A continuum of C^*-norms on $\mathbb{B}(H) \otimes \mathbb{B}(H)$ and related tensor products

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Definition

A pair of C^* algebras (A, B) will be called a nuclear pair if

$$A \otimes_{\min} B = A \otimes_{\max} B.$$

Recall

$$(A, B) \text{ nuclear } \forall B \Leftrightarrow A \text{ nuclear}$$

We will concentrate on two fundamental examples

$$\mathcal{B} = B(\ell_2)$$

$$\mathcal{C} = C^*(\mathbb{F}_\infty)$$

Note that these are both universal but in two different ways
injectively for $\mathcal{B} = B(\ell_2)$
projectively for $\mathcal{C} = C^*(\mathbb{F}_\infty)$
Also both are non-nuclear, also $\mathcal{B} \simeq \bar{\mathcal{B}}$ and $\mathcal{C} \simeq \bar{\mathcal{C}}$

Kirchberg, Invent, 1993
proved the following fondamental

Theorem

$(\mathcal{B},\mathcal{C})$ is a nuclear pair.
Moreover
$(A,\mathcal{C})$ is a nuclear pair $\Leftrightarrow$ A WEP
$(A,\mathcal{B})$ is a nuclear pair $\Leftrightarrow$ A LLP

Kirchberg asked two important questions
1) Is $(\mathcal{B},\mathcal{B})$ a nuclear pair ?
2) Is $(\mathcal{C},\mathcal{C})$ a nuclear pair ? still OPEN
For Question 1 : answer is no (Junge-P, GAFA 1995)

Theorem (Junge-P, GAFA 1995)

If M, N are not nuclear, then the pair (M, N) is not nuclear.

Wassermann (JFA 1976) characterized nuclear von Neumann algebras as finite direct sums of $C \otimes M_n$ with C commutative. Equivalently, he showed
$\forall$ non nuclear von Neumann algebra M

$$\mathbb{B} = \prod_{n \geq 1} M_n \subset M$$

So preceding theorem reduces to the case

$$M = N = \prod_{n \geq 1} M_n$$

or equivalently to the case

$$M = N = \mathcal{B}$$

Let N be a non nuclear von Neumann algebra
Wassermann's result shows that any separable operator space E embeds (completely isometrically) into N
i.e. we can replace $B(H)$ by N (for operator space theory)
we write this

$$E \subset N$$

Note that

Theorem (Haagerup)

C^* algebra A is WEP IFF $A \otimes \bar{A}$ satisfies

$$\forall n \, \forall x_j \in A \quad \| \sum_1^n x_j \otimes \bar{x}_j \|_{\min} = \| \sum_1^n x_j \otimes \bar{x}_j \|_{\max}$$

So this holds for $A = \mathcal{B}$.

But nevertheless

$$\min \neq \max \ \text{ on } \mathcal{B} \otimes \bar{\mathcal{B}} \ (\text{or on } \mathcal{B} \otimes \mathcal{B})$$

Theorem (Ozawa-P)

Let M, N be any pair of von Neumann algebras. If

$$M \otimes_{\min} N \neq M \otimes_{\max} N$$

i.e. if

$$\mathrm{card}\{\mathrm{C}^* - \textit{norms on } M \otimes N\} > 1$$

then:

$$\mathrm{card}\{\mathrm{C}^* - \textit{norms on } M \otimes N\} \geq 2^{\aleph_0}$$

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We say a C^*-norm $\| \cdot \|_\alpha$ on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ is *admissible* if it is invariant under the flip and tensorizes unital completely positive maps

Complement:

When $M = N = \mathcal{B}$

we can find a continuum of *admissible* C^*-norms α on $\mathcal{B} \otimes \mathcal{B}$

This produces a continuum of *injective tensor product functors* in the sense of Kirchberg

by inducing each α on $A \otimes B$

with

$$A \subset \mathcal{B} \quad B \subset \mathcal{B}$$

Theorem (Ozawa-P)

Let

$$\mathbb{B} = \prod_{n\geq 1} M_n$$

(so that

$$\{x \in \mathbb{B} \mid \|x\| \leq 1\} = \prod_{n\geq 1} \{x \in M_n \mid \|x\|_{M_n} \leq 1\})$$

and let

$$M \subset \mathbb{B}(\ell_2)$$

be a non-nuclear von Neumann algebra. Then the set of

$$\{\mathrm{C}^* - \textit{norms on } M \otimes \mathbb{B}\}$$

has cardinality equal to $2^{2^{\aleph_0}}$.

Let

$$OS_n = \{E \subset \mathcal{B} \mid dim(E) = n\}$$

We declare $E = F$ if E, F completely isometric Let

$$(OS_n, d_{cb})$$

be the metric space formed of all n-dimensional operator spaces equipped with the “distance"

$$d_{cb}(E, F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb} \mid u: \ E \to F\}$$

$$\forall E, F, G \in OS_n \quad d_{cb}(E, G) \leq d_{cb}(E, F) d_{cb}(F, G)$$

Key ingredient

Theorem (Junge-P, 1995)

For any $n > 2$
(OS_n, d_{cb}) is not separable
more precisely there is $\delta > 0$ and a family $\{E_i \mid i \in I\} \subset OS_n$
with $card(I) = 2^{\aleph_0}$
such that

$$\forall i \neq j \in I \quad d_{cb}(E_i, E_j) > 1 + \delta$$

We will use a variant:

Theorem

Assume given for any $E \in OS_n$ a separable subset $C_E \subset OS_n$ and let

$$d(E, F) = \max\{d_{cb}(E, C_F), d_{cb}(F, C_E)\}$$

Then the same holds (by extracting a suitable subfamily and changing $\delta > 0$) for the metric d.

Let A be a C^* algebra.
Let $OS_n(A) = \{E \in OS_n \mid E \subset A\}$

Various Remarks:

• If A is separable, $OS_n(A)$ is separable

• for any (possibly uncountable) free group $\mathbb{F}$, if $A = C^*(\mathbb{F})$ then $OS_n(A)$ is separable
because $E \subset C^*(\mathbb{F}) \Rightarrow E \subset C^*(\mathbb{F}_\infty)$ and $C^*(\mathbb{F}_\infty)$ is separable

Notation:

$$d_{SA}(E) = \inf\{d_{cb}(E, F) \mid F \subset A\}$$

C^*-norm

A, B C^*-algebras
Let $I \subset A$ be an ideal in A
C^*-norm on $(A/I) \otimes B$

$$\forall t \in (A/I) \otimes B \qquad \alpha(t) = \|t\|_{(A \otimes_{\min} B)/(I \otimes_{\min} B)}$$

Operator space dual

For any $E \in OS_n$, say $E \subset \mathcal{B}$
there is an embedding $E^* \subset \mathcal{B}$ such that

$$\forall n \quad M_n(E^*) = CB(E, M_n) \text{ isometrically}$$

for any operator space F

$$CB(E, F) = F \otimes_{\min} E^* \subset \mathcal{B} \otimes_{\min} \mathcal{B} \text{ isometrically}$$

Lemma (Key Lemma)

Let $E \in OS_n$
Assume $M = A/I$
Consider $E \subset M$ and $E^* \subset N$
Let $t_E \in E \otimes E^* \subset (A/I) \otimes N$ be associated to Id_E
Then

$$d_{SA}(E) \le \|t_E\|_{(A \otimes_{\min} N)/(I \otimes_{\min} N)}$$

Special case from [JP 1995] $A = \mathcal{C}$

$$t_E \in \mathcal{B} \otimes \mathcal{B}$$

$$d_{SA}(E) = \|t_E\|_{\mathcal{B} \otimes_{\max} \mathcal{B}} = \|t_E\|_{(A \otimes_{\min} \mathcal{B})/(I \otimes_{\min} \mathcal{B})}$$

where we use

$$\mathcal{B} \simeq A/I \text{ with } A = C^*(\mathbb{F})$$

for some large enough free group $\mathbb{F}$

Proof that $\exists$ 2 norms on $M \otimes N$

Pick $E \in OS_n$ such that $d_{SC}(E) > 1$
then set $A = C^*(\mathbb{F})$ so that $A/I = M$ and $E^* \subset N$
so that $t_E \in E \otimes E^* \subset (A/I) \otimes N$

$$\alpha(t) = \|t\|_{(A \otimes_{\min} N)/(I \otimes_{\min} N)}$$

Then

$$\alpha(t_E) \geq d_{SC}(E) > 1 = \|t_E\|_{\min}$$

so

$$\alpha \neq \| \ \|_{\min}$$

and a fortiori

$$\|t_E\|_{\max} > 1$$

$$\| \ \|_{\max} \neq \| \ \|_{\min}$$

Proof that $\exists$ 3 norms on $M \otimes N$

Assume $E \subset M$ and $E^* \subset N$ again
Let $q: \ C^*(\mathbb{F}) \to M$ be onto, let $j_E: \ C^* < E > \to M$ be inclusion
Let now

$$\mathcal{A}_E = C^* < E > * C^*(\mathbb{F})$$

we have a surjection $Q_E = j_E * q: \ \mathcal{A}_E \to M$ so that

$$\mathcal{A}_E / \mathcal{I} = M$$

Again we set $\alpha_E(t) = \|t\|_{(\mathcal{A}_E \otimes_{\min} N)/(\mathcal{I} \otimes_{\min} N)}$
Note $OS_n(A)$ is separable
(because as previously observed can replace $\mathbb{F}$ by $\mathbb{F}_\infty$) Thus we can find F such that $d_{SA}(F) > 1$ and hence on the one hand

$$\alpha_E(t_F) \geq d_{SA}(F) > 1 = \|t_F\|_{\min}$$

so

$$\alpha_E \neq \| \ \|_{\min}$$

but on the other hand $\alpha_E(t_E) = 1$ but we just proved that $\|t_E\|_{\max} > 1$ so $\alpha_E \neq \| \ \|_{\max}$

Proof that $\exists 2^{\aleph_0}$ norms on $M \otimes N$

For any E we associate the class

$$\mathcal{C}_E = \{F \subset C^* < E > *C^*(\mathbb{F})\}$$

We consider a family $\{E_i \mid i \in I\}$ in OS_n with card $2^{\aleph_0}$ and $\delta > 0$ such that

$$\forall i \neq j \in I \quad d(E_i, E_j) > 1 + \delta$$

where

$$d(E, F) = \max\{d_{cb}(E, C_F), d_{cb}(F, C_E)\}$$

Then the preceding reasoning shows that $\forall i \neq j \in I$ we have either

$$\alpha_{E_i}(t_{E_j}) > 1 + \delta \ \text{ but } \ \alpha_{E_j}(t_{E_j}) = 1$$

or

$$\alpha_{E_j}(t_{E_i}) > 1 + \delta \ \text{ but } \ \alpha_{E_i}(t_{E_i}) = 1$$

and hence

$$\alpha_{E_i} \neq \alpha_{E_j}$$