Symmetries of b-manifolds and their generalizations

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Outline

Moment maps in Symplectic Geometry

Definition (Symplectic case)

Let G be a compact Lie group acting symplectically on (M, ω).
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$d\mu^X = \iota_{X^\#}\omega, \quad (1)$$

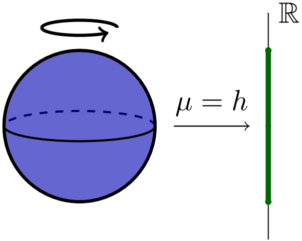
with $\mu^X = <\mu, X>$.
The map μ is called the **moment map**.

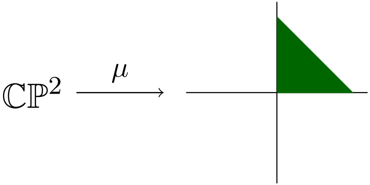
Toric symplectic manifolds

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. The bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



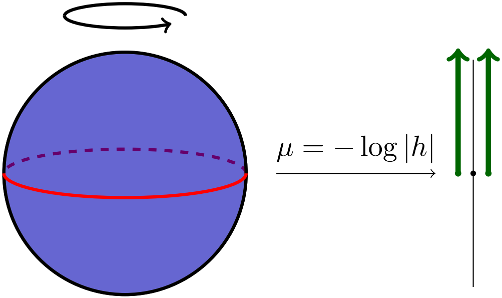


$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

Adding singularities in the picture

$$(S^2, \frac{1}{h} dh \wedge d\theta) \leftrightsquigarrow (S^2, h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}).$$

We want to study generalizations of rotations on a sphere.



b-Symplectic/b-Poisson structures

Definition

Let (M^{2n}, Π) be an oriented Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a **Poisson b-structure** on (M, Z).

Disclaimer

b-symplectic manifolds =log-symplectic manifolds= b-log symplectic manifolds

Symplectic foliation of a Poisson b-manifold

The symplectic foliation has dense symplectic leaves and codimension 2 symplectic leaves whose union is Z.

Examples: Dimension 2

Radko classified b-Poisson structures on compact oriented surfaces giving a list of invariants:

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^{\epsilon}(\Pi) = \int_{|h|>\epsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$.

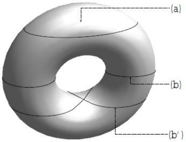


Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') is not admissible.

Higher dimensions: Some compact examples.

- The product of (R, π_R) a Radko compact surface and a (S, π) be a compact symplectic manifold is a b-Poisson manifold.
- Take (N, π) be a regular corank 1 Poisson manifold and let X be a Poisson vector field. Now consider the product $S^1 \times N$ with the bivector field
$$\Pi = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi.$$
This is a b-Poisson manifold as long as,
  1. the function f vanishes linearly.
  2. The vector field X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

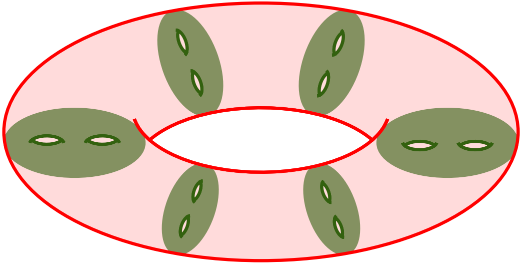
This last example is semilocally the *canonical* picture of a b-Poisson structure.

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field transverse to the symplectic foliation induced on Z.
3. Given a regular corank 1 Poisson structure, there exists a semilocal extension to a b-Poisson structure if an only if two foliated cohomology classes of the symplectic foliation vanish.

The singular hypersurface

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then M is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.



A dual approach...

b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-tangent bundle).

A vector field v is a b-**vector field** if $v_p \in T_pZ$ for all $p \in Z$.

The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

The b-**cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

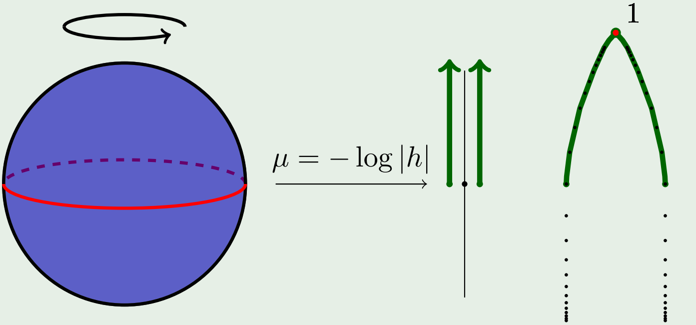
$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

A b-**symplectic form** is a closed, nondegenerate, b-form of degree 2.

This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method since we can play the same tricks as in the symplectic case.

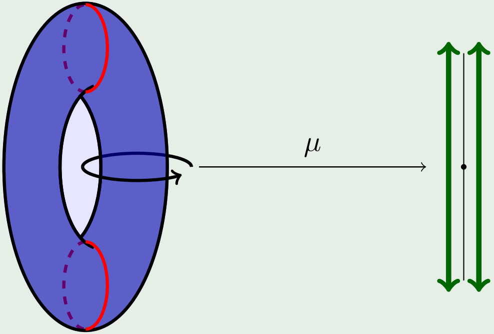
Example

$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the usual $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h, \theta) = \log |h|$.

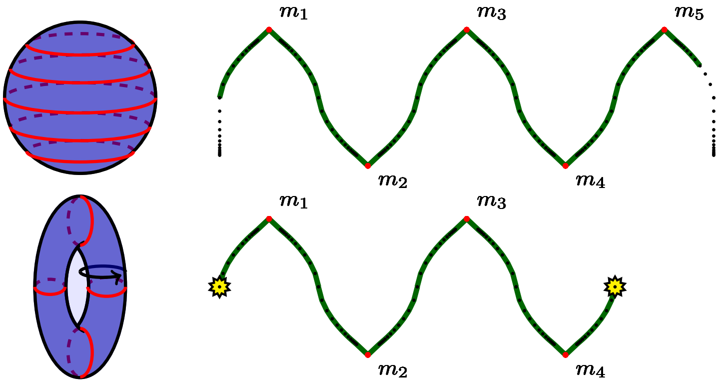


Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \to$ R^2 is given by $\mu(\theta_1, \theta_2) = \log \left|\tan \frac{\theta_1}{2}\right|$.



More generally...



b-toric actions

Definition

An action of $\mathbb{T}^n$ on a b-symplectic manifold (M, ω) is a **Hamiltonian action** if:

- for each $X \in \mathfrak{t}$, the b-one-form $\iota_{X^\#}\omega$ is exact (i.e., has a primitive $H_X \in {}^bC^\infty(M)$)
- for any $X, Y \in \mathfrak{t}$, we have $\omega(X^\#, Y^\#) = 0$.

The action is **toric** if it is effective and the dimension of the torus is half the dimension of M.

b-moment map μ such that

$$< \mu(p), X >= H_X(p),$$

but we will have to allow $\mu(p)$ to take values of $\pm\infty$, so we need to extend the pairing to accommodate that.

The b-line

The b-line is constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern and $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity to prescribe a smooth structure.

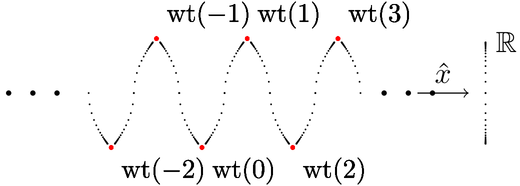


Figure: A weighted b-line with $I = \mathbb{Z}$

The b-**line with weight function** wt is described as a topological space by ${}_{\mathrm{wt}}^{\ b}\mathbb{R} \cong (\mathbb{Z} \times \overline{\mathbb{R}})/\{(a, (-1)^a\infty) \sim (a+1, (-1)^a\infty)\}$. The weights are given by the modular periods associated to each connected component of Z.

Adjacency graph and definition of b-moment map

Theorem (Guillemin, M., Pires, Scott)

Let $(M, Z, \omega, \mathbb{T}^n)$ *be a b-symplectic manifold with an effective Hamiltonian toric action. For an appropriately-chosen* ${}^b\mathfrak{t}^*$ *or* ${}^b\mathfrak{t}^*/\langle a \rangle$*, there is a moment map* $\mu : M \to {}^b\mathfrak{t}^*$ *or* $\mu : M \to {}^b\mathfrak{t}^*/\langle a \rangle$*.*

Example

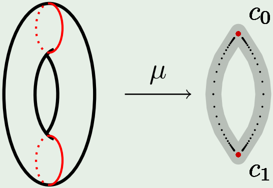
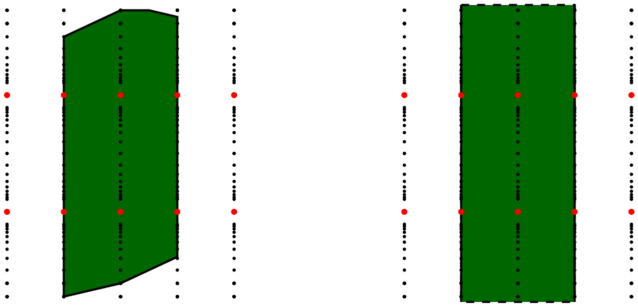


Figure: The moment map μ surjects onto ${}^b\mathfrak{t}^*/\langle 2 \rangle$.

b-Delzant polytopes and local models

Image of the moment map.



We can recover information about the action from a standard Delzant polytope on a mapping torus via *symplectic cutting* in a neighbourhood of the critical hypersurface.

The semilocal model

Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold as

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

A b-Delzant theorem

Theorem (Guillemin,M.,Pires, Scott)

For a fixed primitive lattice vector $v \in \mathfrak{t}^*$ *and weight function* $wt : [1, N] \to \mathbb{R}_{>0}$*, the maps*

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*) \end{array} \right\} \to \left\{ \begin{array}{c} Delzant\ b\text{-}polytopes \\ in\ {}^b\mathfrak{t}^* \end{array} \right\} \quad (2)$$

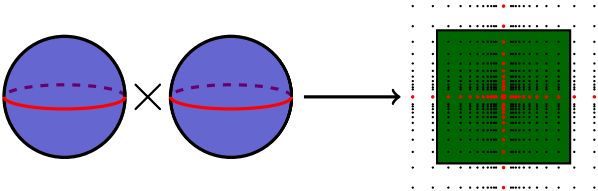
and

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N \rangle) \end{array} \right\} \to \left\{ \begin{array}{c} Delzant\ b\text{-}polytopes \\ in\ {}^b\mathfrak{t}^*/\langle N \rangle \end{array} \right\} \quad (3)$$

that send a b*-symplectic toric manifold to the image of its moment map are bijections.*

Generalization of b-symplectic manifolds

Product of two toric b-spheres. This is a toric c-symplectic manifold (c for "corners ").



These c-manifolds admit Morse-like singularities and a Moser path method seems to work too. Are they topologically constrained?

Simple Poisson structures on S^4

The sphere S^4

- does not admit a symplectic structure.
- does not admit a b-symplectic structure. (Marcut-Osorno and Cavalcanti)
- Using inversion we can construct Poisson structures on S^4 with quadratic type singularities and an isolated singularity (symplectic elsewhere).

Question

Does S^4 admit a c-structure?

What is the common feature?

Example

Consider the projective submodule generated by $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. By Serre-Swan, it has an associated vector bundle E. This example corresponds to isolated elliptic singularities in dimension 2.

E-symplectic manifolds

Goal: Study the Poisson geometry underlying a projective submodule V which is a Lie subalgebra of $Vect(M)$.

We then have a Lie algebroid structure with anchor map $a : T^m M \longrightarrow TM$. The singular locus is the set where the differential is not surjective.