_I(X) \\
\uparrow & & \uparrow \\
\mathbb{K}_{I,J} \otimes_{\min} X & \longrightarrow & \mathbb{K}_{I,J}(X).
\end{array}$$

The complete isometry in the top row coming from (3.8), restricts to a complete isometry in the bottom row, proving (3.9).

Similarly, it follows from the second last paragraph, and from the fact that $B((H_1 \otimes^2 H_2) \otimes^2 H_3) \cong B(H_1 \otimes^2 (H_2 \otimes^2 H_3))$, that $\otimes_{\min}$ is *associative*. That is,

$$(X_1 \otimes_{\min} X_2) \otimes_{\min} X_3 = X_1 \otimes_{\min} (X_2 \otimes_{\min} X_3). \tag{3.10}$$

To see this clearly, suppose that $X_i \subset B(H_i)$, and consider the commutative diagram

$$\begin{array}{ccc} B((H_1 \otimes^2 H_2) \otimes^2 H_3) & \longrightarrow & B(H_1 \otimes^2 (H_2 \otimes^2 H_3)) \\ \uparrow & & \uparrow \\ B(H_1 \otimes^2 H_2) \otimes_{\min} B(H_3) & & B(H_1) \otimes_{\min} B(H_2 \otimes^2 H_3) \\ \uparrow & & \uparrow \\ (X_1 \otimes^{\min} X_2) \otimes^{\min} X_3 & \longrightarrow & X_1 \otimes^{\min} (X_2 \otimes^{\min} X_3). \end{array}$$

The vertical arrows are complete isometries by 3.1.6 and the 'injectivity' of $\otimes_{\min}$. The $*$-isomorphism in the top row, which is a complete isometry, restricts to a complete isometry in the bottom row. Taking completions, and using 3.1.7, gives (completely isometrically)

$$(X_1 \otimes_{\min} X_2) \otimes_{\min} X_3 = \overline{(X_1 \otimes^{\min} X_2) \otimes^{\min} X_3} \cong \overline{X_1 \otimes^{\min} (X_2 \otimes^{\min} X_3)},$$

which equals $X_1 \otimes_{\min} (X_2 \otimes_{\min} X_3)$. This proves the associativity. Accordingly, the space in (3.10) will be merely denoted by $X_1 \otimes_{\min} X_2 \otimes_{\min} X_3$, and the proof above shows that it can be identified with a subspace of $B(H_1 \otimes^2 H_2 \otimes^2 H_3)$. Similarly, one may consider the N-fold minimal tensor product $X_1 \otimes_{\min} \cdots \otimes_{\min} X_N$ of any N-tuple of operator spaces.

Proposition 3.1.9. Let E, F be Banach spaces and let X be an operator space.

(1) $\mathrm{Min}(E) \otimes_{\min} X = E \check{\otimes} X$ as Banach spaces.
(2) $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F) = \mathrm{Min}(E \check{\otimes} F)$ as operator spaces.

Proof. We have isometric embeddings $\mathrm{Min}(E) \otimes_{\min} X \subset CB(X^*, \mathrm{Min}(E))$ and $E \check{\otimes} X \subset B(X^*, E)$ by (3.2) and the Banach space variant of (3.2). However $CB(X^*, \mathrm{Min}(E)) = B(X^*, E)$ by (1.8). Thus both spaces in (1) coincide isometrically with the same subspace of $B(X^*, E)$, which proves (1). The isometry in (2) follows from (1). Thus the complete isometry in (2) will follow if $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F)$ is a minimal operator space. To see this, suppose that $E \subset C(K_1)$ and $F \subset C(K_2)$ isometrically. Then $\mathrm{Min}(E) \subset C(K_1)$ and $\mathrm{Min}(F) \subset C(K_2)$ completely isometrically. So by the 'injectivity' of $\otimes_{\min}$, we have that $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F)$ is contained inside $C(K_1) \otimes_{\min} C(K_2)$ completely isometrically. However, we observed in 3.1.6 that the minimal tensor product of commutative C^*-algebras is a commutative C^*-algebra, and hence is a $C(K)$-space, and is a 'minimal operator space'. □

3.1.10 (Duality of Min and Max) We take the time to prove an item stated earlier, namely: For any Banach space E, we have

$$\mathrm{Min}(E)^* = \mathrm{Max}(E^*) \quad \text{and} \quad \mathrm{Max}(E)^* = \mathrm{Min}(E^*), \tag{3.11}$$

completely isometrically. To see this, note that an element in $M_n(\mathrm{Max}(E)^*)$ may be regarded as a map in $CB(\mathrm{Max}(E), M_n)$ by (1.6). By (1.10) this is exactly the same as a map in $B(E, M_n)$.

On the other hand, $M_n(\mathrm{Min}(E^*)) \cong M_n \check{\otimes} E^* \cong B(E, M_n)$ by Proposition 3.1.9 and **??**. That is, an element in $M_n(\mathrm{Min}(E^*))$ may be regarded as a map in $B(E, M_n)$. These identifications preserve the norm, so that $M_n(\text{Max}(E)^*) = M_n(\text{Min}(E^*))$. That is, $\text{Max}(E)^* = \text{Min}(E^*)$. Therefore also $\text{Max}(E^*)^* = \text{Min}(E^{**})$. However we claim that $\text{Min}(E^{**}) = \text{Min}(E)^{**}$. This claim may be seen by first proving it in the case that $E = C(K)$, for compact K. The claim follows in this case from 1.3.11, since the second dual of a commutative C^*-algebra is a commutative C^*-algebra, and hence is a minimal operator space. Next we use the fact that 'minimal operator spaces' are completely isometric to subspaces of unital commutative C^*-algebras, and the fact that the second dual of a complete isometry is a complete isometry (see 1.3.3). Thus if $\text{Min}(E) \subset C(K)$ completely isometrically, then dualizing this embedding we get a commuting diagram

$$\begin{array}{ccc} C(K)^{**} & \longrightarrow & \text{Min}(C(K))^{**} \\ \uparrow & & \uparrow \\ \text{Min}(E^{**}) & \longrightarrow & \text{Min}(E)^{**} \end{array}$$

where all arrows except possibly the bottom one are complete isometries. Hence the bottom one is a complete isometry, proving the claim.

Finally, $\text{Max}(E^*)$ and $\text{Min}(E)^*$ are two operator space structures on E^* with the same operator space dual, and therefore they are completely isometric, by 1.3.1.

3.1.11 (Haagerup tensor product) Before we define this tensor product, we introduce an intimately related class of bilinear maps. Suppose that X, Y, and W are operator spaces, and that $u \colon X \times Y \to W$ is a bilinear map. For $n, p \in \mathbb{N}$, define a bilinear map $M_{n,p}(X) \times M_{p,n}(Y) \to M_n(W)$ by

$$(x, y) \longmapsto \Big[\sum_{k=1}^{p} u(x_{ik}, y_{kj})\Big]_{i,j},$$

where $x = [x_{ij}] \in M_{n,p}(X)$ and $y = [y_{ij}] \in M_{p,n}(Y)$. If $p = n$ we write this map as u_n. If the norms of these bilinear maps are uniformly bounded over $p, n \in \mathbb{N}$, then we say that u is *completely bounded*, and write the supremum of these norms as $\|u\|_{\text{cb}}$. Sometimes this is called *completely bounded in the sense of Christensen and Sinclair*. It is easy to see (by adding rows and columns of zeroes to make $p = n$) that $\|u\|_{\text{cb}} = \sup_n \|u_n\|$. (Indeed, if $[x_{ij}] \in M_{n,p}(X)$ and $[y_{ij}] \in M_{p,n}(Y)$, and if $m = \max\{n, p\}$, then

$$\Big\|\Big[\sum_{k=1}^{p} u(x_{ik}, y_{kj})\Big]\Big\|_n = \|u_m([x'_{ij}], [y'_{ij}])\|_m \leq \|u_m\|\|[x_{ij}]\|\|[y_{ij}]\|,$$

where $[x'_{ij}]$ and $[y'_{ij}]$ are $m \times m$ matrices obtained from $[x_{ij}]$ and $[y_{ij}]$ by adding rows or columns of zeros.)

We say that u is *completely contractive* if $\|u\|_{\text{cb}} \leq 1$. Completely bounded multilinear maps of three variables have a similar definition (involving the expression $[\sum_{k,l} u(x_{ik}, y_{kl}, z_{lj})]$), and similarly for four or more variables. We remark that if $v \colon X \to B(H)$ and $w \colon Y \to B(H)$ are completely bounded linear maps, then it is easy to see that the bilinear map $(x, y) \mapsto v(x)w(y)$ is completely bounded in the sense above, and has completely bounded norm dominated by $\|v\|_{\rm cb}\|w\|_{\rm cb}$. Indeed, note that

$$\Big\| \Big[\sum_{k=1}^{n} v(x_{ik}) w(y_{kj}) \Big] \Big\|_n \leq \|[v(x_{ij})]\|_n \|[w(y_{ij})]\|_n \leq \|v\|_{\rm cb} \|w\|_{\rm cb} \|[x_{ij}]\|_n \|[y_{ij}]\|_n,$$

if $[x_{ij}] \in M_n(X)$ and $[y_{ij}] \in M_n(Y)$, since $M_n(B(H))$ is a Banach algebra.

Let X, Y be operator spaces. For $n \in \mathbb{N}$ and $z \in M_n(X \otimes Y)$ we define

$$\|z\|_{\rm h} = \inf\{\|x\|\|y\|\}, \tag{3.12}$$

where the infimum is taken over all $p \in \mathbb{N}$, and all ways to write $z = x \odot y$, where $x \in M_{n,p}(X), y \in M_{p,n}(Y)$. Here $x \odot y$ denotes the formal matrix product of x and y using the $\otimes$ sign as multiplication: namely $x \odot y = [\sum_{k=1}^{p} x_{ik} \otimes y_{kj}]$. To make sense of this, we first note that any $z \in M_n(X \otimes Y)$ can be written as such a $x \odot y$. To see this we observe that this is clearly true if z has only one nonzero entry. For example, if this entry were the 1-2 entry, and if $z_{12} = \sum_{k=1}^{p} x_k \otimes y_k$, then $z = x \odot y$ where $x \in M_{np}(X)$ has first row consisting of the x_k and zeros elsewhere, and $y \in M_{pn}(Y)$ has second column consisting of the y_k and zeros elsewhere. Next note that

$$x \odot y + x' \odot y' = [x \,:\, x'] \odot \begin{bmatrix} y \\ y' \end{bmatrix},$$

for matrices x, x', y, y' of appropriate sizes. Similarly for a sum of any (finite) number of terms of the form $x \odot y$. Thus by writing $z \in M_n(X \otimes Y)$ as a sum of n^2 matrices, each of which has only one nonzero entry, and using the facts above, we do indeed have $z = x \odot y$ as desired.

It is clear that $\|\lambda z\|_{\rm h} = |\lambda|\|z\|_{\rm h}$ if $\lambda \in \mathbb{C}$. Next note that the last centered equation actually shows that $\|z + z'\|_{\rm h} \leq \|z\|_{\rm h} + \|z'\|_{\rm h}$ for $z, z' \in M_n(X \otimes Y)$. For suppose that $z = x \odot y$ and $z' = x' \odot y'$, with $\|x\|\|y\| < \|z\|_{\rm h} + \epsilon$ and $\|x'\|\|y'\| < \|z'\|_{\rm h} + \epsilon$. By the trick of writing $x \odot y = tx \odot \frac{1}{t} y$ with $t = \sqrt{\|y\|/\|x\|}$, we can assume that $\|y\| = \|x\|$. Similarly, assume that $\|y'\| = \|x'\|$. Then

$$\|z + z'\|_{\rm h} \leq \|[x \,:\, x']\| \Big\| \begin{bmatrix} y \\ y' \end{bmatrix} \Big\| \leq \sqrt{\|x\|^2 + \|x'\|^2} \sqrt{\|y\|^2 + \|y'\|^2},$$

the last $\leq$ following from the C^*-identity used four times. For example,

$$\|[x \,:\, x']\|^2 = \|xx^* + x'x'^*\| \leq \|xx^*\| + \|x'x'^*\| = \|x\|^2 + \|x'\|^2.$$

Thus

$$\|z + z'\|_{\rm h} \leq \|x\|\|y\| + \|x'\|\|y'\| \leq \|z\|_{\rm h} + \|z'\|_{\rm h} + 2\epsilon.$$

Now let $\epsilon \to 0$, to see that $\|\cdot\|_{\rm h}$ is a seminorm.

Suppose that $u \colon X \times Y \to W$ is a bilinear map which is completely contractive in the sense above. Let $\tilde{u} \colon X \otimes Y \to W$ be the canonically associated linear map. For $z \in M_n(X \otimes Y)$, if $z = x \odot y$ as above, then

$$\|\tilde{u}_n(z)\|_n = \Big\| \Big[\sum_{k=1}^{p} u(x_{ik}, y_{kj}) \Big] \leq \|x\|\|y\|.$$

Taking the infimum over x, y with $z = x \odot y$, we see by the definition of $\|\cdot\|_{\rm h}$ that

$$\|\tilde{u}_n(z)\| \leq \|z\|_{\rm h}, \tag{3.13}$$

where the latter quantity is as defined in (3.12). If φ and ψ are contractive functionals on X and Y respectively, then using 1.2.7 and the fact at the end of the second paragraph of 3.1.11, we see that the bilinear map $(x, y) \mapsto \varphi(x)\psi(y)$ is completely contractive. Thus from (3.13) we see that

$$\Big|\sum_{k=1}^{p} \varphi(x_k)\psi(y_k)\Big| \leq \|z\|_{\rm h}\,, \qquad z = \sum_{k=1}^{p} x_k \otimes y_k.$$

By the definition of the Banach space injective tensor norm of z (see 3.2.2 in the C^*-course), we deduce that the latter norm of an element $z \in X \otimes Y$ is dominated by $\|z\|_{\rm h}$. Hence indeed $\|\cdot\|_{\rm h}$ is a norm.

Proposition 3.1.12. If X and Y are operator spaces, then the completion $X \otimes_h Y$ of $X \otimes Y$ with respect to $\|\cdot\|_{\rm h}$ is an operator space.

Proof. We use Ruan's theorem, in the form of Exercise (6) to Section 2.1. To see (R1)', suppose that $\alpha \in M_{m,n}, \beta \in M_{n,m}, z \in M_n(X \otimes Y)$ with $z = x \odot y$ as above. Then $\alpha z \beta = (\alpha x) \odot (y\beta)$, and so

$$\|\alpha z \beta\|_{\rm h} \leq \|\alpha x\|\|y\beta\| \leq \|\alpha\|\|x\|\|y\|\|\beta\|.$$

Taking the infimum over x, y with $z = x \odot y$, we see by definition of $\|\cdot\|_{\rm h}$ that

$$\|\alpha z \beta\|_{\rm h} \leq \|\alpha\|\|\beta\|\|z\|_{\rm h}.$$

For (R2)', let $z' = x' \odot y' \in M_p(X \otimes Y)$, then $z \oplus z' = (x \oplus x') \odot (y \oplus y')$, and

$$\|z \oplus z'\|_{\rm h} \leq \|x \oplus x'\|\|y \oplus y'\| = \max\{\|x\|, \|x'\|\} \max\{\|y\|, \|y'\|\}.$$

As in the proof that $\|\cdot\|_{\rm h}$ is a norm, we can assume that $\|x\| = \|y\|$ and $\|x'\| = \|y'\|$. Then $\|z \oplus z'\|_{\rm h} \leq \max\{\|x\|\|y\|, \|x'\|\|y'\|\}$, and taking the infimum over such x, x', y, y' gives (R2)'. Note that (R1)' and (R2)' pass to the completion of $X \otimes Y$. So $X \otimes_h Y$ is an operator space. □

This operator space $X \otimes_h Y$ is called the *Haagerup tensor product*. Note that the canonical bilinear map $\otimes \colon X \times Y \to X \otimes_h Y$ is completely contractive in the sense above.

Using (3.13) we see that if $u \colon X \times Y \to W$ is a bilinear map with associated linear map $\tilde{u} \colon X \otimes Y \to W$, then u is completely bounded if and only if $\tilde{u}$ extends to a completely bounded linear map on $X \otimes_h Y$. Moreover we have

$$\|u\|_{\rm cb} = \big\|\tilde{u} \colon X \otimes_h Y \longrightarrow W\big\|_{\rm cb}$$

in that case. The above property means that the Haagerup tensor product *linearizes completely bounded bilinear maps*. A moments thought shows that this is a *universal property*. That is, suppose that (W, μ) is a pair consisting of an operator space W, and a completely contractive bilinear map $\mu \colon X \times Y \to W$, such that the span of the range of μ is dense in W, and which possesses the following property:

Given any operator space Z and given any completely bounded bilinear map $u\colon X \times Y \to Z$, then there exists a linear completely bounded $\tilde{u}\colon W \to Z$ such that $\tilde{u}(\mu(x,y)) = u(x,y)$ for all $x \in X, y \in Y$, and such that $\|\tilde{u}\|_{\rm cb} \leq \|u\|_{\rm cb}$.

Then $X \otimes_h Y \cong W$ via a complete isometry v satisfying $v \circ \otimes = \mu$.

We leave it to the reader to check the above assertions as an exercise.

3.1.13 (More properties of the Haagerup tensor product)

- Since $X \otimes^{\rm h} Y$ is an (uncompleted) operator space, there is a canonical norm on $M_{m,n}(X \otimes_h Y)$, via viewing this space as a subspace of $M_r(X \otimes_h Y)$, for $r = \max\{m,n\}$. It is easy to see that for $z \in M_{m,n}(X \otimes_h Y)$, this canonical norm is still given by the formula (3.12), however with $x \in M_{m,p}(X), y \in M_{p,n}(Y)$ there. There is a canonical linear isomorphism between $C_m(X) \otimes R_n(Y)$ and $M_{m,n}(X \otimes Y)$, taking $[x_i] \otimes [y_i] \to [x_i \otimes y_j]_{(i,j)}$. Using the definition (3.12) it is a very easy exercise to show that this isomorphism is actually an isometry $C_m(X) \otimes^{\rm h} R_n(Y) \cong M_{m,n}(X \otimes^{\rm h} Y)$. Passing to the completion, we have $C_m(X) \otimes_{\rm h} R_n(Y) \cong M_{m,n}(X \otimes_{\rm h} Y)$ isometrically.
- This tensor product is *functorial*. That is, if $u_i : X_i \to Y_i$ are completely bounded maps between operator spaces, then $u_1 \otimes u_2 : X_1 \otimes_h X_2 \to Y_1 \otimes_h Y_2$ is completely bounded, and $\|u_1 \otimes u_2\|_{\rm cb} \leq \|u_1\|_{\rm cb}\, \|u_2\|_{\rm cb}$. Indeed, if $z = x \odot y \in M_n(X_1 \otimes X_2)$, then $(u_1 \otimes u_2)_n(z) = (u_1)_n(x) \odot (u_2)_n(y)$, and so

  $$\|(u_1 \otimes u_2)_n(z)\|_{\rm h} \leq \|(u_1)_n(x)\|\|(u_2)_n(y)\| \leq \|u_1\|_{\rm cb}\, \|u_2\|_{\rm cb} \|x\|\|y\|.$$

  Taking the infimum over such x, y with $z = x \odot y$ gives

  $$\|(u_1 \otimes u_2)_n(z)\|_{\rm h} \leq \|u_1\|_{\rm cb}\, \|u_2\|_{\rm cb} \|z\|_{\rm h}.$$

  Thus $u_1 \otimes u_2$ is continuous on $X_1 \otimes^{\rm h} X_2$, and extends uniquely to $u_1 \otimes u_2 : X_1 \otimes_h X_2 \to Y_1 \otimes_h Y_2$ satisfying $\|u_1 \otimes u_2\|_{\rm cb} \leq \|u_1\|_{\rm cb}\, \|u_2\|_{\rm cb}$.
- The Haagerup tensor product is *projective*, that is, if u_1 and u_2 in the last item are complete quotient maps, then so is $u_1 \otimes u_2$. To see this, note that by the functoriality, the map $u_1 \otimes u_2$ is a complete contraction. Let $z \in M_n(Y_1 \otimes Y_2)$, with $\|z\|_{\rm h} < 1$. By definition, we may write $z = y_1 \odot y_2$, where $y_1 \in M_{n,p}(Y_1)$, $y_2 \in M_{p,n}(Y_2)$ both have norm < 1. Then $y_1 = (u_1)_{n,p}(x_1)$ and $y_2 = (u_2)_{p,n}(x_2)$ for $x_1 \in M_{n,p}(X_1)$, $x_2 \in M_{p,n}(X_2)$, both of norm < 1. Let $w = x_1 \odot x_2 \in M_n(X_1 \otimes_h X_2)$, this matrix has norm < 1, and $(u_1 \otimes u_2)_n(w) = z$. By an obvious density argument, this shows that $u_1 \otimes u_2$ above is a complete quotient map.
- The Haagerup tensor product is not *commutative*. That is, in general $X \otimes_h Y$ and $Y \otimes_h X$ are not isometric. We shall see some examples of this later.
- The Haagerup tensor product is *associative*. That is,

  $$(X_1 \otimes_h X_2) \otimes_h X_3 \cong X_1 \otimes_h (X_2 \otimes_h X_3)$$

  completely isometrically. To see this, we first show it for the uncompleted Haagerup tensor product, where there is an obvious algebraic linear isomorphism $\rho : (X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes (X_2 \otimes X_3)$. If $z \in M_n((X_1 \otimes^{\rm h} X_2) \otimes X_3)$ with $\|z\|_{\rm h} < 1$ then

$z = u \odot w$, where $u \in M_{n,p}(X_1 \otimes^{\mathrm{h}} X_2)$ and $w \in M_{p,n}(X_3)$, both of norm < 1. By the first few lines in 3.1.13, we have $u = x \odot y$ for some $x \in M_{n,k}(X), y \in M_{k,p}(Y)$, both of norm < 1. But then $\rho_n(z) = x \odot (y \odot z)$, and hence it is easy to see that $\|\rho_n(z)\|_{\mathrm{h}} < 1$. So ρ is a complete contraction, and similarly ρ^{-1} is a complete contraction. So ρ is a complete isometry. Taking the completion, just as in the proof of the associativity of $\otimes_{\min}$, gives the associativity of $\otimes_h$. Accordingly, the three-fold tensor product in the last displayed equation will be merely denoted by $X_1 \otimes_h X_2 \otimes_h X_3$. The induced norms on $M_n(X_1 \otimes X_2 \otimes X_3)$ may be described by the '3-variable' version of (3.12). From this one may see that $X_1 \otimes_h X_2 \otimes_h X_3$ has the universal property of 'linearizing' completely bounded trilinear maps (see discussion at the end of 3.1.11). Similar assertions clearly hold for the N-fold Haagerup tensor product $X_1 \otimes_h \cdots \otimes_h X_N$ of any N-tuple of operator spaces.

- There are convenient norm expressions for $\| \cdot \|_{\mathrm{h}}$. Suppose that A and B are C^*-algebras. If X and Y are subspaces of A and B respectively, and if $z \in X \otimes Y$, then to say that $z = x \odot y$, is simply to say that $z = \sum_{k=1}^{p} a_k \otimes b_k$, where a_k is the kth entry in the 'row matrix' x, and b_k is the kth entry in the 'column matrix' y. By the C^*-identity,

$$\|x\|^2 = \|xx^*\| = \Big\|\sum_{k=1}^{p} a_k a_k^*\Big\|.$$

Similarly, $\|y\|^2 = \Big\|\sum_{k=1}^{p} b_k^* b_k\Big\|$. Thus by the definition in 3.1.11 we have

$$\|z\|_{\mathrm{h}} \;=\; \inf \Big\|\sum_{k=1}^{p} a_k a_k^*\Big\|^{\frac{1}{2}} \Big\|\sum_{k=1}^{p} b_k^* b_k\Big\|^{\frac{1}{2}} \tag{3.14}$$

where the infimum is taken over all ways to write $z = \sum_{k=1}^{p} a_k \otimes b_k$ in $X \otimes Y$.

The following shows that the last formula extends to the completed Haagerup tensor product $X \otimes_h Y$, replacing p by ∞ in (3.14).

Proposition 3.1.14.

(1) If $z \in X \otimes_h Y$ with $\|z\|_{\mathrm{h}} < 1$ then we may write z as a norm convergent sum $\sum_{k=1}^{\infty} a_k \otimes b_k$ in $X \otimes_h Y$, with $\| \sum_{k=1}^{\infty} a_k a_k^*\| < 1$ and $\| \sum_{k=1}^{\infty} b_k^* b_k\| < 1$, and where the last two sums converge in *norm*. That is, $[a_1 \ a_2 \ \cdots] \in R(X)$ and $[b_1 \ b_2 \ \cdots]^t \in C(Y)$.

(2) If $x = [a_1 \ a_2 \ \cdots] \in R(X)$ and $y = [b_1 \ b_2 \ \cdots]^t \in C^w(Y)$, that is if $\sum_{k=1}^{\infty} a_k a_k^*$ converges in norm and if the partial sums of $\sum_{k=1}^{\infty} b_k^* b_k$ are uniformly bounded in norm, then $\sum_{k=1}^{\infty} a_k \otimes b_k$ converges in norm in $X \otimes_h Y$. Similarly if $x \in R^w(X)$ and $y \in C(Y)$.

Proof. (1) If z is as stated, choose $w_1 \in X \otimes Y$ with $\|z - w_1\|_{\mathrm{h}} < \frac{\epsilon}{2}$ and $\|w_1\|_{\mathrm{h}} < 1$. By (3.14) we may write $w_1 = \sum_{k=1}^{n_1} x_k \otimes y_k$ with $\sum_k x_k x_k^* \leq 1$ and $\sum_k y_k^* y_k \leq 1$. Repeating this argument, we may choose $w_2 \in X \otimes Y$ with $\|z - w_1 - w_2\|_{\mathrm{h}} < \frac{\epsilon}{2^2}$, and $\|w_2\|_{\mathrm{h}} < \frac{\epsilon}{2}$. By (3.14) we write $w_2 = \sum_{k=n_1+1}^{n_2} x_k \otimes y_k$ with $\sum_k x_k x_k^* \leq \frac{\epsilon}{2}$

and $\sum_k y_k^* y_k \leq \frac{\epsilon}{2}$. Continuing so, we obtain for every $m \in \mathbb{N}$ a finite rank tensor $w_m = \sum_{k=n_{m-1}+1}^{n_m} x_k \otimes y_k$ with $\|z - w_1 - \ldots - w_m\| < \frac{\epsilon}{2^m}$, $\sum_{k=n_{m-1}+1}^{n_m} x_k x_k^* \leq \frac{\epsilon}{2^{m-1}}$, and $\sum_{k=n_{m-1}+1}^{n_m} y_k^* y_k \leq \frac{\epsilon}{2^{m-1}}$. Now it is clear that the partial sums of $\sum_{k=1}^{\infty} x_k x_k^*$ and $\sum_{k=1}^{\infty} y_k^* y_k$ are Cauchy. For example, for any $j > i \geq n_{m-1} + 1$ we have

$$\|\sum_{k=i}^{j} x_k x_k^*\| \leq \|\sum_{k=n_{m-1}+1}^{\infty} x_k x_k^*\| \leq \sum_{k=m-1}^{\infty} \frac{\epsilon}{2^k} = \frac{\epsilon}{2^m} \to 0$$

with m. Hence $\sum_{k=1}^{\infty} x_k x_k^*$ and $\sum_{k=1}^{\infty} y_k^* y_k$ converge in norm to elements with norm $\leq 1 + \epsilon$. Also, the partial sums of $\sum_{k=1}^{\infty} x_k \otimes y_k$ are Cauchy, so that that sum converges in norm (see (2) below). Since a subsequence of these partial sums converges to z, by the first displayed equation in the proof, we have $z = \sum_{k=1}^{\infty} x_k \otimes y_k$ as desired.

(2) To see that the partial sums of $\sum_{k=1}^{\infty} a_k \otimes b_k$ are Cauchy, note that from (3.14) we have $\|\sum_{k=n}^{m} a_k \otimes b_k\|_{\mathrm{h}} \leq \|\sum_{k=n}^{m} a_k a_k^*\|^{\frac{1}{2}} \|\sum_{k=n}^{m} b_k^* b_k\|^{\frac{1}{2}}$. Now use the fact that the partial sums of $\sum_{k=1}^{\infty} a_k a_k^*$ are Cauchy. □

The Haagerup tensor product is *injective* (Theorem 3.1.15 below). In order to establish this, we will need a simple linear algebraic fact about tensors $z \in E \otimes F$. Suppose that X is a closed subspace of E, with $z \in X \otimes F \subset E \otimes F$, and suppose also that $z = \sum_{k=1}^{n} x_k \otimes y_k$, with $\{y_k\}$ a linearly independent subset of F. Then we claim that $x_k \in X$ for all $k = 1, \cdots, n$. To prove this, choose by the Hahn-Banach theorem functionals $\varphi_k \in F^*$ with $\varphi_k(y_i) = \delta_{i,k}$, Kronecker's delta. Then $(I_E \otimes \varphi_k)(z) = x_k$. However, since $z \in X \otimes Y$ we must have $(I_E \otimes \varphi_k)(z) \in X$. So $x_k \in X$.

Theorem 3.1.15. If $u_i : X_i \to Y_i$ are completely isometric maps between operator spaces, then $u_1 \otimes u_2 : X_1 \otimes_h X_2 \to Y_1 \otimes_h Y_2$ is a complete isometry.

Proof. We may assume that $X_i \subset Y_i$, and u_i is the inclusion. By a two-step argument, as in the proof of the injectivity of $\otimes_{\min}$, we can assume that $X_2 = Y_2$, and u_2 is the identity map. Also, it is enough to prove the result for the uncompleted tensor products. Of course $u_1 \otimes u_2$ is a complete contraction, by the functoriality of $\otimes_h$. To prove that $u_1 \otimes u_2$ is an isometry, it suffices to show that if $z \in X_1 \otimes X_2$, and that if z viewed as an element of $Y_1 \otimes Y_2$ has Haagerup norm < 1, then $z \in \mathrm{Ball}(X_1 \otimes^{\mathrm{h}} X_2)$. Thus suppose that $z \in X_1 \otimes X_2$, with $z = x \odot y = \sum_{k=1}^{n} x_k \otimes y_k$, where $x = [x_k] \in R_n(Y_1), y = [y_k] \in C_n(X_2)$, with $\|x\|\|y\| < 1$. If $\{b_k\} \subset \{y_k\}$ is a basis for $\mathrm{Span}\{y_k\}$, and if $b = [b_k] \in C_m(X_2)$, then there is a matrix β of scalars with $y = \beta b$. One can in fact choose β so that its first few rows form a copy of the identity matrix. Let $\beta = u\alpha$ be a polar decomposition of β, where $\alpha = (\beta^* \beta)^{\frac{1}{2}}$, and u is an isometry. In fact, it is a simple linear algebraic exercise to see that $\beta^* \beta \geq I$, so that α is invertible. Then $z = x \odot y = xu \odot \alpha b$, and

$$\|xu\|\|\alpha b\| \leq \|x\|\|u^* u \alpha b\| = \|x\|\|u^* y\| < 1.$$

Since α is invertible, the entries of αb are linearly independent. By the fact above the theorem, $xu \in R_m(X_1)$. Thus indeed $z \in \mathrm{Ball}(X_1 \otimes^{\mathrm{h}} X_2)$.

Finally, to see that $u_1 \otimes I$ is a complete isometry, we note that by the last paragraph we have that $C_n(X_1) \otimes^{\mathrm{h}} R_n(X_2) \subset C_n(Y_1) \otimes^{\mathrm{h}} R_n(X_2)$ isometrically. By the last part of the first 'bullet' in 3.1.13, we conclude that $M_n(X_1 \otimes^{\mathrm{h}} X_2) \subset M_n(Y_1 \otimes^{\mathrm{h}} X_2)$ isometrically. That is, $(u_1 \otimes I)_n$ is an isometry, so that $u_1 \otimes I$ is a complete isometry. $\square$

3.1.16 (Operator space projective tensor product) As with the Haagerup tensor product, it is convenient to first define an intimately related class of bilinear maps. Suppose that X, Y, and W are operator spaces and that $u\colon X \times Y \to W$ is a bilinear map. We say that u is *jointly completely bounded* if there exists a constant $K \geq 0$ such that

$$\|[u(x_{ij}, y_{kl})]_{(i,k),(j,l)}\| \leq K\|[x_{ij}]\|\|[y_{kl}]\|$$

for all m, n and $[x_{ij}] \in M_n(X)$, and $[y_{kl}] \in M_m(Y)$. Here, as usual, the matrix is indexed on rows by i and k, and on columns by j and l. The least such K is written as $\|u\|_{\mathrm{jcb}}$. We say that u is *jointly completely contractive* if $\|u\|_{\mathrm{jcb}} \leq 1$. Jointly completely bounded multilinear maps of three or more variables are defined similarly. Any completely contractive (in the sense of 3.1.11) bilinear map u is jointly completely contractive. This is immediate from the simple relation $[u(x_{ij}, y_{kl})] = u_{nm}([x_{ij}] \otimes I_m, I_n \otimes [y_{kl}])$, where u_{mn} is as defined at the start of 3.1.11. Indeed, for $[x_{ij}] \in M_n(X)$, and $[y_{kl}] \in M_m(Y)$ we have

$$\|[u(x_{ij}, y_{kl})]\| = \|u_{nm}([x_{ij}]\otimes I_m, I_n\otimes[y_{kl}])\| \leq \|[x_{ij}]\otimes I_m\|\|I_n\otimes[y_{kl}]\| = \|[x_{ij}]\|\|[y_{kl}]\|.$$

Conceptually, perhaps the simplest way to define the operator space projective tensor product $X \widehat{\otimes} Y$ of two operator spaces X and Y, is to identify it (completely isometrically) with a subspace of $CB(X, Y^*)^*$, via the map θ that takes $x \otimes y$ to the functional $T \mapsto T(x)(y)$ on $CB(X, Y^*)$. This gives $X \otimes Y$ an (incomplete) operator space structure, and the completion is an operator space completely isometric to a subspace of $CB(X, Y^*)^*$. One then can immediately verify results like (3.16), (3.17), and (3.18) below. This was the approach taken in [8], and the reader might like to try these as an exercise. However this approach does not yield the following explicit 'internal formula' for the tensor norm: For $z \in M_n(X \otimes Y)$ and $n \in \mathbb{N}$, define

$$\|z\|_\frown = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\|\}, \tag{3.15}$$

the infimum taken over $p, q \in \mathbb{N}$, and all ways to write $z = \alpha(x \otimes y)\beta$, where $\alpha \in M_{n,pq}, x \in M_p(X), y \in M_q(Y)$, and $\beta \in M_{pq,n}$. Here we wrote $x \otimes y$ for the 'tensor product of matrices', namely $x \otimes y = [x_{ij} \otimes y_{kl}]_{(i,k),(j,l)}$, indexed on rows by i and k, and on columns by j and l. Suppose that $z \in M_n(X \otimes Y)$, and that we have a jointly completely contractive bilinear map $u\colon X \times Y \to W$. Let $\tilde{u} : X \otimes Y \to W$ be the associated linear map. Write $z = \alpha(x \otimes y)\beta$ as above. A simple calculation shows that

$$\|\tilde{u}_n(z)\| = \|\alpha[u(x_{ij}, y_{kl})]\beta\| \leq \|\alpha\|\|[u(x_{ij}, y_{kl})]\|\|\beta\| \leq \|\alpha\|\|[x_{ij}]\|\|[y_{kl}]\|\|\beta\|.$$

Taking the infimum over such ways to write z, we see from the definitions that

$$\|\tilde{u}_n(z)\| \leq \|z\|_\frown\,. \tag{3.16}$$

From the observation at the end of the first paragraph of 3.1.16, it follows that this is also true if u is completely contractive. Taking $u = \otimes \colon X \times Y \to X \otimes_h Y$, in which case $\tilde{u}$ is the identity map, we deduce from (3.16) that $\|z\|_{\rm h} \le \|z\|_{\frown}$.

We leave it as an exercise similar to the analogous statement for the Haagerup tensor product (or see [17]), that the quantities in (3.15) define an operator space structure on $X \otimes Y$ (to see that these are norms as opposed to seminorms, use the fact at the end of the last paragraph). Thus the completion of $X \otimes Y$ with respect to these matrix norms is an operator space, which we call the *operator space projective tensor product*, and write as $X \widehat{\otimes} Y$.

By (3.16) we see that if $u \colon X \times Y \to W$ is a bilinear map with associated linear map $\tilde{u} \colon X \otimes Y \to W$, and if u is jointly completely contractive, then $\tilde{u}$ is completely contractive with respect to $\| \cdot \|_{\frown}$, and extends further to a complete contraction $\tilde{u} : X \widehat{\otimes} Y \to W$. Conversely, if $v : X \widehat{\otimes} Y \to W$ is completely contractive, and if $u \colon X \times Y \to W$ is the associated bilinear map, then u is jointly completely contractive. To see this, note that if $[x_{ij}] \in \text{Ball}(M_n(X)), [y_{kl}] \in \text{Ball}(M_m(Y))$, then $[x_{ij} \otimes y_{kl}] \in \text{Ball}(M_{mn}(X \widehat{\otimes} Y))$ (take $\alpha = \beta = I_{mn}$ in (3.15)). Thus

$$\|[u(x_{ij}, y_{kl})]\| = \|v_{mn}([x_{ij} \otimes y_{kl}])\| \le 1.$$

Writing $JCB(X, Y; W)$ for the space of jointly completely bounded maps, we have shown that $JCB(X, Y; W) \cong CB(X \widehat{\otimes} Y, W)$ isometrically, via the canonical map. In other words, the operator space projective tensor product *linearizes jointly completely bounded bilinear maps.* As for the Haagerup tensor product this is a universal property, and characterizes $X \widehat{\otimes} Y$ uniquely up to complete isometry.

If $u \colon X \times Y \to W$ is bilinear and jointly completely bounded, write $u^{\#}$ for the map from X to the set of functions from Y to W defined by

$$u^{\#}(x)(y) = u(x, y), \qquad x \in X, y \in Y.$$

Then $u^{\#} \in CB(X, CB(Y, W))$: indeed,

$$\begin{aligned} \|u^{\#}\|_{cb} &= \sup\{\|[u^{\#}(x_{ij})]\| : [x_{ij}] \in \text{Ball}(M_n(X)), n \in \mathbb{N}\} \\ &= \sup\{\|[u^{\#}(x_{ij})(y_{kl})]\| : [x_{ij}] \in \text{Ball}(M_n(X)), [y_{kl}] \in \text{Ball}(M_m(Y))\} \\ &= \sup\{\|[u(x_{ij}, y_{kl})]\| : [x_{ij}] \in \text{Ball}(M_n(X)), [y_{kl}] \in \text{Ball}(M_m(Y))\} \\ &= \|u\|_{jcb}. \end{aligned}$$

Conversely, if $v \in CB(X, CB(Y, W))$ and if $u(x, y) = v(x)(y)$, then reversing the last argument shows that u is jointly completely bounded. Indeed, we have shown that $CB(X, CB(Y, W)) \cong JCB(X, Y; W) \cong CB(X \widehat{\otimes} Y, W)$ isometrically via the canonical map. In fact this is a complete isometry, as may be seen by the common trick of replacing W by $M_n(W)$ in the isometric identity, thus $CB(X \widehat{\otimes} Y, M_n(W)) \cong CB(X, CB(Y, M_n(W)))$. Using (1.5) we then have a string of isometries

$$\begin{aligned} M_n(CB(X \widehat{\otimes} Y, W)) &\cong CB(X \widehat{\otimes} Y, M_n(W)) \cong CB(X, CB(Y, M_n(W))) \\ &\cong CB(X, M_n(CB(Y, W))) \cong M_n(CB(X, CB(Y, W))). \end{aligned}$$

The isometry which is the composition of all these isometries is easily seen to just be the nth amplification of the map $CB(X \widehat{\otimes} Y, W) \to CB(X, CB(Y, W))$ above, and hence the latter is a complete isometry. A similar argument works for $CB(Y, CB(X, W))$, and thus we have

$$CB(X \widehat{\otimes} Y, W) \cong CB(X, CB(Y, W)) \cong CB(Y, CB(X, W)) \qquad (3.17)$$

completely isometrically. In particular,

$$(X \widehat{\otimes} Y)^* \cong CB(X, Y^*) \cong CB(Y, X^*) \quad \text{completely isometrically.} \qquad (3.18)$$

Corollary 3.1.17. For any operator spaces X, Y, the space $CB(X, Y^*)$ is a dual operator space, with predual $X \widehat{\otimes} Y$.

We now list a sequence of properties of the operator space projective tensor product. We leave it as an exercise, copying the analoguous proofs in 3.1.13, that $\widehat{\otimes}$ is *functorial*, and *projective*. We use these words in the sense that we have used them for the other two tensor products. We show next that $\widehat{\otimes}$ is *commutative*, that is, $X \widehat{\otimes} Y \cong Y \widehat{\otimes} X$ completely isometrically. To see this we will use Exercise (9) to Section 2.1. Indeed if $\theta : X \otimes Y \to Y \otimes X$, then it is easy to check that the map $\varphi \mapsto \varphi \circ \theta$ on $(Y \widehat{\otimes} X)^*$ is exactly the composition of the complete isometries in the sequence

$$(Y \widehat{\otimes} X)^* \cong CB(X, Y^*) \cong (X \widehat{\otimes} Y)^*$$

provided by (3.18).

To show that $\widehat{\otimes}$ is *associative*, that is, $(X \widehat{\otimes} Y) \widehat{\otimes} Z \cong X \widehat{\otimes} (Y \widehat{\otimes} Z)$ completely isometrically, two methods come to mind. First, one could show that each of these two spaces has the universal property of linearizing jointly completely bounded trilinear maps, and hence they must be the same. A second method is to mimic the proof just given for commutativity, since these spaces have the same duals:

$$((X \widehat{\otimes} Y) \widehat{\otimes} Z)^* \cong CB(X \widehat{\otimes} Y, Z^*) \cong CB(X, CB(Y, Z^*)),$$

and

$$(X \widehat{\otimes} (Y \widehat{\otimes} Z))^* \cong CB(X, (Y \widehat{\otimes} Z)^*) \cong CB(X, CB(Y, Z^*)),$$

using (3.17) several times.

3.1.18 (Properties of $\hat{\otimes}$) The Banach space projective tensor product $X \hat{\otimes} Y$ can be defined just as we defined $\widehat{\otimes}$, but in the Banach category. Thus identify $X \hat{\otimes} Y$ (isometrically) with a subspace of $B(X, Y^*)^*$, via the map θ that takes $x \otimes y$ to the functional $T \mapsto T(x)(y)$ on $B(X, Y^*)$. This gives $X \otimes Y$ an (incomplete) operator space structure, and the completion is $X \hat{\otimes} Y$. It is the 'linearizer' of bounded bilinear functions $f : X \times Y \to Z$. Simpler versions of all the computations above give the analogous properties for $\hat{\otimes}$ (which are well known). For example, $(X \hat{\otimes} Y)^* \cong B(Y, X^*) \cong B(X, Y^*)$ isometrically, for Banach spaces X and Y.

Proposition 3.1.19. Let E, F be Banach spaces and let Y be an operator space.

(1) $\mathrm{Max}(E) \widehat{\otimes} Y = E \hat{\otimes} Y$ isometrically.

(2) $\mathrm{Max}(E \hat{\otimes} F) = \mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ completely isometrically.

Proof. The first item follows as in the proof of the commutativity of $\widehat{\otimes}$ above, by computing the duals of these two tensor products, using (3.18), (1.10), and the Banach variant of (3.18):

$$(\mathrm{Max}(E) \widehat{\otimes} Y)^* \cong CB(\mathrm{Max}(E), Y^*) = B(E, Y^*) \cong (E \hat{\otimes} Y)^*.$$

Then (2) follows from (1) if we can show that $\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ is a maximal operator space, or equivalently that any contractive map on it is completely contractive. To do this, observe that

$$B(\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F), W) = B(E \hat{\otimes} F, W) = B(E, B(F, W)),$$

isometrically for any operator space W, using (1). The latter space equals

$$CB(\mathrm{Max}(E), CB(\mathrm{Max}(F), W)) = CB(\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F), W)$$

by (1.10) and (3.17). Thus $\mathrm{Max}(E) \widehat{\otimes} \mathrm{Max}(F)$ is 'maximal'. $\square$

Proposition 3.1.20. (Comparison of tensor norms) If X and Y are operator spaces then the various tensor norms on $X \otimes Y$ are ordered as follows:

$$\| \cdot \|_\vee \ \leq \ \| \cdot \|_{\min} \ \leq \ \| \cdot \|_{\mathrm{h}} \ \leq \ \| \cdot \|_\frown \ \leq \ \| \cdot \|_\wedge.$$

Indeed the 'identity' is a complete contraction $X \widehat{\otimes} Y \to X \otimes_h Y \to X \otimes_{\min} Y$.

Proof. The first inequality follows easily for example from (3.5) and the definition of $\check{\otimes}$ in the C^*-course. The fact that $\| \cdot \|_\frown \leq \| \cdot \|_\wedge$ follows from the universal property of $\hat{\otimes}$ in the C^*-course. Indeed, since the bilinear map $\otimes \colon X \times Y \to X \widehat{\otimes} Y$ is jointly completely contractive, and hence contractive, it induces a contraction $X \otimes_\wedge Y \to X \widehat{\otimes} Y$. We saw in 3.1.16 the complete contraction $X \widehat{\otimes} Y \to X \otimes_h Y$. For the remaining relation, consider unital C^*-algebras A and B containing X and Y respectively, and the following commutative diagram of uncompleted tensor products

$$\begin{array}{ccc} A \otimes^h B & \longrightarrow & A \otimes^{\min} B \\ \uparrow & & \uparrow \\ X \otimes^h Y & \longrightarrow & X \otimes^{\min} Y, \end{array}$$

where the horizontal arrows are the identity map, and the vertical arrows are complete isometries by the injectivity of these tensor norms. Thus the bottom arrow will be a complete contraction if the top one is. Now $A \otimes_{\min} B$ is a C^*-algebra as observed in 3.1.6. Moreover the maps

$$\pi : A \to A \otimes_{\min} B : a \to a \otimes 1 \ \ , \ \ \rho : B \to A \otimes_{\min} B : b \to 1 \otimes b$$

are (completely contractive) $*$-homomorphisms. The bilinear map $(a, b) \mapsto \pi(a)\rho(b) = a \otimes b$ from $A \times B$ to $A \otimes_{\min} B$ is therefore completely contractive in the sense of 3.1.11. By the universal property of $\otimes_h$, this bilinear map induces a linear complete contraction from $A \otimes^h B$ to $A \otimes^{\min} B$. But the latter map is clearly the identity map, which proves the desired relation. $\square$

Proposition 3.1.21. If X, Y are operator spaces, if H, K are Hilbert spaces, and if $m, n \in \mathbb{N}$, then we have the following complete isometries:

(1) $H^r \otimes_h X = H^r \widehat{\otimes} X$, and $X \otimes_h H^c = X \widehat{\otimes} H^c$.
(2) $H^c \otimes_h X = H^c \otimes_{\min} X$, and $X \otimes_h H^r = X \otimes_{\min} H^r$.
(3) $C_n(X) \cong C_n \otimes_h X = C_n \otimes_{\min} X$, and $R_n(X) \cong X \otimes_h R_n = X \otimes_{\min} R_n$.
(4) $(\bar{H}^r \widehat{\otimes} X \widehat{\otimes} K^c)^* = (\bar{H}^r \otimes_h X \otimes_h K^c)^* \cong CB(X, B(K, H))$.
(5) $S^\infty(K, H) \cong H^c \otimes_{\min} \bar{K}^r$ and $S^\infty(K, H) \otimes_{\min} X \cong H^c \otimes_h X \otimes_h \bar{K}^r$.
(6) $M_{m,n}(X) \cong C_m \otimes_h X \otimes_h R_n$.
(7) $M_{m,n}(X \otimes_h Y) \cong C_m(X) \otimes_h R_n(Y)$.
(8) $H^c \widehat{\otimes} K^c = H^c \otimes_h K^c = H^c \otimes_{\min} K^c = (H \otimes^2 K)^c$, and similarly for row Hilbert spaces.
(9) $S^1(K, H) \cong \bar{K}^r \widehat{\otimes} H^c$.
(10) $CB(S^1(\ell^2_I, \ell^2_J), X) \cong \mathbb{M}_{I,J}(X)$, if I, J are sets.

Proof. We will prove (1) last, although we use it to prove some of the others. To prove (3), note that by (3.9) we have $C_n \otimes_{\min} X \cong C_n(X)$. By the last proposition there is a complete contraction $C_n \otimes_h X \to C_n \otimes_{\min} X \cong C_n(X)$. The inverse of the latter map is the map $u : C_n(X) \to C_n \otimes_h X$ defined by $u(\vec{x}) = \sum_{k=1}^n \vec{e_k} \otimes x_k$, where $\vec{x} = [x_k] \in C_n(X)$. By definition of the Haagerup tensor norm, namely (3.12), we have

$$\Big\|\sum_{k=1}^n \vec{e_k} \otimes x_k\Big\|_{\mathrm{h}} \le \|I_n\|\|\vec{x}\| = \|\vec{x}\|.$$

Thus u is a contraction, and a similar argument at the matrix level shows that it is a complete contraction. Thus $C_n \otimes_h X \cong C_n(X)$. A similar argument proves the other relation in (3).

It suffices to prove (2) for the uncompleted tensor products: $H^c \otimes^{\mathrm{h}} X = H^c \otimes^{\min} X$. Let us examine the norm on both sides. If $z = \sum_{k=1}^m \zeta_k \otimes x_k \in H^c \otimes X$, let $K = \mathrm{Span}(\{\zeta_k\}) \subset H^c$. By the injectivity of $\otimes_h$ (resp. $\otimes_{\min}$) the norm $\|z\|_{\mathrm{h}}$ (resp. $\|z\|_{\min}$) is the same whether computed in $K \otimes X$ or in $H^c \otimes X$. Thus we can assume that H is finite dimensional. A similar argument lets us make this same assumption if $z \in M_n(H^c \otimes X)$. Now K is a Hilbert column space, and is isometrically, and hence completely isometrically by (1.12), isomorphic to C_n for some $n \in \mathbb{N}$. By (3), we have

$$K \otimes_h X \cong C_n \otimes_h X \cong C_n \otimes_{\min} X \cong K \otimes_{\min} X.$$

It is clear from the above discussion that we now have proved (2).

The first equality in (4) is clear from (1), and the rest is clear from the complete isometries

$$(\bar{H}^r \widehat{\otimes} X \widehat{\otimes} K^c)^* \cong CB(X \widehat{\otimes} K^c, H^c) \cong CB(X, CB(K^c, H^c)),$$

which equals $CB(X, B(K, H))$. Here we have used (3.18), (3.17), (1.13), and (1.12).

For the first equality in (5), note that the canonical map $X \otimes Y \to B(Y^*, X)$ has range which is precisely the set of finite rank operators. Thus the canonical complete isometry

$$H^c \otimes_{\min} \bar{K}^r \to CB((\bar{K}^r)^*, H^c) \cong CB(K^c, H^c) = B(K, H)$$

has range that is the closure of the set of finite rank operators (we have used (3.2), (1.13), and (1.12) here). But this closure in $B(K, H)$ is $S^\infty(K, H)$. For the second equality, note that $S^\infty(K, H) \otimes_{\min} X = H^c \otimes_{\min} X \otimes_{\min} \bar{K}^r$ by commutativity of $\otimes_{\min}$ and the first part of (5), and so the second part of (5) follows from (2).

Item (6) follows from (5) and (3.9), and (7) follows from (6) by (3) and the associativity of the Haagerup tensor product:

$$C_m(X) \otimes_h R_n(Y) \cong C_m \otimes_h (X \otimes_h Y) \otimes_h R_n \cong M_{m,n}(X \otimes_h Y).$$

The middle equality in (8) follows from (2), and the first equality from (1). Writing $H^c = C_J$ and $\bar{K}^r = R_I$ for sets I, J, we have $H \otimes^2 K = \ell^2(I \times J)$, so that $(H \otimes^2 K)^c = C_{I \times J}$. Then the last equality in (8) may be seen from (3.9):

$$H^c \otimes_{\min} K^c = C_I \otimes_{\min} C_J = C_I(C_J) = C_{I \times J}.$$

To see (9) note that $(\bar{K}^r \widehat{\otimes} H^c)^* \cong CB(H^c, K^c) = B(H, K)$ (as in the proof of (4), for example). Thus $\bar{K}^r \widehat{\otimes} H^c$ is the unique predual $S^1(K, H)$ of $B(H, K)$. Lastly, for (10), $CB(S^1(\ell^2_I, \ell^2_J), X)$ is completely isometric to

$$CB(R_I \widehat{\otimes} C_J, X) \cong CB(C_J, CB(R_I, X)) \cong R^w_J(C^w_I(X)) \cong \mathbb{M}_{I,J}(X),$$

using (9), (3.17), 1.2.25 twice, and 1.2.23 (8).

Finally, for (1), there is a direct proof in [17], for example, but we give an indirect one. We prove that $X \otimes_h H^c = X \widehat{\otimes} H^c$ completely isometrically, the other relation being similar. Clearly it suffices, by Proposition 3.1.20, to show that $I\colon X \otimes_h H^c \to X \widehat{\otimes} H^c$ is completely contractive. This will follow if we can show that any jointly completely contractive map $u\colon X \times H^c \to B(L)$ is completely contractive (in the sense of 3.1.11). For if the latter statement was true, take $u = \otimes :$ $X \times H^c \to X \widehat{\otimes} H^c$, this is jointly completely contractive, so completely contractive, and thus linearizes to a completely contractive linear map $I : X \otimes_h H^c \to X \widehat{\otimes} H^c$.

We may assume that $H = \ell^2_J$ for some set J. Let $v\colon X \to R^w_J(B(L)) = B(L^{(J)}, L)$ be the linear map defined by $v(x) = (u(x, e_i))_{i \in J}$ for any $x \in X$. As in the proof of (10), there is a sequence of isometries

$$CB(X \widehat{\otimes} C_J, B(L)) = CB(X, CB(C_J, B(L))) = CB(X, R^w_J(B(L)))$$

provided by (3.17) and Proposition 1.2.25, and it is easy to check that the composition of these maps takes u (viewed as an element of $CB(X \widehat{\otimes} C_J, B(L))$) to v. Thus $\|v\|_{\rm cb} = \|u\|_{\rm jcb} \leq 1$. Then we define a map $w\colon C_J \to C_J(B(L)) \subset B(L, L^{(J)})$ by $w(\zeta) = (\zeta_j I_L)$ for $\zeta = (\zeta_j) \in \ell^2_J$. It is clear that $\|w\|_{\rm cb} = 1$. Also, we have a factorization

$$u(x,\zeta) = \sum_j u(x,e_i)\zeta_i I_L = u(x, \sum_j e_i\zeta_i) = v(x)w(\zeta), \qquad x \in X, \zeta = (\zeta_i) \in H = \ell^2_J.$$

But any such product of two completely contractive linear maps, is clearly a completely contractive bilinear map in the sense of 3.1.11) (this is also the easy part of the later result Theorem 3.2.6). Thus u is completely contractive. ☐

Historical notes: This section is an expansion of [4, Section 1.5]; historical attributions are given there. The proof given here of the injectivity of the Haagerup tensor product is from [8]; the original source is [24].

Exercises.

(1) Show that the following map $\rho : R_n \otimes_h X^* \otimes_h C_n \to M_n(X)^*$ is a surjective complete isometry:

$$\rho(\vec{r} \otimes \varphi \otimes \vec{c})([x_{ij}]) = \vec{r}\,[\varphi(x_{ij})]\,\vec{c}, \qquad \vec{r} \in R_n, \vec{c} \in C_n, \varphi \in X^*, [x_{ij}] \in M_n(X).$$

[Hint: Show that ρ is a complete contraction, and that ρ^* is the composition of the canonical complete isometries $(R_n \otimes_h X^* \otimes_h C_n)^* \cong CB(X^*, M_n) \cong M_n(X^{**}) \cong M_n(X)^{**}$.]

3.2 Properties of completely bounded maps

We recall that the Fourier algebras $A(G)$ and $B(G)$ are *completely contractive Banach algebras.* We recall that a completely contractive Banach algebra is a Banach algebra and an operator space for which the multiplication map yields a complete contraction $A \widehat{\otimes} A \to A$. Equivalently,

$$\|[a_{ij}b_{kl}]\| \leq \|[a_{ij}]\|\,\|[b_{kl}]\|.$$

For such algebras the usual version of left and right multipliers are not relevant: one needs them also to be completely bounded.

Similarly, the operator space versions of Banach modules and bimodules are important. These satisfy similar norm conditions. Note however that it is very important which tensor product you use. If one uses the Haagerup tensor product one gets a completely different class, which is just as important. Note that a unital Banach algebra and an operator space for which the multiplication map yields a complete contraction $A \otimes_h A \to A$, are exactly (up to completely isometric isomorphism) the *operator algebras*–the closed subalgebras of operators on a Hilbert space. This is the *Blecher-Ruan-Sinclair theorem.*

Theorem 3.2.1. (The Wittstock Hahn-Banach extension theorem) If X is a closed subspace of an operator space Y, if H, K are Hilbert spaces, and if $u : X \to B(K, H)$ is completely contractive, then there exists a completely contractive $\hat{u} : Y \to B(K, H)$ with $\hat{u}_{|X} = u$.

Proof. We identify $K \cong \ell^2_J$ and $H \cong \ell^2_I$, for sets I, J, so that $B(K, H) \cong \mathbb{M}_{I,J}$. Now $CB(X, \mathbb{M}_{I,J}) \cong (R_I \otimes^{\rm h} X \otimes^{\rm h} C_J)^*$ isometrically, via the map that takes $v \in CB(X, \mathbb{M}_{I,J})$ to the functional taking $r \otimes x \otimes c$ to $rv(x)c$, for $r \in R_I, c \in C_J, x \in X$ (see 3.1.21 (4)). Thus u corresponds to a contractive functional φ in the latter space. By the injectivity of $\otimes_h$, we have $R_I \otimes^{\rm h} X \otimes^{\rm h} C_J \subset R_I \otimes^{\rm h} Y \otimes^{\rm h} C_J$. By the usual Hahn-Banach theorem, φ extends to a contractive functional $\hat{\varphi} \in (R_I \otimes^{\rm h} Y \otimes^{\rm h} C_J)^*$, and by the above, this corresponds to a complete contraction $\hat{u} : Y \to \mathbb{M}_{I,J}$. We have

$$r\hat{u}(x)c = \hat{\varphi}(r \otimes x \otimes c) = \varphi(r \otimes x \otimes c) = ru(x)c, \qquad r \in R_I, c \in C_J, x \in X.$$

Thus, $\hat{u}(x) = u(x)$ for $x \in X$. □

3.2.2 (Injective spaces) An operator space Z is said to be *injective* if for any completely bounded linear map $u\colon X \to Z$ and for any operator space Y containing X as a closed subspace, there exists a completely bounded extension $\hat{u}\colon Y \to Z$ such that $\hat{u}_{|X} = u$ and $\|\hat{u}\|_{\rm cb} = \|u\|_{\rm cb}$. A similar definition exists for Banach spaces. Thus an operator space (resp. Banach space) is injective if and only if it is an 'injective object' in the category of operator (resp. Banach) spaces and completely contractive (resp. contractive) linear maps.

We remark that one version of the Hahn–Banach theorem may be formulated as the statement that $\mathbb{C}$ is injective (as a Banach space). It follows from Theorem 3.2.1 that:

Theorem 3.2.3. If H and K are Hilbert spaces then $B(K, H)$ is an injective operator space.

Corollary 3.2.4. An operator space is injective if and only if it is linearly completely isometric to the range of a completely contractive idempotent map on $B(H)$, for some Hilbert space H.

Proof. ($\Rightarrow$) Supposing $X \subset B(H)$, extend I_X to a complete contraction $P : B(H) \to X$. Clearly $P \circ P = P$.
($\Leftarrow$) Follows from 3.2.3 and an obvious diagram chase (we leave the details as an exercise). □

Theorem 3.2.5. (Representation of completely bounded maps) Suppose that X is a subspace of a C^*-algebra B, that H and K are Hilbert spaces, and that $u\colon X \to B(K, H)$ is a completely bounded linear map. Then there exists a Hilbert space L, a $*$-representation $\pi\colon B \to B(L)$ (which may be taken to be unital if B is unital), and bounded operators $S\colon L \to H$ and $T\colon K \to L$, such that $u(x) = S\pi(x)T$ for all $x \in X$. Moreover this can be done with $\|S\|\|T\| = \|u\|_{\rm cb}$.

Conversely, any linear map u of the form $u = S\pi(\cdot)T$ as above, is completely bounded with $\|u\|_{\rm cb} \le \|S\|\|T\|$.

Proof. We may suppose that u is completely contractive. In the notation of Lemma 2.1.12, u is a 'corner' of a completely positive unital map Θ defined from a subsystem of $M_2(B)$ into $B(H \oplus K)$. By the extension theorem of Wittstock 3.2.1, one can extend Θ to a unital completely positive map $M_2(B) \to B(H \oplus K)$. This in turn, by Stinespring's theorem, equals $V^*\pi(\cdot)V$, for a unital representation π of $M_2(B)$ on another Hilbert space. It is quite easy algebra to see that any unital representation of $M_2(B)$ on a Hilbert space E gives rise to a unitary operator U from that Hilbert space onto $L \oplus L$, for a subspace L of E, and a unital representation π of B on L, such that the first representation equals $[a_{ij}] \mapsto U^*[\pi(a_{ij})]U$, for $[a_{ij}] \in M_2(B)$. In our case, we obtain

$$\begin{bmatrix} 0 & u(x) \\ 0 & 0 \end{bmatrix} = \Theta\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = V^*U^*\begin{bmatrix} 0 & \pi(x) \\ 0 & 0 \end{bmatrix}UV = W'\pi(x)W,$$

where $W = [0 \;\; I]UV$, with a similar formula defining W'. Pre- and post-multiplying by the projection from $H \oplus H$ onto H, and the inclusion from H into $H \oplus H$, gives $u = S\pi(\cdot)T$, for appropriate contractions S, T.

The last assertion we leave as an easy exercise using Proposition 1.2.6, and Exercise (2) of Section 2.1. ☐

The following important result states that any completely bounded bilinear map may be factorized as a product of two completely bounded linear maps. It is due to Christensen and Sinclair (the C^*-algebra case), and Paulsen and Smith (the general case. Note that their injectivity of the Haagerup tensor product (Theorem 3.1.15) immediately reduces the general case to the C^*-algebra case, as we shall see).

Theorem 3.2.6. Suppose that X and Y are operator spaces, and that $u\colon X \times Y \to B(K, H)$ is a bilinear map.

(1) u is completely contractive (as a bilinear map) if and only if there is a Hilbert space L, and there are completely contractive linear maps $v\colon X \to B(L, H)$ and $w\colon Y \to B(K, L)$, with $u(x, y) = v(x)w(y)$ for all $x \in X$ and $y \in Y$.

(2) If X and Y are subspaces of unital C^*-algebras A and B respectively, and if the conditions in (1) hold, then there exist Hilbert spaces K_1 and K_2, unital $*$-representations $\pi\colon A \to B(K_1)$ and $\rho\colon B \to B(K_2)$, and contractions $T \in B(K, K_2), S \in B(K_2, K_1)$ and $R \in B(K_1, H)$, such that

$$u(x, y) \;=\; R\pi(x)S\rho(y)T, \qquad x \in X,\, y \in Y.$$

There are half a dozen or more proofs of this result in the literature, which we describe some of. First, note that the second part of this result follows immediately from the first part and 3.2.5. Second, note that one may assume that X and Y are C^*-algebras. To see this, suppose that X and Y are subspaces of C^*-algebras A and B respectively. If $\tilde{u} : X \otimes_h Y \to B(K, H)$ is the associated linear map, then since $X \otimes_h Y \subset A \otimes_h B$, by Wittstock's extension theorem 3.2.1 we can extend $\tilde{u}$ to a completely contractive linear map on $A \otimes_h B$. This yields a completely contractive bilinear map on $A \times B$. If the C^*-algebra case holds, then this gives the desired result for A and B, and by restriction, for X and Y.

The next observation (used in some proofs), is that one may replace the 'target space' $B(K,H)$ by $\mathbb{C}$, by the same trick used in the proof of Theorem 3.2.1. Indeed, as in that proof we have

$$CB(X \otimes_h Y, B(K,H)) \cong (R_I \otimes_h (X \otimes_h Y) \otimes_h C_J)^* \cong (X' \otimes_h Y')^*,$$

where $X' = R_I \otimes_h X$ and $Y' = Y \otimes_h C_J$. If the result were true in the scalar valued case (and by the last paragraph we may assume that X', Y' are C^*-algebras), then we see that u corresponds to a product $v(x)w(y)$, for complete contractions $w : Y \otimes_h C_J \to B(\mathbb{C}, L) = L^c$, and $v : R_I \otimes_h X \to B(L, \mathbb{C}) = \bar{L}^r$. Using Proposition 3.1.21 (1), (3.18), and (1.12) and the matching 'row space' formula found a few paragraphs below that reference, w induces a complete contraction $w' : Y \to CB(C_J, L^c) \cong B(H,L)$, and v induces a complete contraction $v' : X \to CB(R_I, \bar{L}^r) \cong B(L,H)$. It is easy to check that $u(x,y) = v'(x)w'(y)$, which proves the result.

At this point, proofs of Theorem 3.2.6 that use the route of the last paragraph now have to characterize elements of $(A \otimes_h B)^*$. Two different approaches to this may be found in [17, Section 9.4] and [9]. The former uses a geometric Hahn-Banach separation argument similar to the proof of Lemma 1.2.10 (appearing first in unpublished work of Haagerup [19]). The latter crucially uses a very useful notion due to Roger Smith called *strong independence* of vectors in a Hilbert space [30]. Other proofs may be found in [23, 27, 29], and of course the original papers.

3.2.7 (Completely bounded bilinear functionals) If we have a completely contractive bilinear $u \colon X \times Y \to \mathbb{C}$, by Theorem 3.2.6 we may write $u(x,y) = v(x)w(y)$. Here H is a Hilbert space, and $w \colon Y \to B(\mathbb{C},H) = H^c$ and $v : X \to B(H,\mathbb{C}) = \bar{H}^r$ are completely contractive linear maps. Regarding these as mapping into H and $\bar{H}$ respectively, we have

$$u(x,y) = \langle w(y), \overline{v(x)} \rangle_H, \qquad x \in X, y \in Y.$$

Or, if $H = \ell^2_I$, then we may regard w, v as complete contractions $Y \to C_I$ and $v : X \to R_I$ respectively, and then $u(x,y) = v(x)w(y)$, where the multiplication occurring here is simply multiplying a row matrix by a column matrix.

This can be rewritten in another important way. Note that $v^* : B(H,\mathbb{C})^* \to X^*$ is a complete contraction, and hence so is $s = v^* \circ \theta$, where $\theta : H^c \to B(H,\mathbb{C})^*$ is the canonical complete isometry (see (1.13)). Also,

$$s(w(y))(x) = \theta(w(y))(v(x)) = v(x)w(y) = u(x,y) = \tilde{u}(y)(x),$$

where $\tilde{u} : Y \to X^*$ is the canonical linear map associated with u. Thus $\tilde{u} = s \circ w$ is a 'factorization' of $\tilde{u}$ 'through' H^c:

$$Y \longrightarrow H^c \longrightarrow X^*.$$

The steps here are reversible, that is, if $u \colon X \times Y \to \mathbb{C}$ and if $\tilde{u} = s \circ w$, where H is a Hilbert space, and $w \colon Y \to H^c$ and $s : H^c \to X^*$ are completely contractive linear maps, then u is a completely contractive bilinear functional. That is, u corresponds in the usual way to an element of $\mathrm{Ball}((X \otimes_h Y)^*)$. This is a characterization of

Ball($(X \otimes_h Y)^*$), or, if you like, of $(X \otimes_h Y)^*$. Summarizing, such functionals are 'nothing but' the maps $X \to Y^*$ which 'factor completely contractively through' H^c.

3.2.8 (Further remarks on Theorem 3.2.6) An analoguous result to (1) of Theorem 3.2.6 holds for multilinear completely bounded maps. Thus if $X_1, \ldots, X_m$ are operator spaces and if $v_j : X_j \to B(H_j, H_{j-1})$ are completely contractive linear maps then the N-linear mapping taking $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$ to $v_1(x_1) \cdots v_m(x_m) \in B(H_m, H_0)$ is easily seen to be completely contractive in the sense of 3.1.11. Conversely, any completely contractive m-linear map $u : X_1 \times X_2 \times \ldots \times X_m \to B(K, H)$ has this form. The proof of this latter assertion proceeds by induction on m. Assume that it is true for $k = 2$ and $k = m - 1$. We have an associated completely bounded linear map defined on $X_1 \otimes_h X_2 \otimes_h \ldots \otimes_h X_m \cong X_1 \otimes_h (X_2 \otimes_h \ldots \otimes_h X_m)$ (the latter by a fact from the discussion on associativity in 3.1.13). By the $k = 2$ case, this map may be factorized as: $x_1 \otimes (x_2 \otimes \cdots \otimes x_m) \mapsto v_1(x_1) w(x_2 \otimes \cdots \otimes x_m)$. By the $k = m - 1$ case, $w(x_2 \otimes \cdots \otimes x_m) = v_2(x_2) \cdots v_m(x_m)$. Thus u is of the required form.

Likewise, (2) of Theorem 3.2.6 has an analogous formulation for multilinear maps, which follows immediately from (1) and Theorem 3.2.5.

Historical notes: The proof of Wittstock's Theorem 3.2.1 given here is the modification from [8] of a proof due to Effros. Theorem 3.2.5 was first proved by Haagerup in unpublished work [18] from 1980. There are some interesting historical anecdotes in this handwritten manuscript concerning this result, and related topics. The first published proof was the simple one that Paulsen found [22], and this is the one given here.

3.3 Duality and tensor products of dual spaces

3.3.1 (Mapping spaces as duals) If Y is a dual operator space then we saw in Corollary 3.1.17 that so is $CB(X, Y)$, for any operator space X. Indeed by (3.18) an explicit predual for $CB(X, Y)$ is $X \widehat{\otimes} Y_*$. From this, together with the density of the finite rank tensors in $X \widehat{\otimes} Y_*$, and the general Banach space convergence principle in the proof of Lemma 1.3.8, it follows that a bounded net $(u_t)_t$ in $CB(X, Y)$ converges in the w^*-topology to a $u \in CB(X, Y)$ if and only if $u_t(x)(z) \to u(x)(z)$ for all $x \in X, z \in Y_*$. That is, if and only if $u_t(x)$ converges in the w^*-topology to $u(x)$ in Y for all $x \in X$. Next, suppose that $Y = B(K, H)$ for Hilbert spaces H, K. Since the latter net is bounded, it follows from the fact that the weak* topology coincides with the WOT on bounded sets in $B(K, H)$, that the above equivalent conditions are also equivalent to

$$\langle u_t(x)\zeta, \eta\rangle \;\to\; \langle u(x)\zeta, \eta\rangle \quad \text{for all} \quad x \in X, \zeta \in K, \eta \in H. \tag{3.19}$$

It is easy to see that the latter condition is equivalent to the same condition, but with η, ζ arbitrary elements of a orthonormal basis for H and for K.

3.3.2 (Dual matrix spaces) If X is a dual operator space then so is $M_n(X)$. Indeed by (3.18) and (1.6) we have

$$(S_n^1 \widehat{\otimes} X_*)^* \cong CB(X_*, M_n) \cong M_n(X).$$

More generally the same proof, but substituting 1.2.23 (12) for (1.6), shows that for sets I, J, $\mathbb{M}_{I,J}(X)$ is a dual operator space with operator space predual $S^1(\ell_I^2, \ell_J^2) \widehat{\otimes} X_*$, and also $\mathbb{M}_{I,J}(X) \cong CB(X_*, \mathbb{M}_{I,J})$. Alternatively, note that by (3.18) and 3.1.21 (10), we have

$$(S^1(\ell_I^2, \ell_J^2) \widehat{\otimes} X_*)^* \cong CB(S^1(\ell_I^2, \ell_J^2), X) \cong \mathbb{M}_{I,J}(X).$$

If I, J are sets, and if I_0 and J_0 are finite subsets of I and J respectively, write $\Delta = I_0 \times J_0$. The set Λ of such Δ is a directed set under the usual ordering. For such Δ, and for $x \in \mathbb{M}_{I,J}(X)$, we write x_Δ for the matrix x, but with entries x_{ij} switched to zero if $(i,j) \notin \Delta$. Then $(x_\Delta)_\Delta$ is a net indexed by $\Delta \in \Lambda$, which we call *the net of finite submatrices of* x.

Corollary 3.3.3. (Effros and Ruan) Let X be a dual operator space, and let I, J be sets.

(1) If $(x_t)_t$ is a bounded net in $\mathbb{M}_{I,J}(X)$, then $x_t \to x \in \mathbb{M}_{I,J}(X)$ in the w^*-topology in $\mathbb{M}_{I,J}(X)$, if and only if each entry in x_t converges in the w^*-topology in X to the corresponding entry in x.
(2) If Y is a dual operator space, and if $u\colon X \to Y$ is a w^*-continuous completely bounded map, then the amplification $u_{I,J}\colon \mathbb{M}_{I,J}(X) \to \mathbb{M}_{I,J}(Y)$ is w^*-continuous.
(3) $\mathbb{M}_{I,J}^{\text{fin}}(X)$ is w^*-dense in $\mathbb{M}_{I,J}(X)$. Indeed if I, J are sets, and $x \in \mathbb{M}_{I,J}(X)$, then the net of finite submatrices of x converges to x in the w^*-topology.

Proof. As we said in (3.3.2), $\mathbb{M}_{I,J}(X) = CB(X_*, \mathbb{M}_{I,J}) = CB(X_*, B(\ell_J^2, \ell_I^2))$. By (3.19) and the remark after it, it follows that a bounded net $x^s \to x \in \mathbb{M}_{I,J}(X) \cong CB(X_*, B(\ell_J^2, \ell_I^2))$ if and only if

$$\langle [x_{i,j}^s(\varphi)] e_j, e_i \rangle = x_{i,j}^s(\varphi) \to \langle [x_{i,j}(\varphi)] e_j, e_i \rangle = x_{i,j}(\varphi), \qquad \varphi \in X_*,$$

that is, if and only if $x_{i,j}^s \to x_{i,j}$ weak*, for all $i \in I, j \in J$. This is (1).

Items (2) and (3) follow immediately from (1). For example, if $u\colon X \to Y$ is w^*-continuous, and if we have a bounded net $x^s \to x \in \mathbb{M}_{I,J}(X)$, then by (1) each entry of x^s converges weak* to the corresponding entry of x. Also, $(u_{I,J}(x^s))$ is a bounded net in $\mathbb{M}_{I,J}(X)$, and it converges to $u_{I,J}(x)$ by (1) again, since each entry of $u_{I,J}(x^s)$ converges weak* to the corresponding entry of $u_{I,J}(x)$. Thus $u_{I,J}$ is w^*-continuous as a consequence of the Krein-Smulian theorem (namely, a linear bounded map $u\colon E \to F$ between dual Banach spaces is w*-continuous if and only if whenever $x_t \to x$ is a bounded net converging in the w*-topology in E, then $u(x_t) \to u(x)$ in the w*-topology). □

Note that in 3.3.3 (2), if u is also a complete isometry then by the consequence of the Krein-Smulian theorem stated in the proof of Lemma 1.3.8 above, we see that $u_{I,J}$ is a w^*-homeomorphism onto its range, which is w^*-closed. As a corollary we see that for a w^*-closed subspace $X \subset B(K,H)$, one may *define* $\mathbb{M}_{I,J}(X)$ to be the w^*-closure of $\mathbb{M}^{\text{fin}}_{I,J}(X)$ in $\mathbb{M}_{I,J}(B(K,H)) = B(K^{(J)}, H^{(I)})$. Indeed, taking u to be the embedding $X \to B(K,H)$, we see that $\mathbb{M}_{I,J}(X)$ is w^*-homeomorphically completely isometric to a w^*-closed subspace of $\mathbb{M}_{I,J}(B(K,H))$. Applying (3) of the last result we deduce the statement about $\mathbb{M}^{\text{fin}}_{I,J}(X)$.

We turn next to a characterization of dual operator spaces:

Theorem 3.3.4. Let X be an operator space with a given weak* topology (coming from a predual Banach space). The following are equivalent:

(i) X with its given weak* topology is a dual operator space.
(ii) $M_n(X)$ is a dual Banach space, and the n^2 canonical inclusion maps from X into $M_n(X)$ are w^*-continuous, for all $n \geq 2$.
(iii) Whenever $(x^s)_s$ is a net in $\text{Ball}(M_n(X))$, $x \in M_n(X)$, and the i-j entry of x^s converges in the weak* topology to the i-j entry of x for all $i, j = 1, \ldots, n$, then $x \in \text{Ball}(M_n(X))$.

Proof. Write $\epsilon_{ij} : X \to M_n(X)$ for the 'i-j inclusion map'.

(i) $\Rightarrow$ (ii) If $x_t \to x$ weak* in X then by Theorem 3.3.3 (i) we have $\epsilon_{ij}(x_t) \to \epsilon_{ij}(x)$ weak*. So ϵ_{ij} is w^*-continuous.

(ii) $\Rightarrow$ (iii) If x^s, x are as in (iii), with $x^s_{ij} \to x_{ij}$ weak* for all i, j, then by (ii) we have $\epsilon_{ij}(x^s_{ij}) \to \epsilon_{ij}(x_{ij})$ weak*, so that

$$x^s = \sum_{i,j=1}^{n} \epsilon_{ij}(x^s_{ij}) \to \sum_{i,j=1}^{n} \epsilon_{ij}(x_{ij}) = x$$

weak* in $M_n(X)$. If $x^s \in \text{Ball}(M_n(X))$, then since the latter ball is weak* closed, it follows that $x \in \text{Ball}(M_n(X))$.

(iii) $\Rightarrow$ (i) We may suppose that the predual Banach space $W \subset X^*$. We will always regard W as an operator space by giving it the inherited matrix norms from X^*. We will use Exercise (1) of Section 2.4, namely that the following canonical map $\rho : R_n \otimes_h X^* \otimes_h C_n \to M_n(X)^*$ is a surjective complete isometry:

$$\rho(\vec{r} \otimes \varphi \otimes \vec{c})([x_{ij}]) = \vec{r}[\varphi(x_{ij})]\vec{c}, \qquad \vec{r} \in R_n, \vec{c} \in C_n, \varphi \in X^*, [x_{ij}] \in M_n(X).$$

Alternatively, this fact can be proved from the later result (3.21), since using that result and 3.1.21 (6), we have $R_n \otimes_h X^* \otimes_h C_n \cong (C_n \otimes_h X \otimes_h R_n)^* \cong M_n(X)^*$. Note that

$$\rho(\vec{e_k} \otimes \varphi \otimes \vec{e_l})([x_{ij}]) = \varphi(x_{kl}), \qquad \varphi \in X^*, [x_{ij}] \in M_n(X), k, l \in \{1, \ldots, n\}.$$

Since $\otimes_h$ is injective we deduce that $R_n \otimes_h W \otimes_h C_n \subset R_n \otimes_h X^* \otimes_h C_n \cong M_n(X)^*$ isometrically. Define $\theta : M_n(X) \to (R_n \otimes_h W \otimes_h C_n)^*$ by $\theta(x)(u) = \rho(u)(x)$, for

$x \in M_n(X), u \in R_n \otimes_h W \otimes_h C_n$. Note that $\theta(x)(\vec{e_k} \otimes w \otimes \vec{e_l}) = w(x_{kl})$, for $w \in W$, by the last centered equation. From this it is clear that θ is one-to-one, and it is easy to see that it is onto. Note that $\theta(\mathrm{Ball}(M_n(X)))$ is w^*-closed by hypothesis, for if $\theta(x^s) \to \theta(x)$ weak* in $(R_n \otimes_h W \otimes_h C_n)^*$, with $\|x^s\| \leq 1$, then by the last line, $w(x_{kl}^s) \to w(x_{kl})$ for all $w \in W$, and k, l. That is, $x_{kl}^s \to x_{kl}$ weak*, so that $\|x\| \leq 1$ by (iii).

If $u \in R_n \otimes_h W \otimes_h C_n$ then $\|u\| = \|\rho(u)\|$, which equals

$$\sup\{\|\rho(u)(x)\| : x \in \mathrm{Ball}(M_n(X))\} = \sup\{\|\theta(x)(u)\| : x \in \mathrm{Ball}(M_n(X))\}.$$

Thus the pre-polar $\theta(\mathrm{Ball}(M_n(X)))_\circ$ equals the unit ball of $R_n \otimes_h W \otimes_h C_n$. Therefore by the bipolar theorem, $\theta(\mathrm{Ball}(M_n(X))) = \mathrm{Ball}((R_n \otimes_h W \otimes_h C_n)^*)$. That is, θ is an isometry. The composition of θ with the canonical isometries $(R_n \otimes_h W \otimes_h C_n)^* \cong CB(W, M_n) = M_n(W^*)$ from 3.1.21 (4) and (1.6), is the nth amplification of the isometry $X \to W^*$. Since this holds for any $n \geq 1$, the latter map is a complete isometry. Thus $W^* = X$ completely isometrically. $\square$

3.3.5 (Normal spatial tensor product) If X and Y are dual operator spaces, with operator space preduals X_* and Y_*, then $CB(Y_*, X)$ is the dual operator space of $X_* \widehat{\otimes} Y_*$ by 3.3.1. As in (3.6), we regard $X \otimes_{\min} Y \hookrightarrow CB(Y_*, X)$, and we define the *normal minimal tensor product* $X \overline{\otimes} Y$ to be the w^*-closure of $X \otimes Y$ (or of $X \otimes_{\min} Y$) in $CB(Y_*, X)$. Equivalently, if X and Y are w^*-closed subspaces of $B(H)$ and $B(K)$ respectively, then we may define $X \overline{\otimes} Y$ to be the w^*-closure in $B(H \otimes^2 K)$ of the copy of $X \otimes Y$. This is sometimes referred to as the *normal spatial tensor product*. If M and N are W^*-algebras, then $M \overline{\otimes} N$ as described above is clearly a von Neumann algebra; and indeed $M \overline{\otimes} N$ is just the usual von Neumann algebra tensor product. In particular, $B(H) \overline{\otimes} B(K) = B(H \otimes^2 K)$, since the former contains as a weak* dense subset $\mathbb{K}(H) \otimes_{min} \mathbb{K}(K)$ (or even the tensor product of the finite rank operators). To see that these two definitions of $X \overline{\otimes} Y$ are the same (up to w^*-homeomorphic complete isometry), we use the following argument. Since X and Y are w^*-closed subspaces of $B(H)$ and $B(K)$ respectively, we know by 1.3.7 that X_* and Y_* are quotients of $S^1(H)$ and $S^1(K)$ respectively. By the 'projectivity' property of $\widehat{\otimes}$, we obtain a complete quotient map $Q \colon S^1(H) \widehat{\otimes} S^1(K) \to X_* \widehat{\otimes} Y_*$. Using the identification (3.18) we see that Q^* may be viewed as a w^*-continuous completely isometric embedding

$$CB(Y_*, X) \;\hookrightarrow\; CB(S^1(K), B(H)) \;\cong\; B(H \otimes^2 K),$$

the last relation from the first paragraph of 3.1.6. Via the canonical identification of $X \otimes Y$ with a subset of $CB(Y_*, X)$, it is easy to argue that the w^*-closure of $X \otimes Y$ in $B(H \otimes^2 K)$ may be identified with the w^*-closure of $X \otimes Y$ in $CB(Y_*, X)$.

In general, $CB(Y_*, X)$ (or equivalently, $(X_* \widehat{\otimes} Y_*)^*$) is not equal to $X \overline{\otimes} Y$; nonetheless they do coincide in many cases of interest. In fact, we have

Theorem 3.3.6. (Blecher, Ruan) For any dual operator spaces X and Y we have

$$(X_* \widehat{\otimes} Y_*)^* \cong CB(Y_*, X) \cong X \otimes_{\mathfrak{F}} Y,$$

where the latter is the *normal Fubini product* of X and Y.

The normal Fubini product is defined in terms of 'slice maps', and thus the question of whether $X \overline{\otimes} Y = CB(Y_*, X)$ holds, is related to *slice map properties.*

Corollary 3.3.7. (Effros-Ruan von Neumann tensor product formula) If M and N are von Neumann algebras then the operator space predual of $M \overline{\otimes} N$ is $M_* \widehat{\otimes} N_*$. Thus

$$M \overline{\otimes} N \cong (M_* \widehat{\otimes} N_*)^* \cong CB(N_*, M).$$

Proof. (Sketch) By Theorem 3.3.6 this is equivalent to showing that $M \overline{\otimes} N = M \otimes_{\mathfrak{F}} N$. The latter is defined in terms of 'slice maps', and the crux of the proof is applying Tomiyama's slice map theorem for von Neumann algebras, which in turn is a consequence of the deep fact $(M \overline{\otimes} N)' = M' \overline{\otimes} N'$. □

3.3.8 (Application to Fourier algebras) The last theorem has a very important influential application to the Fourier algebra $A(G)$, of a compact group G say, that was an early triumph of operator space theory. Quoting from Nico Spronk's course: "Let us recall the happy fact that $A(G)^* = VN(G)$. Now we have a unitary equivalence $L^2(G) \otimes^2 L^2(K) \cong L^2(G \times K)$ which intertwines $\lambda_G \times \lambda_H \cong \lambda_{G \times H}$, and hence gives us a spatial equivalence

$$VN(G) \bar{\otimes} VN(K) \cong VN(G \times K).$$

Hence the Effros-Ruan tensor product formula gives us (completely isometrically)

$$A(G) \widehat{\otimes} A(K) \cong A(G \times K).$$

We discuss two more examples of when $X \overline{\otimes} Y = CB(Y_*, X)$ holds. We have $X \overline{\otimes} Y = CB(Y_*, X)$ holds when $X = B(H, K)$. Indeed, for any dual operator space Y and sets I, J,

$$\mathbb{M}_{I,J} \overline{\otimes} Y \cong \mathbb{M}_{I,J}(Y) \tag{3.20}$$

as dual operator spaces. This follows by the remark after 3.3.3, and an argument similar to the one used for (3.8). Also, $\mathbb{M}_{I,J}(Y) \cong CB(Y_*, \mathbb{M}_{I,J})$ by 1.2.23 (12) (setting one of the spaces there equal to $\mathbb{C}$). Thus $\mathbb{M}_{I,J} \overline{\otimes} Y \cong CB(Y_*, \mathbb{M}_{I,J})$. Note that taking J singleton, and $Y = \bar{K}_r$, gives $H^c \overline{\otimes} \bar{K}^r \cong B(K, H)$.

Finally the relation also holds when one of X or Y is a commutative von Neumann algebra. We omit the simple proof (see e.g. [4, Chapter 1]).

We leave it as an exercise that the normal spatial tensor product is 'associative', and 'functorial' for w^*-continuous completely bounded maps.

3.3.9 (W^*-continuous extensions of bilinear maps) Let X, Y be operator spaces, let W be a dual operator space, and let $u \colon X \times Y \to W$ be a completely contractive bilinear map. We claim that there is a unique separately w^*-continuous extension $\tilde{u} \colon X^{**} \times Y^{**} \to W$ of u, and this extension is completely contractive too. To prove this, we may assume by Lemma 1.3.8 that W is a w^*-closed subspace of some $B(H)$. By the Theorem 3.2.6, there exists a Hilbert space L and two completely contractive maps $v \colon X \to B(L, H)$ and $w \colon Y \to B(H, L)$ such that $u(x, y) = v(x)w(y)$ for all $x \in X$, $y \in Y$. By 1.3.9, v and w have w^*-continuous completely contractive extensions $\tilde{v}\colon X^{**} \to B(L,H)$ and $\tilde{w}\colon Y^{**} \to B(H,L)$. Define $\tilde{u}\colon X^{**} \times Y^{**} \to B(H)$ by setting $\tilde{u}(\eta,\nu) = \tilde{v}(\eta)\tilde{w}(\nu)$, for $\eta \in X^{**}$, $\nu \in Y^{**}$. Then the easy part of Theorem 3.2.6 ensures that $\tilde{u}$ is completely contractive, and it clearly is separately w^*-continuous, and extends u. Note that for any separately w^*-continuous extension $\tilde{u}\colon X^{**} \times Y^{**} \to B(H)$ of u, we must have

$$\tilde{u}(\eta,\nu) = \lim_s \lim_t u(x_t, y_s), \qquad \text{if}\, x_t \to \eta, y_s \to \nu,$$

where all the limits here are in the weak* topology. From this we see that $\tilde{u}(\eta,\nu) \in W$, and also the uniqueness of the extension.

3.3.10 (Self-duality of $\otimes_h$) Let X and Y be operator spaces. Then

$$X^* \otimes_h Y^* \hookrightarrow (X \otimes_h Y)^* \quad \text{completely isometrically} \tag{3.21}$$

via the canonical map J (that is, $J(\varphi \otimes \psi)(x \otimes y) = \varphi(x)\psi(y)$). To prove this, we first assume that X and Y are finite-dimensional. In this case, J is a surjection, by linear algebra. An element U in the ball of $M_n((X \otimes_h Y)^*)$ corresponds by (1.6) to a complete contraction $u : X \otimes_h Y \to M_n$. By 3.2.6 (1), there exist a Hilbert space L and two complete contractions $v\colon X \to B(L,\mathbb{C}^n)$ and $w\colon Y \to B(\mathbb{C}^n, L)$ such that $u(x \otimes y) = v(x)w(y)$ for any $x \in X$ and $y \in Y$. We may assume that L is finite dimensional, by replacing L by its finite dimensional subspace $[w(Y)\mathbb{C}^n]$, and restricting $v(x)$ to that subspace. Thus we may assume that $v\colon X \to M_{n,p}$ and $w\colon Y \to M_{p,n}$, for an integer $p \geq 1$. We let $\varphi = [\varphi_{ij}] \in M_{n,p}(X^*)$ and $\psi = [\psi_{ij}] \in M_{p,n}(Y^*)$ be the two matrices corresponding to v and w respectively (by a variant of (1.6), these have norm ≤ 1. Then

$$u(x \otimes y) = v(x)w(y) = [\varphi_{ij}(x)][\psi_{ij}(y)] = \Big[\sum_k \varphi_{ik}(x)\psi_{kj}(y)\Big], \qquad x \in X, y \in Y.$$

If $z = \varphi \odot \psi$ then it is easy to see that $J_n(z) = U$, and $\|z\|_{\mathrm{h}} \leq \|\varphi\|\|\psi\| = \|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 1$. The converse inequality $\|U\|_{\mathrm{cb}} \leq \|z\|_{\mathrm{h}}$ may be obtained by reversing the arguments.

In the general case, fix $[u_{ij}] \in M_n(X^* \otimes Y^*)$. Write each $u_{ij} \in X^* \otimes Y^*$ in the form $\sum_{k=1}^m \varphi_k \otimes \psi_k$, for functionals $\varphi_k \in X^*, \psi_k \in Y^*$. Let W (resp. Z) be the span of all these (finite number of) functionals in X^* (resp. Y^*), over all i and j too. Then $W \cong (X/W_\perp)^*$ and $Z \cong (Y/Z_\perp)^*$ isometrically. The canonical maps $X \to X/W_\perp$ and $Y \to Y/Z_\perp$ induce a complete quotient map $X \otimes_h Y \to (X/W_\perp) \otimes_h (Y/Z_\perp)$, by the projectivity of $\otimes_h$ (see the third bullet in 3.1.13). By 1.3.3, the last map dualizes to give a weak* continuous complete isometry $((X/W_\perp) \otimes_h (Y/Z_\perp))^* \to (X \otimes_h Y)^*$. On the other hand, by the last paragraph, $((X/W_\perp) \otimes_h (Y/Z_\perp))^* \cong W \otimes_h Z$ completely isometrically. Thus we have the following diagram of completely isometries (the vertical arrow coming from the injectivity of $\otimes_h$, see 3.1.13):

$$\begin{array}{ccccc} M_n(X^* \otimes Y^*) & & & & \\ \uparrow & & & & \\ M_n(W \otimes_h Z) & \longrightarrow & M_n(((X/W_\perp) \otimes_h (Y/Z_\perp))^*) & \longrightarrow & M_n((X \otimes_h Y)^*). \end{array}$$

We may view $[u_{ij}]$ in $M_n(W \otimes_h Z)$. The composition of the maps in the last sequence is easily seen to take $[u_{ij}]$ to $[J(u_{ij})] \in M_n((X \otimes_h Y)^*)$, and so we are done.

3.3.11 (The dual of the Haagerup tensor product) If X and Y are operator spaces then $w \in (X \otimes_h Y)^*$ if and only if there exist $[\varphi_i] \in R_I^w(X^*)$ and $[\psi_i] \in C_I^w(Y^*)$ such that w may be written as

$$w(x \otimes y) = \sum_{i \in I} \varphi_i(x)\psi_i(y), \qquad x \in X, y \in Y. \tag{3.22}$$

The last sum converges absolutely in $\mathbb{C}$, as may be seen by the Cauchy–Schwarz inequality:

$$\sum_{i \in I} |\varphi_i(x)\psi_i(y)| \leq \big(\sum_{i \in I} |\varphi_i(x)|^2\big)^{\frac{1}{2}} \big(\sum_{i \in I} |\psi_i(y)|^2\big)^{\frac{1}{2}} = \|\varphi(x)\| \|\psi(y)\|,$$

where $\varphi : X \to R_I$ and $\psi : Y \to C_I$ are the canonical maps which are associated with $[\varphi_i] \in R_I^w(X^*)$ and $[\psi_i] \in C_I^w(Y^*)$. Indeed, by 1.2.23 (12), for example, φ and ψ are completely contractive iff $[\varphi_i]$ and $[\psi_i]$ have norm ≤ 1. Note that (3.22) may be rewritten as $w(x \otimes y) = \varphi(x)\psi(y)$, and now this relation may be seen to be a restatement of the discussion in 3.2.7. Indeed, this argument shows that $\|w\| \leq 1$ iff φ, ψ above may be chosen to be complete contractions, and iff $[\varphi_i]$ and $[\psi_i]$ in (3.22) have norm ≤ 1.

Thus every $w \in (X \otimes_h Y)^*$ 'is' a sum of rank one tensors

$$w = \sum_{i \in I} \varphi_i \otimes \psi_i \qquad \varphi_i \in X^* \, , \psi_i \in Y^*,$$

with $[\varphi_i] \in R_I^w(X^*), [\psi_i] \in C_I^w(X^*)$. Viewing $(X \otimes_h Y)^*$ as a tensor product of X^* and Y^* in this way, leads to the *weak* Haagerup tensor product*, often called the *extended Haagerup tensor product*, which we shall not discuss further here. It has properties analogous to the Haagerup tensor product, and was studied by Blecher and Smith, Effros and Ruan, Spronk, and others.

If time permits we will cover more on the weak∗/extended Haagerup tensor product, including some material from Spronk, Proc London Math Soc **89** (2004), 161-192.

Historical note: Nearly all the facts about infinite matrices in 3.3.2 and 3.3.3 are explicitly in [20, 13, 14]. The result 3.3.4 is due to Le Merdy [21]. See e.g. [17] for more on the normal spatial tensor product and the Fubini tensor product. The selfduality relation (3.21) was proved in full in [16]. Different proofs appear in [3, 9]. The dual of the Haagerup tensor product was first explored by Effros and Kishimoto [11] following unpublished work of Haagerup [19], viewing this dual as a tensor product originates in [9].

END OF COURSE

For more details on topics in this course, or other more advanced aspects of the theory of operator spaces, see the basic operator space texts, which can be found in the reference list below.

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