Pfaffian groupoids

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December 10, 2013

Motivation

Understand the work of Cartan on Lie Pseudogroups, and the theory of PDE's using the language of Lie groupoids and Lie algebroids.

Definition of Pfaffian groupoid

Definition

A Pfaffian groupoid $(\mathcal{G}, \theta)$ consists of:

- $\mathcal{G} \rightrightarrows M$ Lie groupoid,
- $\theta \in \Omega^1(\mathcal{G}, t^*E)$ point-wise surjective, $E \to M \in \mathrm{Rep}(\mathcal{G})$, with $\ker \theta \cap \ker ds$ involutive,

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with the property that θ is multiplicative:

$$m^*\theta_{(g,h)} = g \cdot pr_1^*\theta_{(g,h)} + pr_2^*\theta_{(g,h)},$$

$m, pr_1, pr_2 : \mathcal{G}_2 \subset \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$

Examples

Example (Rotations on the plane)

For the standard action of S^1 on $\mathbb{R}^2$ by rotations, we have the action groupoid over $\mathbb{R}^2$

$$\mathcal{G} := S^1 \ltimes \mathbb{R}^2,$$

$s(\alpha, z) = z, t(\alpha, z) = \alpha \cdot z$, and

$$\theta = d\alpha \in \Omega^1(\mathcal{G}).$$

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$$\theta = d\alpha \in \Omega^1(\mathcal{G}).$$

A bisection β of $\mathcal{G}$ (i.e. $\beta : \mathbb{R}^2 \to \mathcal{G}$, $s \circ \beta = id$ and $t \circ \beta$-diffeo) belongs to

$$\mathrm{Sol}(\mathcal{G}, \theta) = \{\beta \mid \beta^*\theta = 0\} \quad \text{iff} \quad \alpha : \mathbb{R}^2 \to S^1 \text{ is constant.}$$

$$\mathrm{Diff}(\mathbb{R}^2) \supset \Gamma^{\mathrm{naive}} = t \circ \mathrm{Sol}(\mathcal{G}, \theta) = \{\text{rotations of the plane}\}$$

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Example (Jet groupoids and the Cartan form)

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$$\mathcal{G} = J^1(M \times M) = \{ \text{ first jets of local diffeos } (= \text{bisections})\}.$$

The Cartan form $\theta^1 \in \Omega^1(\mathcal{G}; t^*TM)$ at $X \in T_{j^1_x\phi} J^1(M \times M)$ is:

$$dpr_1(X) - d_x\phi(dpr_2(X)).$$

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$$\mathrm{Sol}(\mathcal{G}, \theta^1) = \{\beta : M \to J^1(M \times M) \mid \beta = j^1 f, f \text{ a local diffeo}\}$$

correspond to VB-iso $F : TM \to TM$ over f s.t $F_x = d_x f$.

Definition of Lie-Prolongation

Definition

Let $(\mathcal{G}, \theta)$ be a Pfaffian groupoid. A Lie-prolongation of $(\mathcal{G}, \theta)$ is a Pfaffian groupoid $(\mathcal{G}', \theta')$ together with a Lie groupoid morphism

$$p : (\mathcal{G}', \theta') \longrightarrow (\mathcal{G}, \theta), \quad p \text{ surjective}$$

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$$\ker \theta' \cap \ker ds' = \ker \theta' \cap \ker dt',$$

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$$\ker \theta' \cap \ker ds' = \ker \theta' \cap \ker dt',$$
- $Lie(p) = \theta'|_{A'}$, $A' = Lie(\mathcal{G}')$,
- $dp(\ker \theta') \subset \ker \theta$,
- for $X, Y \in \ker \theta'$, $\delta\theta(dp(X), dp(Y)) = 0$,

Definition of the Classical Lie-prolongation

Definition

The classical Lie-prolongation space $P(\mathcal{G},\theta)$ of $(\mathcal{G},\theta)$ consists of $j^1_x\beta \in J^1\mathcal{G}$ with the property that for any $X, Y \in T_xM$

$$\theta(d_x\beta(X)) = 0 \quad \text{and} \quad \delta\theta(d_x\beta(X), d_x\beta(Y)) = 0.$$

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Proposition

Whenever $P(\mathcal{G},\theta) \subset J^1\mathcal{G}$ smooth and $pr : P(\mathcal{G},\theta) \to \mathcal{G}$ is a submersion,

$$(P(\mathcal{G},\theta), \theta^{(1)} = \theta^1|_{P(\mathcal{G},\theta)})$$

is a Lie-prolongation of $(\mathcal{G},\theta)$.

Examples

Example (Rotations on the plane)

For $\mathcal{G} = S^1 \ltimes R^2$, a bisection $\beta : \mathbb{R}^2 \longrightarrow \mathcal{G}$ is of the form $\beta = (\alpha, id)$, with $(x, y) \mapsto \alpha \cdot (x, y)$ a diffeo.

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$$P(\mathcal{G}, \theta) = \{j^1_{(x,y)}\beta \mid \frac{\partial \alpha}{\partial x}|_{(x,y)} = \frac{\partial \alpha}{\partial y}|_{(x,y)} = 0\}$$

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Example (Jet groupoid and the Cartan form)

For $J^1(M \times M)$ and the Cartan form θ^1,

$$P(J^1(M \times M), \theta^1) = J^2(M \times M), \quad \text{and} \quad (\theta^1)^{(1)} = \theta^2,$$

where $J^2(M \times M)$ is the second jets of local diffeos, and θ^2 is the Cartan form.

Definition of Spencer operator

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$$D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E),\ (X, \alpha) \mapsto D_X(\alpha)$$

together with a surjective V.B-map $l : A \to E$, which is $C^\infty(M)$-linear in X, satisfies the Leibniz identity relative to l:

$$D_X(f\alpha) = fD_X(\alpha) + L_X(f)l(\alpha),$$

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together with a surjective V.B-map $l : A \longrightarrow E$, which is $C^\infty(M)$-linear in X, satisfies the Leibniz identity relative to l:

$$D_X(f\alpha) = fD_X(\alpha) + L_X(f)l(\alpha),$$

and the following two compatibility conditions:

$$D_{\rho(\alpha)}(\alpha') = \nabla_{\alpha'}(l(\alpha)) + l([\alpha, \alpha'])$$

$$D_X[\alpha, \alpha'] = \nabla_\alpha(D_X\alpha') - D_{[\rho(\alpha), X]}\alpha' - \nabla_{\alpha'}(D_X\alpha) + D_{[\rho(\alpha'), X]}\alpha.$$

Integrability result for Pfaffian groupoids

Theorem

Let $E \in Rep(\mathcal{G})$ and $A = Lie(\mathcal{G})$. Then any multiplicative form $\theta \in \Omega^1(\mathcal{G}, t^*E)$, making $(\mathcal{G}, \theta)$ Pfaffian, induces a Spencer operator on A with coefficient on E, given by

$$D_X(\alpha) = \text{``}L_{\alpha^r}\theta(X)'', \quad \textit{and} \quad l(\alpha) = \theta(\alpha),$$

with the property that $\ker l \subset A$ *is a Lie subalgebroid.*

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with the property that $\ker l \subset A$ *is a Lie subalgebroid.*
If $\mathcal{G}$ is source simply connected, then this construction defines a 1-1 correspondence.

Examples

Example (Rotations on the plane)

The infinitesimal action of $S^1 \curvearrowright \mathbb{R}^2$ is defined by
$a : \mathbb{R} \longrightarrow \mathfrak{X}(\mathbb{R}^2), 1 \mapsto a(1)_{(x,y)} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

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$$Lie(S^1 \ltimes \mathbb{R}^2) = \mathbb{R} \ltimes \mathbb{R}^2.$$

The associated Spencer operator $D : \mathbb{R}^2 \times \Gamma(\mathbb{R} \ltimes \mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, $l : \mathbb{R} \ltimes \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$ of $d\alpha \in \Omega^1(S^1 \ltimes R^2)$ is then

$$l(r) = r, \quad \text{and} \quad D_X(f) = fdf(X).$$

Examples

Example (Jet algebroids and the classical Spencer Operator)

For a V.B $A \longrightarrow M$, one has a decomposition of vector spaces

$$\Gamma(J^1 A) \simeq \Gamma(A) \oplus \Omega^1(M, A),$$

coming from the exact sequence

$$0 \longrightarrow T^*M \otimes A \longrightarrow J^1 A \stackrel{pr}{\longrightarrow} A \longrightarrow 0.$$

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$l = pr : J^1A \to A$ is the projection to the 2nd component.

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The classical Spencer operator $D^{clas} : \Gamma(J^1A) \to \Omega^1(M, A)$,
$l = pr : J^1A \to A$ is the projection to the 2nd component.

- If A is a Lie algebroid D^{clas} is a Spencer Operator,
- if $A = Lie(\mathcal{G})$, D^{clas} is the Spencer Operator of the Cartan form $\theta^1 \in \Omega^1(\mathcal{G}, t^*A)$.

Definition of compatible Spencer Operators

Definition

Let

$$\begin{aligned} &\tilde{D} : \mathfrak{X}(M) \times \Gamma(A') \to \Gamma(A), && D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E) \\ &\tilde{l} : A' \to A && l : A \to E \end{aligned}$$

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- $$D \circ \tilde{l} - l \circ \tilde{D} = 0$$
  $$D_X \tilde{D}_Y - D_Y \tilde{D}_X - l \circ \tilde{D}_{[X,Y]} = 0$$

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- l' is a Lie algebroid map.

Definition of the classical Lie prolongation

Definition

The classical Lie prolongation space $P_D(A)$ of $D : \mathfrak{X} \times \Gamma(A) \longrightarrow \Gamma(E)$, $l : A \longrightarrow E$, consists of elements $(\alpha, \omega)_x \in J^1A$ with the property that for any $X, Y \in \mathfrak{X}(M)$,

$$D(\alpha)(x) = l(\omega_x)$$

$$D_X(\omega(Y))(x) - D_Y(\omega(X))(x) - l \circ \omega[X, Y]_x = 0.$$

Proposition

Whenever $P_D(A)$ smooth and $pr : P_D(A) \to A$ surjective,

$$(P_D(A), D^{(1)} = D^{clas}|_{P_D(A)})$$

is compatible with D.

Proposition

Whenever $P_D(A)$ smooth and $pr : P_D(A) \to A$ surjective,

$$(P_D(A), D^{(1)} = D^{clas}|_{P_D(A)})$$

is compatible with D. If D is the Spencer operator of $(\mathcal{G}, \theta)$, then $P(\mathcal{G}, \theta)$ is smooth iff $P_D(A)$ is smooth, and

$$Lie(P(\mathcal{G}, \theta)) = P_D(A).$$

Integrability result for Lie-Prolongations

Theorem

Let $\mathcal{G}'$ be Lie groupoid, and $(\mathcal{G}, \theta)$ a Pfaffian groupoid, and let $D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E)$ be the Spencer Operator of $(\mathcal{G}, \theta)$.

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Theorem

Let $\mathcal{G}'$ be Lie groupoid, and $(\mathcal{G},\theta)$ a Pfaffian groupoid, and let $D:\mathfrak{X}(M)\times\Gamma(A)\longrightarrow\Gamma(E)$ be the Spencer Operator of $(\mathcal{G},\theta)$. If $\mathcal{G}'$ is s-simply connected and $\mathcal{G}$ s-connected, then there is a 1-1 correspondence

- *Lie prolongations $p:(\mathcal{G}',\theta')\longrightarrow(\mathcal{G},\theta)$, and*
- *Spencer operators $D:\mathfrak{X}(M)\times\Gamma(A')\longrightarrow\Gamma(A)$ compatible with D.*

In this correspondence D' is the Spencer Operator of θ'.

Maurer-Cartan equation

Out of $D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E)$, one has an antysimmetric bilinear map

$$\{\cdot,\cdot\}_D : A \times A \to E, \quad \frac{1}{2}\{\alpha,\beta\}_D = D_{\rho(\alpha)}(\beta) - D_{\rho(\beta)}(\alpha) - l[\alpha,\beta]$$

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and a differential operator

$$d_D : \Omega^1(\mathcal{G}', t^*A) \to \Omega^2(\mathcal{G}', t^*E),$$

$d_D\theta'(X,Y) = D^t_X(\theta'(Y)) - D^t_Y(\theta'(X)) - l(\theta'[X,Y])$, where D^t is the pullback of D via $t : \mathcal{G}' \to M$.

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$$MC(\theta',\theta) := d_D\theta' - \frac{1}{2}\{\theta',\theta'\}_D.$$

Lie Prolongations and MC

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If $\mathcal{G}'$ is s-connected and p is a submersion with $Lie(p) = \theta'|_{A'}$, then the converse also holds.

Thank you

Thank you for your attention.