Totally Disconnected L.C. Groups: The Scale and Minimizing Subgroups

George Willis
The University of Newcastle

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Lecture 1: The scale and minimizing subgroups for an endomorphism

- Compact open subgroups; Examples
- The scale of an endomorphism
- The scale function on G
- The structure of minimizing subgroups

Lecture 2: Tidy subgroups and the scale

Lecture 3: The contraction group and the nub

Lecture 4: Flat groups of automorphisms

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Topological background

Definition (Inductive dimension)

The empty set is defined to have inductive dimension -1. The topological space $(X, \mathfrak{T})$ has *inductive dimension n* if it does not have dimension less than n and there is a base for $\mathfrak{T}$ comprising sets whose boundary has inductive dimension at most $n - 1$.
The inductive dimension of X is denoted $ind(X)$.

A locally compact space is totally disconnected if and only if it has inductive dimension 0, in which case the topology has a base of compact open subsets.

Totally disconnected locally compact groups are sometimes referred to as *0-dimensional groups*.

Compact open subgroups

Theorem (van Dantzig)

Let G be a totally disconnected locally compact groups and $\mathscr{O} \ni e$ be a neighbourhood of the identity. Then there is a compact open subgroup $V \subseteq \mathscr{O}$.

The compact open subgroup need not be normal.

Corollary

Every compact t.d. group is profinite, that is, a projective limit of finite groups. Conversely, every profinite group is compact with the profinite topology.

The set of all compact open subgroups of t.d.l.c. group G will be denoted by $\mathcal{B}(G)$.

Examples of tdlc groups

Examples

1. $F^{\mathbb{Z}}$, where F is a finite group, with the product topology.
   This group is compact.
2. $(\mathbb{F}_p((t)), +)$, the additive group of the field of formal Laurent series over the field of order p.
   $t^n\mathbb{F}_p[[t]] \cong \mathbb{F}_p^{\mathbb{N}}$ is a compact open subgroup for each $n \in \mathbb{Z}$.
3. Aut(T_q), the automorphism group of the regular tree with every vertex having valency q.
   The stabilizer of any finite subtree is compact and open.
4. $SL(n, \mathbb{Q}_p)$, the special linear group over the field of p-adic numbers.
   $SL(n, \mathbb{Z}_p)$ is a compact open subgroup.

The scale of an endomorphism

Let G be a t.d.l.c. group.

- An *endomorphism* of G is a continuous homomorphism $\alpha : G \to G$.
- An *automorphism* of G is an endomorphism that is a bijection whose inverse is continuous.
- The automorphism α is *inner* if there is $g \in G$ such that $\alpha = \alpha_g$ where $\alpha_g(x) = gxg^{-1}$ for every $x \in G$.

Definition

Let G be a t.d.l.c. group and α be an endomorphism of G. The *scale* of α is

$$s(\alpha) = \min \{[\alpha(V) : \alpha(V) \cap V] \mid V \in \mathcal{B}(G)\}.$$

Any V at which the minimum is attained is *minimizing for* α.

Properties of the scale

Theorem

Let G be a t.d.l.c. group and $\alpha \in End(G)$. Then

- $s(\alpha^n) = s(\alpha)^n$ *for every* $n \geq 0$.

If α is an automorphism, then

- $s(\alpha) = 1 = s(\alpha^{-1})$ *if and only if there is* $V \in \mathcal{B}(G)$ *such that* $\alpha(V) = V$*; and*
- $\Delta(\alpha) = s(\alpha)/s(\alpha^{-1})$, *where* $\Delta(\alpha)$ *is the module of* α.
- *The function* $s : G \to \mathbb{Z}^+$ *defined by* $s(g) = s(\alpha_g)$ *is continuous.*

These properties will be explained in the first two lectures for the case when α is an automorphism.

The set of topologically periodic elements is closed

An element $g \in G$ is *topologically periodic* if $\overline{\langle g \rangle}$ is compact. Denote the set of topologically periodic elements in G by $P(G)$. K. H. Hofmann asked whether $P(G)$ is closed. The answer uses only that there is a function on G with the properties of s.

Proposition

$P(G)$ is closed.

Proof.

1. $g \in P(G)$ implies that $s(g) = 1$.
2. $s(h) = 1$ for every $h \in \overline{P(G)}$.
3. If $h \in \overline{P(G)}$, then h normalizes some $V \in \mathcal{B}(G)$.
4. There is $g \in P(G) \cap hV$. Then g normalizes V and $h \in gV$. Hence $\langle h \rangle \leq \langle g \rangle V$ and has compact closure.

The scale on G as a spectral radius

Proposition (R. G. Möller)

Let $\alpha \in \text{Aut}(G)$ and $V \in \mathcal{B}(G)$. Then

$$s(\alpha) = \lim_{n\to\infty} [\alpha^n(V) : \alpha^n(V) \cap V]^{1/n}.$$

For each $V \in \mathcal{B}(G)$ define $w_V(g) = [\alpha_g(V) : \alpha_g(V) \cap V]$.

Proposition

- w_V is submultiplicative, that is, $w_V(gh) \leq w_V(g)w_V(h)$.
- All weights w_V are equivalent, that is, given $U, V \in \mathcal{B}(G)$ there is $K > 0$ such that

$$K^{-1}w_V(g) \leq w_U(g) \leq Kw_V(g) \text{ for all } g \in G.$$

The scale on G as a spectral radius 2

Equivalence of the weights implies that the weighted L^1-space

$$L^1(G, w_V) = \left\{ \varphi \in L^1(G) \mid \int_G |\varphi| w_V < \infty \right\}$$

is independent of the weight w_V.
Submultiplicativity of the weights implies that $L^1(G, w_V)$ is a Banach algebra under convolution.

For $g \in G$, let L_g denote the operator on $L^1(G, w_V)$ of left translation by g. $L_g : \varphi \mapsto \varphi_g$, where $\varphi_g(x) = \varphi(gx)$. Then L_g is a bounded operator and

$$s(g) = r(L_g) \ (= \text{ the spectral radius of } L_g).$$

Subgroups tidy for an endomorphism

Let $\alpha \in \mathrm{End}(G)$ and $V \in \mathcal{B}(G)$. Define

$$V_+ = \{v \in V \mid \exists\{v_n\}_{n\geq 0} \subset V \text{ with } v_0 = v \text{ and } \alpha(v_{n+1}) = v_n\}$$

$$\text{and } V_- = \{v \in V \mid \alpha^n(v) \in V\ \forall n \geq 0\}.$$

Theorem

The subgroup $V \in \mathcal{B}(G)$ is minimizing for $\alpha \in End(G)$ iff

TA(α) $V = V_+V_-$;

TB1(α) $V_{++} := \bigcup_{n\geq 0} \alpha^n(V_+)$ *is closed; and*

TB2(α) $\left\{[\alpha^{n+1}(V_+) : \alpha^n(V_+)]\right\}_{n\geq 0}$ *is constant.*

In this case, $s(\alpha) = [\alpha(V_+) : V_+]$.

V is *tidy above* for α if it satisfies TA(α) and *tidy below* if it satisfies TB1(α) and TB2(α).

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Subgroups tidy for an endomorphism 2

OUTLINE OF PROOF

1. Given $V \in \mathcal{B}(G)$, reduce to a subgroup U that satisfies TA(α).
   $w_U(\alpha) \leq w_V(\alpha)$, *with equality iff V satisfies TA(α)*.
2. Given $V \in \mathcal{B}(G)$ satisfying TA(α), augment V to obtain a subgroup U satisfying TB(α) as well.
   $w_U(\alpha) \leq w_V(\alpha)$, *with equality iff V satisfies TB(α)*.
3. Show that, if U and V are both tidy for α, then $w_U(\alpha) = w_V(\alpha)$.

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Achieving tidiness above

From now on, α will be an automorphism.

In this case, $V_+ = \bigcap_{n\geq 0} \alpha^n(V)$ and $V_- = \bigcap_{n\geq 0} \alpha^{-n}(V)$.

Lemma

Let $V \in \mathcal{B}(G)$. There is $N \geq 0$ such that the subgroup $U := \bigcap_{k=0}^{N} \alpha^k(V)$ satisfies

$$\alpha(U) \subseteq \alpha(U_+)U. \tag{1}$$

Achieving tidiness above 2

Lemma

Suppose that $\alpha(V) \subseteq \alpha(V_+)V$. *Then* $V = V_+V_-$.
Conversely, if $V = V_+V_-$ *then* $\alpha(V) \subseteq \alpha(V_+)V$.

Lemma

Let V *be any compact open subgroup of* G. *Then*

$$w_V(\alpha) = [\alpha(V) : \alpha(V) \cap V] \geq [\alpha(V_+) : V_+], \tag{2}$$

with equality if and only if $\alpha(V) \leq \alpha(V_+)V$.

Summary: Given $V \in \mathcal{B}(G)$, there is $N \geq 0$ such that $U := \bigcap_{k=0}^{N} \alpha^k(V)$ is tidy above for α.
The subgroup U satisfies $w_U(\alpha) \leq w_V(\alpha)$ with equality if and only if V is already tidy above.

Tidiness above in the examples 2

Examples

1. Let $G = F^{\mathbb{Z}}$, let α be the shift automorphism and $V = \{g \in F^{\mathbb{Z}} \mid g(-1) = e = g(1) = g(2)\}$. Then $N = 1$ and $U = \{g \in F^{\mathbb{Z}} \mid g(n) = e \text{ if } |n| < 3\}$.
2. Let $G = (\mathbb{F}_p((t)), +)$, let α be multiplication by t^{-1} and $V = \text{span}\{t^{-4}, t^{-3}, t^{-2}\} + \mathbb{F}_p[[t]]$. Then $N = 3$ and $U = \mathbb{F}_p[[t]]$.

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Tidiness above in the examples

Examples

3. Let $G = \mathrm{Aut}(T_q)$, let α be the inner automorphism α_g where g is a translation with axis ℓ, and $V = \mathrm{Fix}(a)$, where a is a vertex distance 4 from ℓ. Then $N = 1$ and $U = \mathrm{Fix}([a, g.a])$, where $[a, g.a]$ is the path of length 9 from a to $g.a$ (which intersects ℓ in an edge).
4. Let $G = SL(n, \mathbb{Q}_p)$, let α conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, and $V = SL(n, \mathbb{Z}_p)$. Then $N = 1$ and $U = \left\{ \begin{pmatrix} a_{11} & pa_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}_p \right\}$.

References

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