Property T for locally compact quantum groups

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(joint works with Xiao Chen and Anthony To Ming Lau)

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• M. Daws, P. Fima, A. Skalski and S. White gave the definition of property T for general locally compact quantum groups, in their paper on the Haagerup property.

• If G is a locally compact group and $\mu : G \to \mathcal{L}(\mathfrak{H})$ is a continuous unitary representation, then a net $\{\xi_i\}_{i\in I}$ of unit vectors in $\mathfrak{H}$ is called an *almost invariant unit vector (for μ)* if

$$\sup_{t\in K} \|\mu_t(\xi_i) - \xi_i\| \to 0,$$

for any compact subset $K \subseteq G$.

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• $\widehat{G}$ is the topological space of irreducible unitary representations of G.

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(T1) $C^*(G) \cong \ker \pi_{\mathbf{1}_G} \oplus \mathbb{C}$ canonically.

(T2) $\exists$ minimal projection $p \in M(C^*(G))$ such that $\pi_{\mathbf{1}_G}(p) = 1$.

(T3) $\mathbf{1}_G$ is an isolated point in $\widehat{G}$.

(T4) All fin. dim. representations in $\widehat{G}$ are isolated points.

(T5) $\exists$ a fin. dim. representation in $\widehat{G}$ which is an isolated point.

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- The equivalences of prop. T with T1-T5 for discrete quantum groups are some of the main results in the paper by Kyed and Soltan.

- (Delorme-Guichardet theorem) If G is σ-compact, then it has property T if and only if $H^1(G, \mu) = (0)$ for any continuous unitary representation μ of G.

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  - $\mu \subset \nu$ if $\exists$ isom. $V : \mathfrak{H} \to \mathfrak{K}$ s.t. $\mu(a) = V^*\nu(a)V, \ \forall a \in A$;
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- $\widehat{A} \subseteq \mathrm{Rep}(A)$: irred. rep. (equip with Fell top.)

Lemma

Let A be a C^*-algebra and $(\mu, \mathfrak{H}) \in \widehat{A}$.
(a) The following statements are equivalent.

1. *μ is an isolated point in $\widehat{A}$.*
2. *$\forall (\pi, \mathfrak{K}) \in \mathrm{Rep}(A)$ with $\mu \prec \pi$, one has $\mu \subset \pi$.*
3. *$A = \ker \mu \oplus \bigcap_{\nu \in \widehat{A} \setminus \{\mu\}} \ker \nu$.*

(b) If $\dim \mathfrak{H} < \infty$, then $\{\mu\}$ is a closed subset of $\widehat{A}$.

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• $V_{\mathbb{G}}^{\mathrm{u}} \in M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$: unitary that implements the bij. corresp. between unitary coreprens. of $C_0(\mathbb{G})$ and non-deg. *-repns. of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ through $U = (\pi_U \otimes \mathrm{id})(V_{\mathbb{G}}^{\mathrm{u}})$.

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(b) A net $\{\xi_i\}_{i \in \mathfrak{I}}$ of unit vectors in $\mathfrak{H}$ is called an *almost U-invariant unit vector* if $\|U(\xi_i \otimes \eta) - \xi_i \otimes \eta\| \to 0, \forall \eta \in L^2(\mathbb{G})$.

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- U has a non-zero invariant vector $\Leftrightarrow \ \pi_{\mathbf{1}_{\mathbb{G}}} \subset \pi_U$.
- U has almost invariant vectors $\Leftrightarrow \ \pi_{\mathbf{1}_{\mathbb{G}}} \prec \pi_U$.

Definition

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The following extends some results in the paper by Kyed and Soltan:

Proposition

The following statements are equivalent.

(0) $\mathbb{G}$ *has property T*

(1) $C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \cong \ker \pi_{\mathbf{1}_{\mathbb{G}}} \oplus \mathbb{C}$.

(2) $\exists$ *a proj.* $p_{\mathbb{G}} \in M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ *with* $p_{\mathbb{G}} C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) p_{\mathbb{G}} = \mathbb{C} p_{\mathbb{G}}$ *and* $\pi_{\mathbf{1}_{\mathbb{G}}}(p_{\mathbb{G}}) = 1$.

(3) $\pi_{\mathbf{1}_{\mathbb{G}}}$ *is an isolated point in* $\widehat{C_0^{\mathrm{u}}(\widehat{\mathbb{G}})}$.

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• If V is another unit. corepn. of $C_0(\mathbb{G})$ on a Hil. sp. $\mathfrak{K}$, then $U \circledT V := U_{13}V_{23} \in M\big(\mathcal{K}(\mathfrak{H} \otimes \mathfrak{K}) \otimes C_0(\mathbb{G})\big)$ is a unit. corepn.

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• Suppose $\mathbb{G}$ is of Kac type and $\kappa : C_0(\mathbb{G}) \to C_0(\mathbb{G})$ is the antipode. If $\tau : \mathcal{K}(\mathfrak{K}) \to \mathcal{K}(\bar{\mathfrak{K}})$ is the canon. anti-isom., then $\bar{V} := (\tau \otimes \kappa)(V) \in M(\mathcal{K}(\bar{\mathfrak{K}}) \otimes C_0(\mathbb{G}))$ is a unit. corepn.

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Lemma

Suppose that $\mathbb{G}$ is of Kac type. $T \in \mathfrak{H} \otimes \bar{\mathfrak{K}}$ is $U \textcircled{\scriptsize T} \bar{V}$-invariant if and only if $U(T \otimes 1)V^* = T \otimes 1$ (as operators from $\mathfrak{K} \otimes L^2(\mathbb{G})$ to $\mathfrak{H} \otimes L^2(\mathbb{G})$).

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Proposition

$\pi_{\mathbf{1}_{\mathbb{G}}} \subset \pi_{U \textcircled{\scriptsize{T}} \bar{V}}$ if and only if $\exists$ a fin. dim. unit. corepn. W s.t. $\pi_W \subset \pi_U$ and $\pi_W \subset \pi_V$.

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Theorem

Then property T of $\mathbb{G}$ is also equivalent to the following statements.

(4) *All fin. dimen. irred. repn. of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ are isolated points in $\widehat{C_0^{\mathrm{u}}(\widehat{\mathbb{G}})}$.*

(5) $C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ *for a C^*-algebra B and $n \in \mathbb{N}$.*

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• If $\mathbb{G}$ is a discrete group Γ, then $\mathrm{Pol}(\widehat{\mathbb{G}}) = \mathbb{C}[\Gamma]$.

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If $Z^1(G; \mu) := \{\text{1-cocycles}\}$ and $B^1(G; \mu) := \{\text{1-coboundaries}\}$, then $H^1(G; \mu) := Z^1(G; \mu)/B^1(G; \mu)$ is called the *first cohomology of G with coefficient in* $(\mu, \mathfrak{H})$.

• Problem: loc. comp. quant. groups "may not have point"

• Suppose $\mathbb{G}$ is discrete and $(\mathrm{Pol}(\widehat{\mathbb{G}}), \Delta, S, \varepsilon)$ is the Hopf *-alg. of matrix coefficents of $\widehat{\mathbb{G}}$. Kyed defined a cohom. theory of $(\mathrm{Pol}(\widehat{\mathbb{G}}), \varepsilon)$ with coeff. in *-repn of $\mathrm{Pol}(\widehat{\mathbb{G}})$, and showed that property T of $\mathbb{G}$ is equiv. to the vanishing of all such cohom.

• If $\mathbb{G}$ is a discrete group Γ, then $\mathrm{Pol}(\widehat{\mathbb{G}}) = \mathbb{C}[\Gamma]$.

• How about locally compact groups?

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(b) *continuously locally bounded*, if $\exists$ a family $\{\kappa_K\}_{K \in \mathfrak{K}(G)}$ in $\mathbb{R}_+$:
  - $\|\theta(f)\|_{\mathfrak{H}} \leq \kappa_K \|f\|_{L^1(G)}$ $(K \in \mathfrak{K}(G); f \in C_K(G))$,
  - $\forall \epsilon > 0, \exists$ a compact neigh. U of e s.t. $\kappa_K < \epsilon$ if $K \subseteq U$.

Proposition

If $\pi : C_c(G) \to \mathcal{L}(\mathfrak{H})$ is a non-deg. *-repn., $\exists$ a stronly cont. unit. repn. μ of G on $\mathfrak{H}$ with $\pi = \tilde{\mu}$ if and only if π is loc. bdd.

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Definition

$B_{\mathcal{P}}$: *loc. convex* **-alg.;* $\varepsilon : B \to \mathbb{C}$ *a* $\mathcal{P}$*-cont.* **-character. We say that* $(B_{\mathcal{P}}, \varepsilon)$ *has property (FH) if* $\forall \mathcal{P}$*-cont. non-deg.* **-repn* $\pi : B \to \mathcal{L}(\mathfrak{H})$*, any* $\mathcal{P}$*-cont.* 1*-cocycle for* (π, ε) *is a* 1*-coboundary.*

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- If $\mathbb{G}$ is an amen. loc. comp. quant. gp. of Kac type, then $(L^1(\mathbb{G})_{\|\cdot\|_{L^1(\mathbb{G})}}, \pi_{\mathbf{1}_{\mathbb{G}}})$ has prop. (FH).

• Consider $\mathfrak{T}$ to be the loc. convex inductive top. on $C_c(G)$ induced by the sys. $\big\{(C_K(G), \|\cdot\|_{L^1(G)})\big\}_{K\in\mathfrak{K}(G)}$ of normed sp.

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(a) If G is 2nd countable, then $(C_c(G)_{\mathcal{T}}, \varepsilon_G)$ has prop. (FH) $\Leftrightarrow$ G has prop. T.

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- One may also define char. amen. for loc. conv. *-alg. If G is 2nd count., $C_c(G)_{\mathfrak{T}}$ is ε_G-amen. if and only if G is compact.

Reference:
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Thanks