Totally Disconnected L.C. Groups: Tidy subgroups and the scale

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Lecture 1: The scale and minimising subgroups for an endomorphism

Lecture 2: Tidy subgroups and the scale

- The tidy below condition
- Index the same for all tidy subgroups
- Continuity of the scale function on G

Lecture 3: The contraction group and the nub

Lecture 4: Flat groups of automorphisms

Structure of minimising subgroups

Let $\alpha \in \mathrm{End}(G)$ and $V \in \mathcal{B}(G)$. Define

$$V_+ = \{v \in V \mid \exists\{v_n\}_{n\geq 0} \subset V \text{ with } v_0 = v \text{ and } \alpha(v_{n+1}) = v_n\}$$

$$\text{and} \quad V_- = \{v \in V \mid \alpha^n(v) \in V\ \forall n \geq 0\}.$$

Theorem

The subgroup $V \in \mathcal{B}(G)$ is minimising for $\alpha \in End(G)$ iff

- TA(α) $V = V_+ V_-$;
- TB1(α) $V_{++} := \bigcup_{n\geq 0} \alpha^n(V_+)$ *is closed; and*
- TB2(α) $\left\{[\alpha^{n+1}(V_+) : \alpha^n(V_+)]\right\}_{n\geq 0}$ *is constant.*

In this case, $s(\alpha) = [\alpha(V_+) : V_+]$.

V is *tidy above* for α if it satisfies TA(α) and *tidy below* if it satisfies TB1(α) and TB2(α).

Structure of minimising subgroups 2

OUTLINE OF PROOF

1. Given $V \in \mathcal{B}(G)$, reduce to a subgroup U that satisfies TA(α).
   $w_U(\alpha) \leq w_V(\alpha)$, *with equality iff* V *satisfies TA(*α*)*.
2. Given $V \in \mathcal{B}(G)$ satisfying TA(α), augment V to obtain a subgroup U satisfying TB(α) as well.
   $w_U(\alpha) \leq w_V(\alpha)$, *with equality iff* V *satisfies TB(*α*)*.
3. Show that, if U and V are both tidy for α, then $w_U(\alpha) = w_V(\alpha)$.

Definition of the subgroup $L_{\alpha,V}$

Definition

Let V be tidy above for α. Put

$$\mathscr{L}_{\alpha,V} = \{v \in G \mid \alpha^n(v) \in V \text{ for almost every } n \in \mathbb{Z}\}$$

and $L_{\alpha,V} = \overline{\mathscr{L}_{\alpha,V}}$.

Then $L_{\alpha,V}$ is a closed subgroup of G and the orbit $\{\alpha^n(v)\}_{n\in\mathbb{Z}}$ has compact closure for each $v \in \mathscr{L}_{\alpha,V}$.

Proof of compactness of $L_{\alpha,V}$

For $v \in \mathscr{L}_{\alpha,V}$ and not in $V_0 := \bigcap_{k\in\mathbb{Z}} \alpha^k(V)$, write:

$\mathfrak{m}(v)$ for the largest m such that $\alpha^m(v) \in V_+$,

and $\mathfrak{M}(v)$ for the smallest m such that $\alpha^m(v) \in V_-$.

Define $\mathfrak{e}(v) = \mathfrak{M}(v) - \mathfrak{m}(v) - 1$ and $\mathfrak{e}(v) = 0$ if $v \in V_0$.

Let $v_1, \dots, v_r$ be representatives chosen from the V_+-cosets in $(\alpha(V_+) \setminus V_+) \cap \mathscr{L}_{\alpha,V}$ such that $\mathfrak{e}(v_j)$ is minimised. Note that $\mathfrak{m}(v_j) = -1$ and $\mathfrak{e}(v_j) = \mathfrak{M}(v_j)$ for each v_j.

Lemma

Let $v \in \mathscr{L}_{\alpha,V}$. Then

$$v = v_0\alpha^{m_1}(v_{j_1}) \dots \alpha^{m_l}(v_{j_l}), \tag{1}$$

where $v_0 \in V_0$ and $v_{j_i} \in \{v_1, \dots, v_r\}$ for each $i \in \{1, \dots, l\}$ and $m_1 < m_2 < \cdots < m_l$.

Proof of compactness of $L_{\alpha,V}$ 2

Lemma

Put $\mathfrak{M} = \max\{\mathfrak{M}(v_j) \mid j \in \{1, \dots, r\}\}$. *Then*

$$\mathscr{L}_{\alpha,V} \subseteq \alpha^{\mathfrak{M}}(V_+)V_-.$$

Proposition

Let $\alpha \in \mathrm{Aut}(G)$ and V be a compact open subgroup of G that is tidy above for α. Then $L_{\alpha,V}$ is compact.

Joining $L_{\alpha,V}$ to V

Proposition

Let $\alpha \in \mathrm{Aut}(G)$ and V be tidy above for α. Then

$$V' := \left\{ v \in V \mid vL_{\alpha,V} \subseteq L_{\alpha,V}V \right\}$$

is an open subgroup of V. Then $U := V'L_{\alpha,V}$ is a compact open subgroup of G that satisfies TA(α) and TB(α). Furthermore,

$$w_U(\alpha) = [\alpha(U) : \alpha(U) \cap U] \leq [\alpha(V) : \alpha(V) \cap V] = w_V(\alpha)$$

with equality if and only if $\mathscr{L}_{\alpha,V} \leq V$.

Tidiness below in examples

Examples

1. Let $G = F^{\mathbb{Z}}$, let α be the shift automorphism and $V = \{g \in F^{\mathbb{Z}} \mid g(n) = 1 \text{ if } |n| < 3\}$.
   Then $\mathscr{L}_{\alpha,V} = \{g \in F^{\mathbb{Z}} \mid g \text{ has finite support}\}$ and $L_{\alpha,V} = G = U$.
2. Let $G = (\mathbb{F}_p((t)), +)$, let α be multiplication by t^{-1} and $V = \mathbb{F}_p[[t]]$. Then $\mathscr{L}_{\alpha,V}$ and $L_{\alpha,V}$ are trivial, and $U = V$.

Tidiness below in examples 2

Examples

3. Let $G = \mathrm{Aut}(T_q)$, let α be the inner automorphism α_g, where g is a translation with axis ℓ, and $V = \mathrm{Fix}([a, g.a])$, where a is a vertices distance 4 from ℓ.
   Then $\mathscr{L}_{\alpha,V}$ comprises all automorphisms fixing all but finitely many of the vertices $g^n.a$ (and all vertices on ℓ).
   Furthermore $U = \mathrm{Fix}([c, d])$ where c and d are the projections of a and $g.a$ onto ℓ.
4. Let $G = SL(n, \mathbb{Q}_p)$, let α conjugation by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and V be any subgroup tidy above for α. Then $\mathscr{L}_{\alpha,V} = V_0$ and $V = U$ is tidy for α.

V is minimising if and only if tidy

Theorem

Let U and V be tidy for α. Then $U \cap V$ is tidy for α.

Lemma

Let U and V be tidy for α. Then

$$[\alpha(U) : \alpha(U) \cap U] = [\alpha(V) : \alpha(V) \cap V].$$

Theorem

Let $\alpha \in \mathrm{Aut}(G)$. Then the compact open subgroup $V \leq G$ is minimising for α if and only if tidy for α.

Corollary

$s(\alpha^n) = s(\alpha)^n$ *for every* $n \geq 0$.

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Stability of tidiness

Lemma

Let $g \in G$ and $V \in \mathcal{B}(G)$ be tidy above for g. Then for every $v \in V$ there are $s \in V_-$ and $t \in V_+$ such that

$$s^{-1}(gv)^{-k}s \in Vg^{-k} \text{ and } t^{-1}(gv)^{k}t \in Vg^{k} \text{ for every } k \geq 0. \quad (2)$$

Proposition

Let $g \in G$ and $V \in \mathcal{B}(G)$ be tidy above for g. Then there is $w \in V$ such that, for every $k \geq 0$,

$$w\left(g^{\pm k}V_{g\pm}g^{\mp k}\right)w^{-1} = (gv)^{\pm k}V_{(gv)\pm}(gv)^{\mp k}. \quad (3)$$

Stability of tidiness 2

Theorem

Let $g \in G$ and $V \in \mathcal{B}(G)$ be tidy for g. Then, for every $v \in V$, V is tidy for gv and $s(gv) = s(g)$.

Corollary

The scale function $s : G \to \mathbb{Z}^+$ is continuous.

References

1. G. Willis, 'The structure of totally disconnected, locally compact groups', *Math. Annalen*, **300** (1994), 341–363.
2. G. Willis, 'Further properties of the scale function on totally disconnected groups', *J. Algebra*, **237** (2001), 142–164.