Statistical topological data analysis using persistence landscapes applied to brain arteries

CANSSI–SAMSI Workshop: Geometric Topological

and Graphical Model Methods in Statistics

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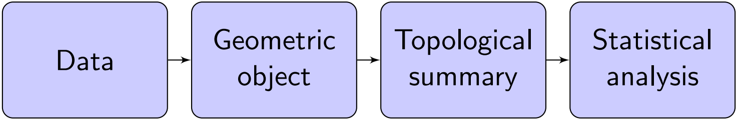
Statistical topological data analysis

The plan:



Brain arteries

Joint work with Ezra Miller (Duke/SAMSI), J.S. Marron (UNC-CH), Paul Bendich (Duke) and Sean Skwerer (UNC-CH).

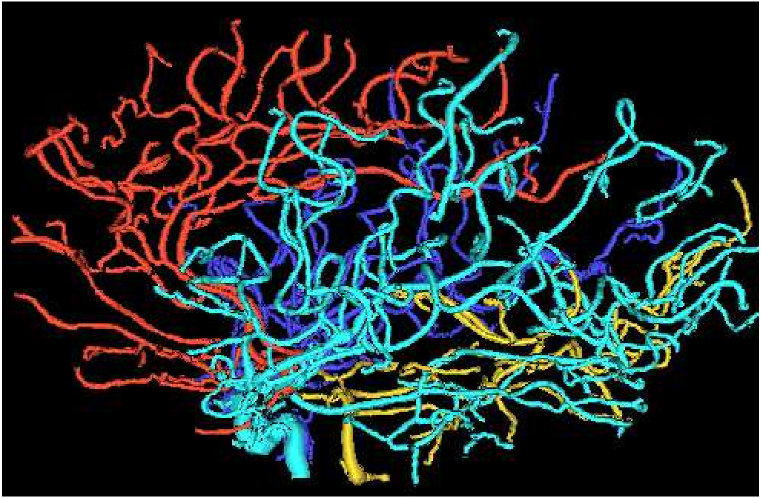
Brain arteries

Goal: Analyze the shape of brain arteries in order to

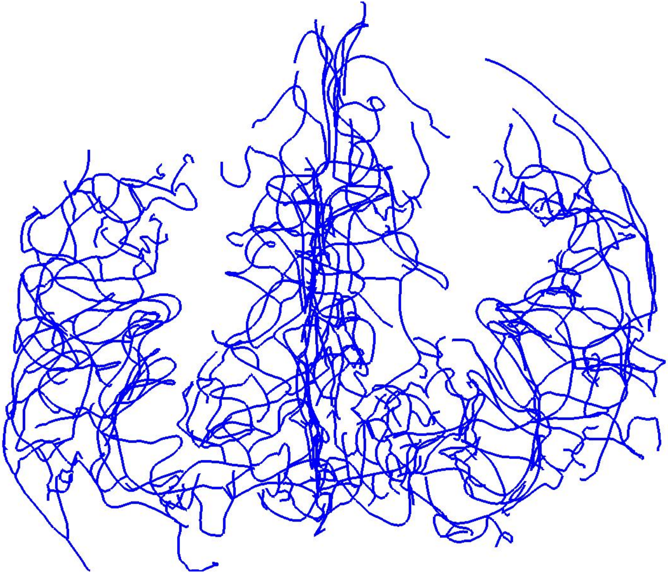
- understand normal changes with respect to age
- detect and locate pathology (tumors)
- predict stroke risk

The data

Bullitt and Aylward (2002) MRA $\rightarrow$ Tubes



Filling the arteries – increasing sublevel sets



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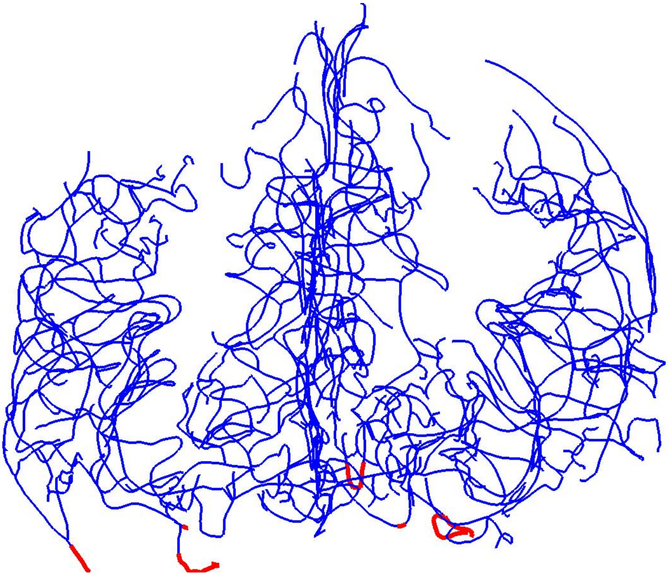

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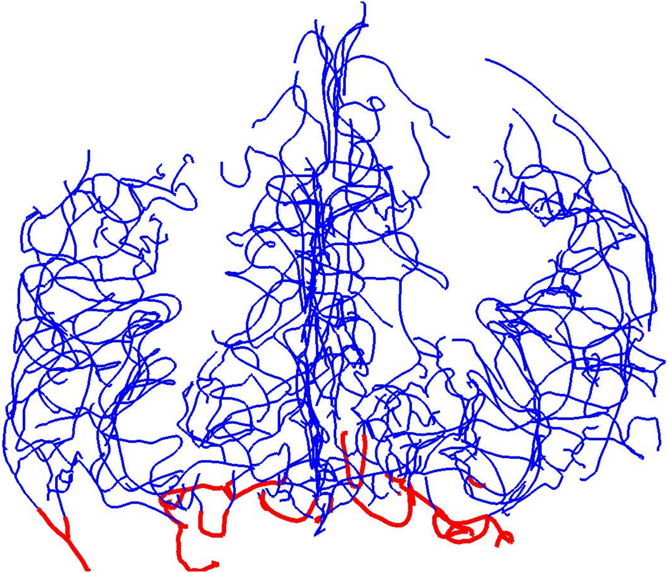

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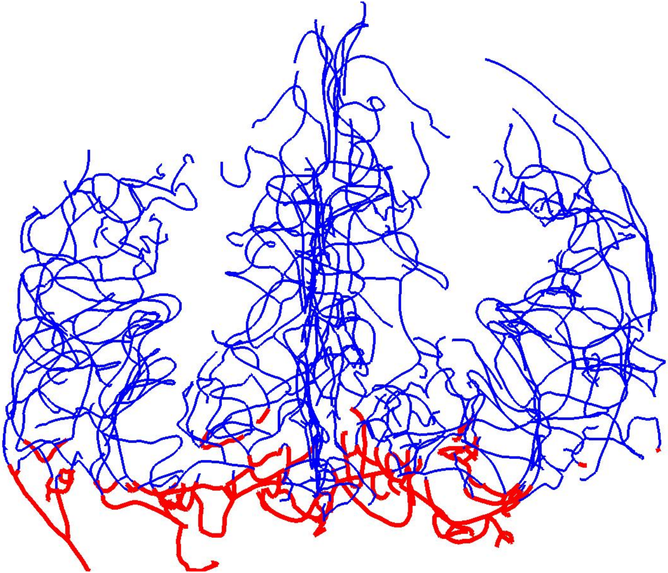

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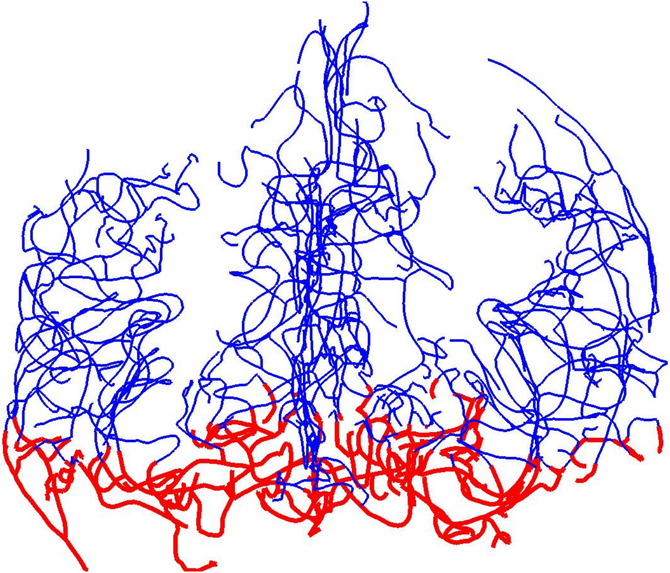

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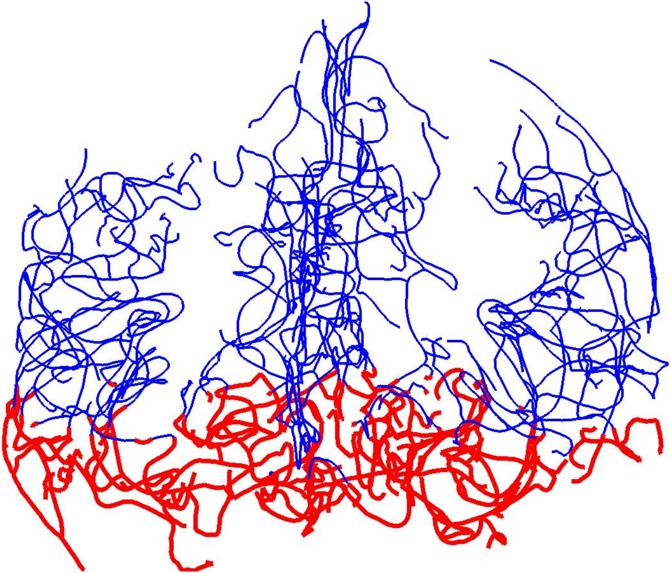

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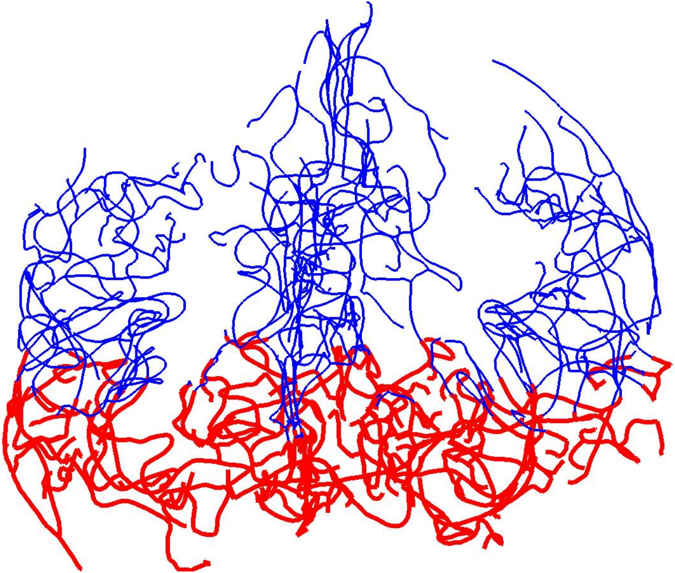

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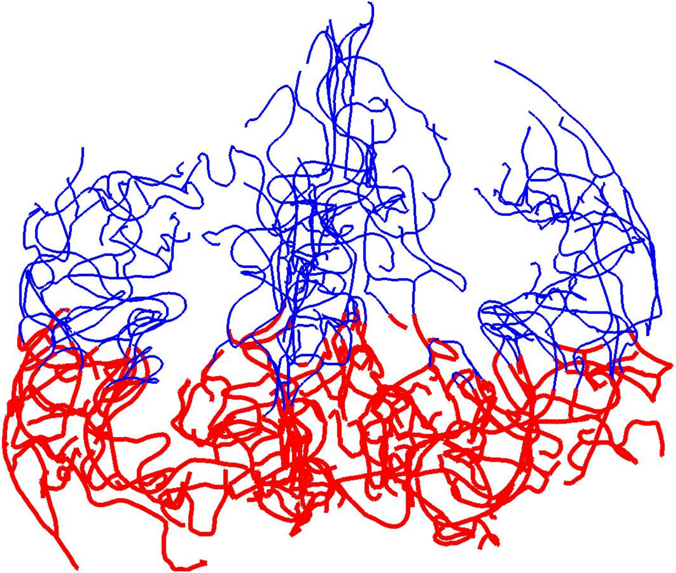

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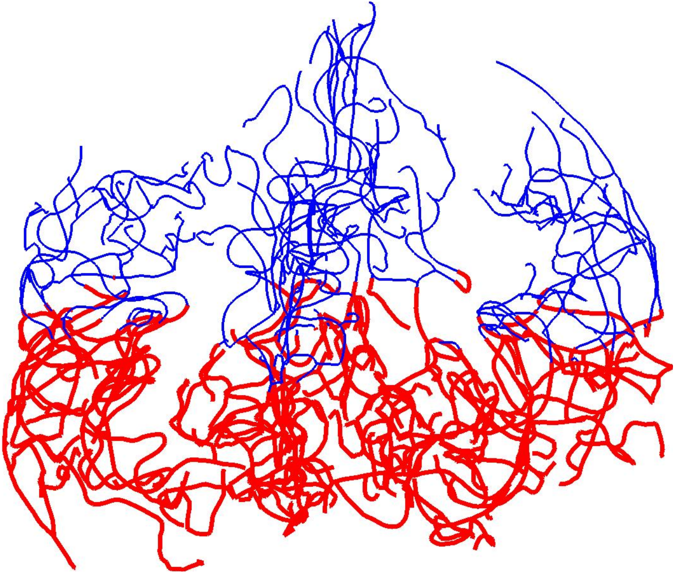

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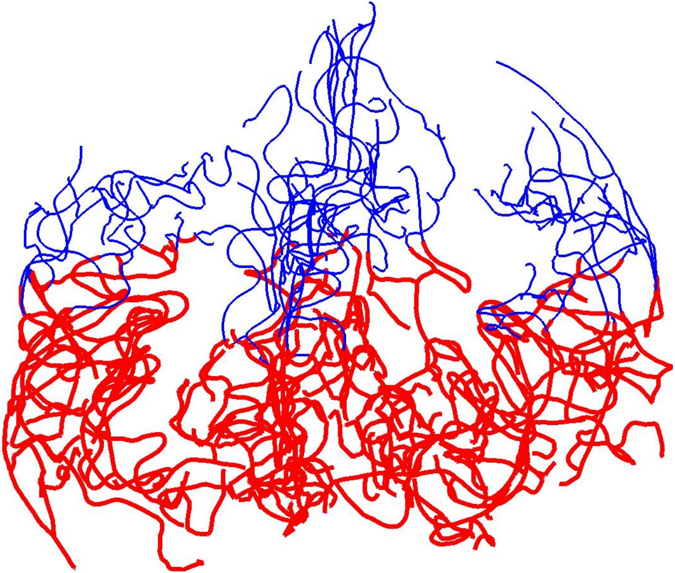

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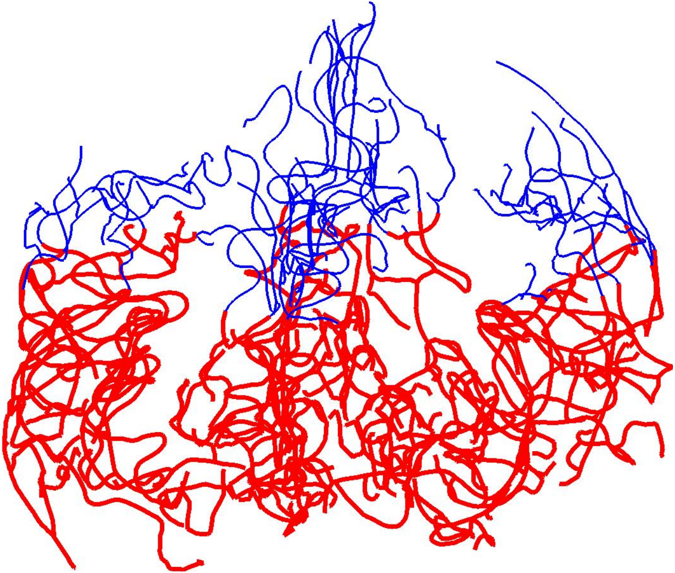

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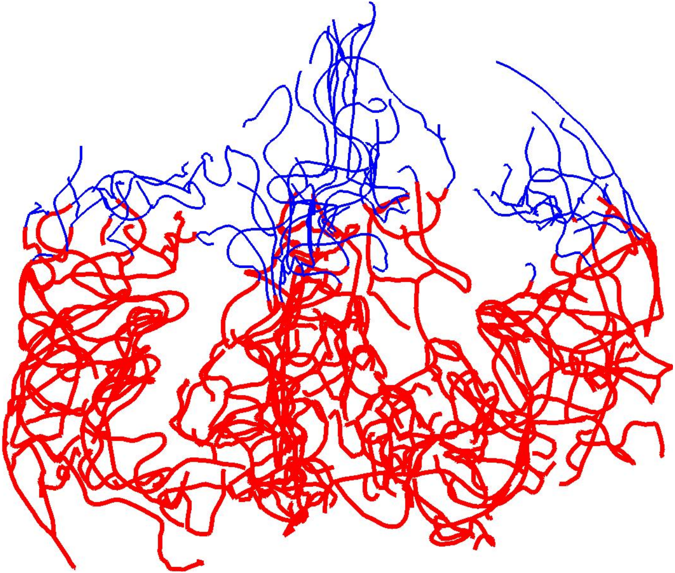

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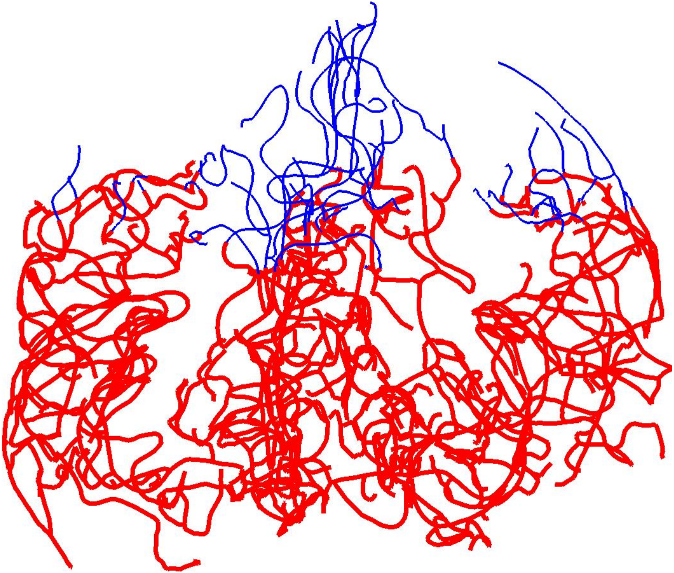

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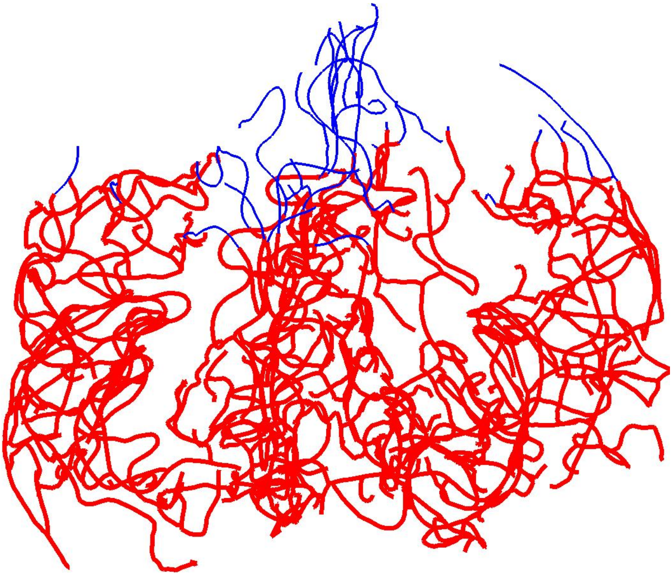

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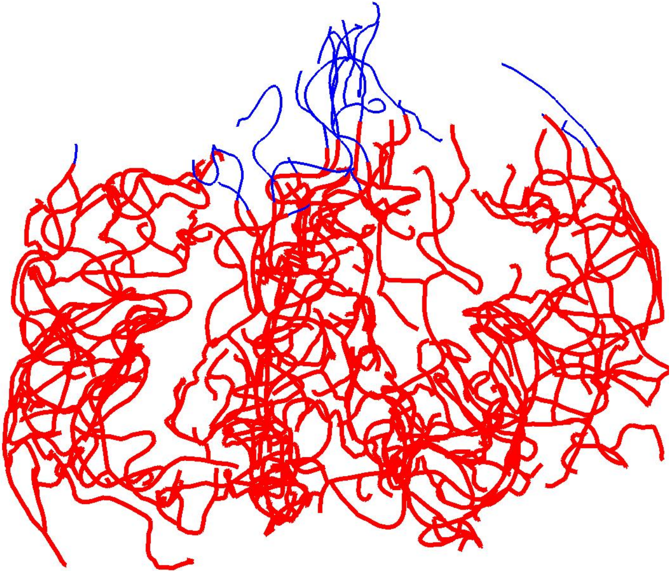

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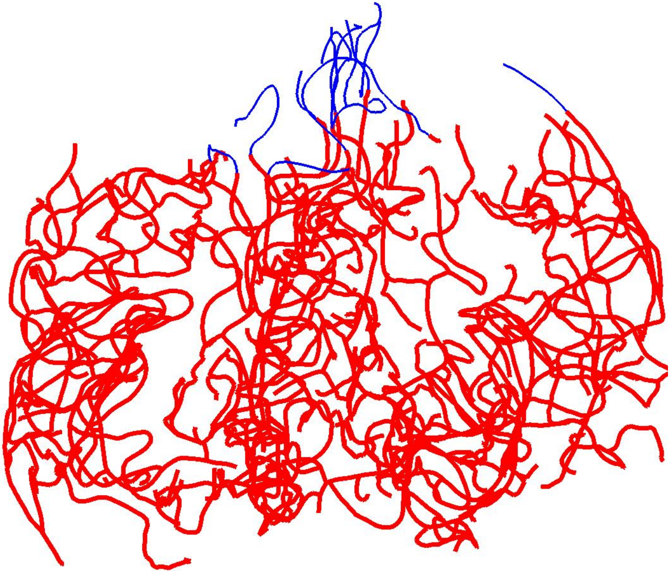

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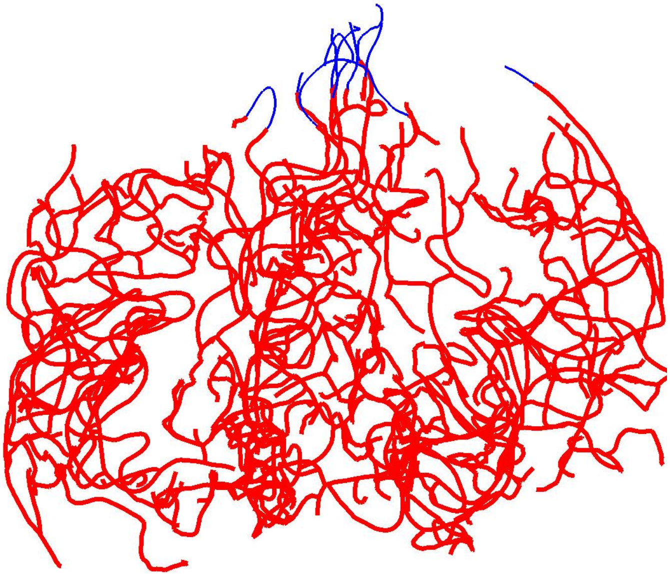

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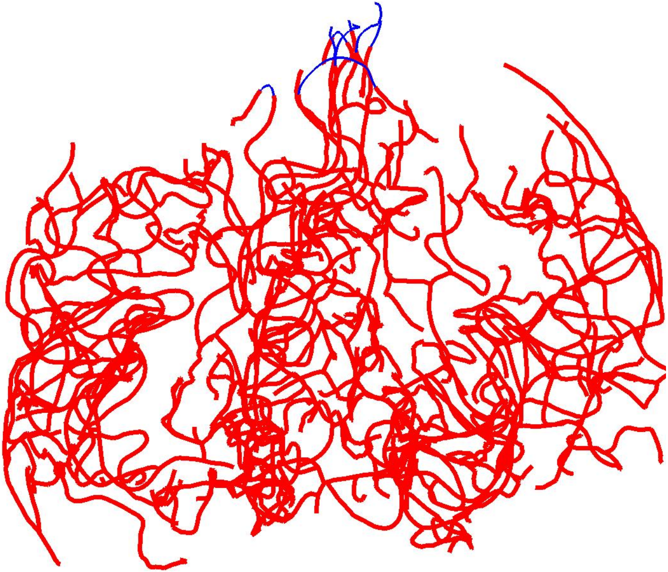

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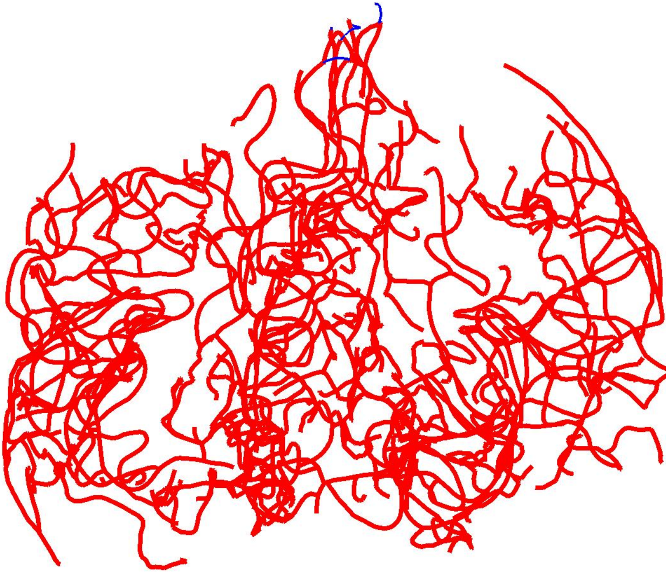

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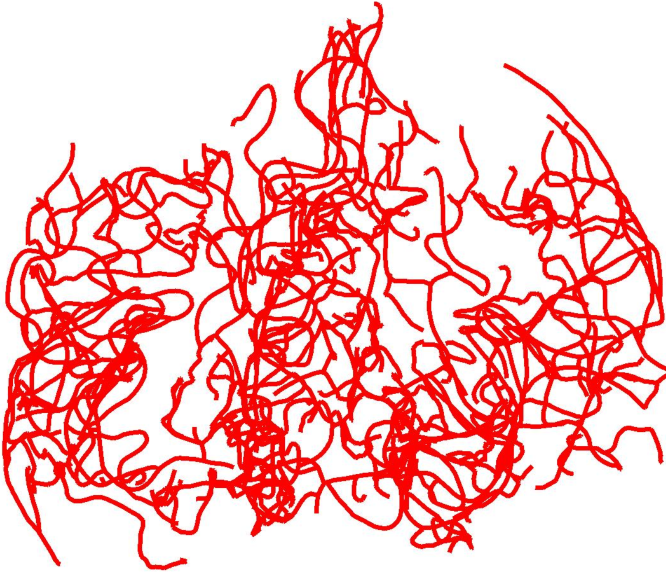

Mathematical viewpoint

Let X be a graph representing the brain arteries of one subject:

- vertices with (x, y, z, r) coordinates
- edges connecting adjacent vertices

Mathematical viewpoint

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- vertices with (x, y, z, r) coordinates
- edges connecting adjacent vertices

Let X_t denotes the full subgraph on the vertices with z coordinate at most t.

$$\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_N = X$$

Take homology in degree 0.

$$H_0(X_0) \to H_0(X_1) \to H_0(X_2) \to \cdots \to H_0(H_N)$$

More general setup

For each t, have
- a simplicial complex X_t
- a vector space $H(X_t)$

For $t \leq t'$, have
- an inclusion $X_t \subseteq X_{t'}$
- a linear map $H(X_t) \to H(X_{t'})$

Persistent homology is the image of this map.

This set of vector spaces and linear maps is called a persistence module.

We want a summary of the persistence module that is amenable to statistical analysis.

Persistence landscape

Recall that the persistence module consisted of linear maps

$$H(X_t) \to H(X_{t'}), \text{ for } t \leq t'.$$

For $k = 1, 2, 3, \ldots$, define $\lambda_k : \mathbb{R} \to \mathbb{R}$ by

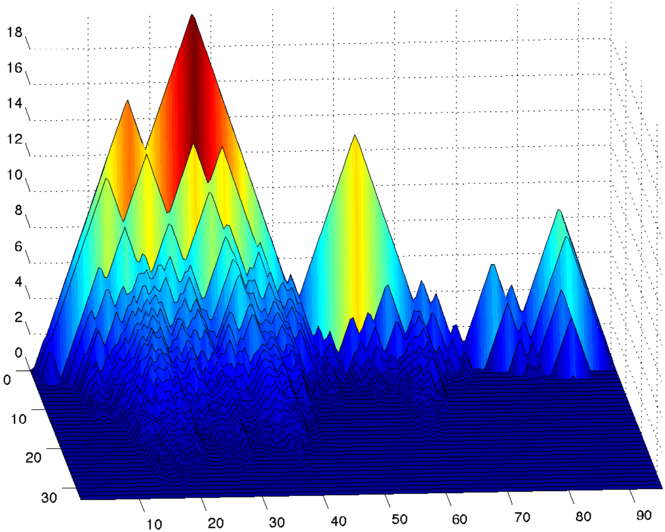
$$\lambda_k(t) = \max(\ h \ \mid \ \text{rank}(H(X_{t-h}) \to H(X_{t+h}) \geq k\ )$$

We can combine these to get one function

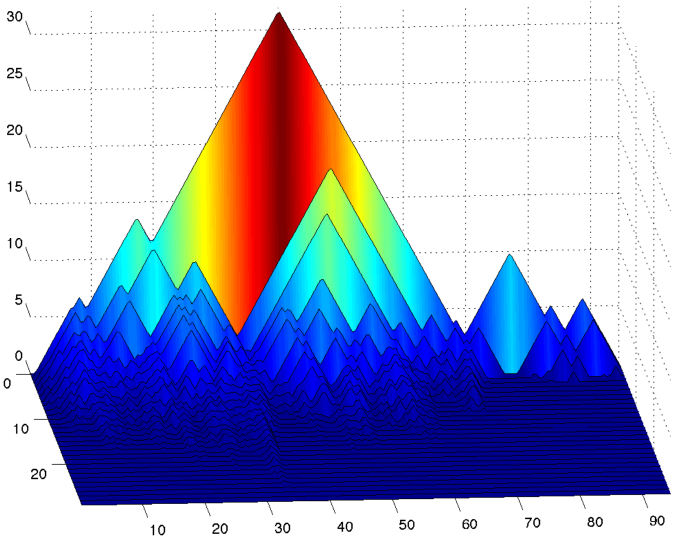
$$\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R},$$

where $\lambda(k, t) = \lambda_k(t)$.

Persistence landscape examples



Persistence landscape examples



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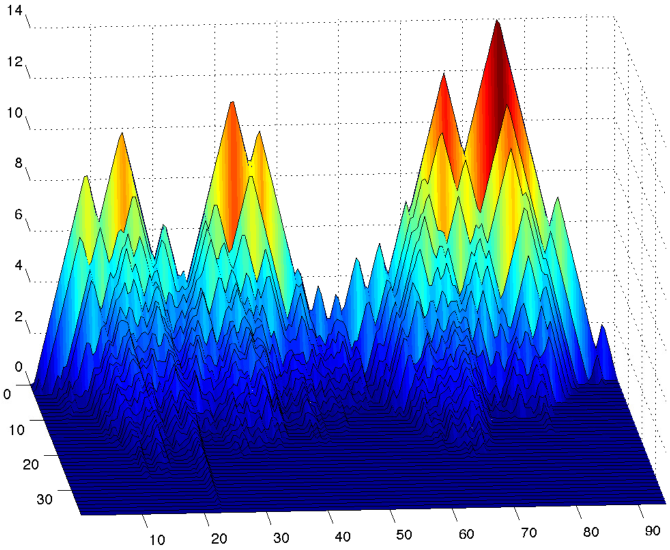

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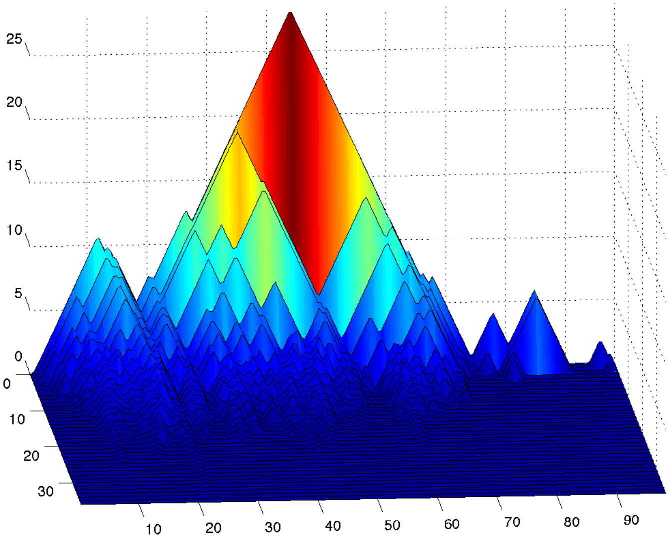

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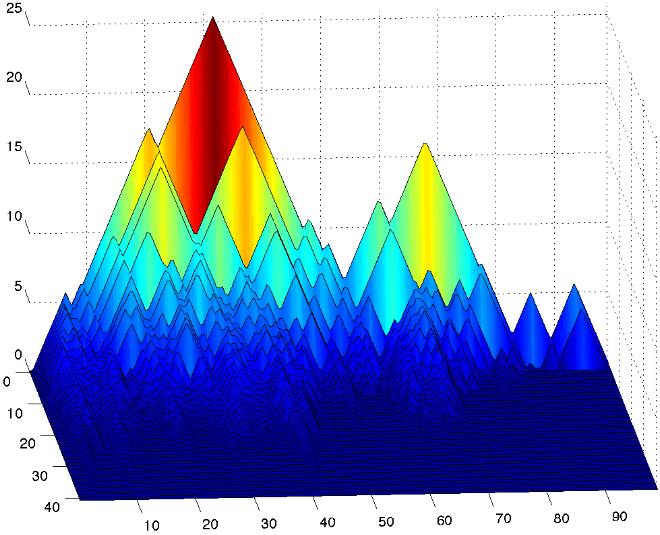


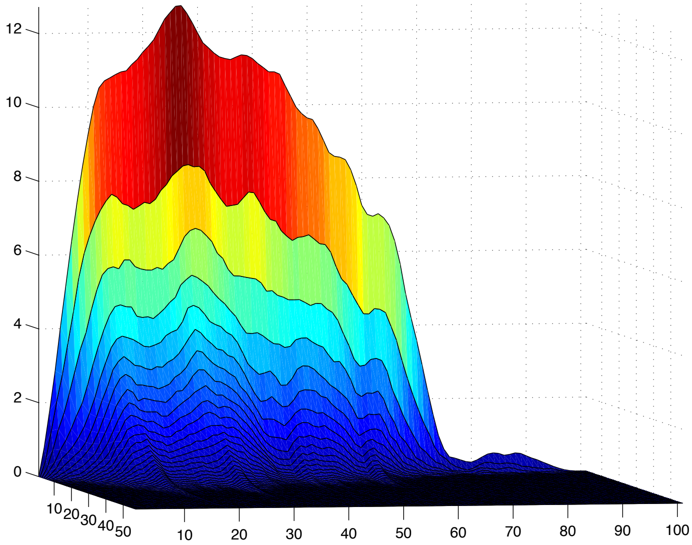
Mean landscapes

Persistence landscapes, $\lambda^{(1)}, \ldots, \lambda^{(n)}$, have mean, $\overline{\lambda} = \frac{1}{n}\sum_{i=1}^{n} \lambda^{(i)}$.

That is,

$$\overline{\lambda}_k(t) = \frac{1}{n}\sum_{i=1}^{n} \lambda_k^{(i)}(t)$$

Mean landscape for brain arteries



Summary space

Let $1 \leq p < \infty$. Then $\|\lambda\|_p = \left(\sum_k \int {\lambda_k}^p \right)^{\frac{1}{p}}$.

We assume $\|\lambda\| := \|\lambda\|_p < \infty$. That is, $\lambda \in L^p(\mathbb{N} \times \mathbb{R})$.

So λ is a random variable with values in a Banach space.

Asymptotics

$\lambda \in L^p(\mathbb{N} \times \mathbb{R})$, $\|\lambda\|$ is a real random variable.

If $E\|\lambda\| < \infty$ then there exists $E(\lambda) \in L^p(\mathbb{N} \times \mathbb{R})$ such that $E(f(\lambda)) = f(E(\lambda))$ for all continuous linear functionals f.

Theorem (Strong Law of Large Numbers (SLLN))

$\overline{\lambda}^{(n)} \to E(\lambda)$ *almost surely if and only if* $E\|\lambda\| < \infty$.

Theorem (Central Limit Theorem (CLT))

Assume $p \geq 2$. *If* $E\|\lambda\| < \infty$ *and* $E(\|\lambda\|^2) < \infty$ *then* $\sqrt{n}[\overline{\lambda}^{(n)} - E(\lambda)]$ *converges weakly to a Gaussian random variable with the same covariance structure as* λ.

Weighted norms

Recall that $\|\lambda\|_p = \left(\sum_k \int \lambda_k{}^p \right)^{\frac{1}{p}}$.

Fix $i \leq j$. Define $\|\lambda\|_{p,i,j} = \left(\sum_{k=i}^{j} \int \lambda_k{}^p \right)^{\frac{1}{p}}$.

The previous SLLN and CLT also apply to this weighted norm.

Correlation with age

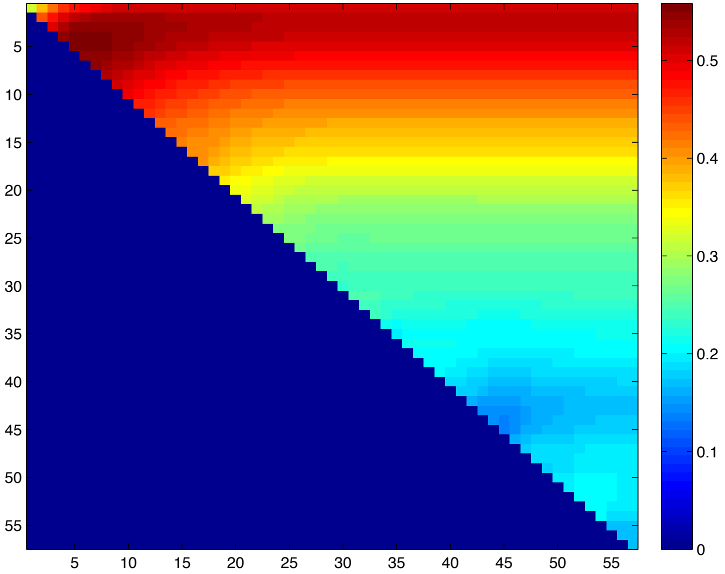
Pearson's correlation coefficient of
age with statistics derived from the brain arteries

Previous study without topology:
Dan Shen et al (2014) $r = 0.25$

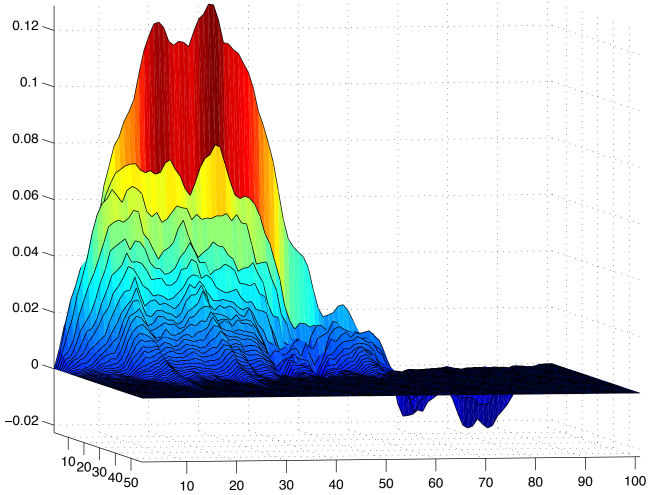
Using persistence landscape:

topological statistic	r
$\|\lambda\|_1$	0.5077
$\|\lambda\|_{1,2,57}$	0.5214
$\|\lambda\|_{1,5,5}$	0.5582

Correlation of age with $\|\lambda\|_{1,i,j}$



Principal Component Analysis



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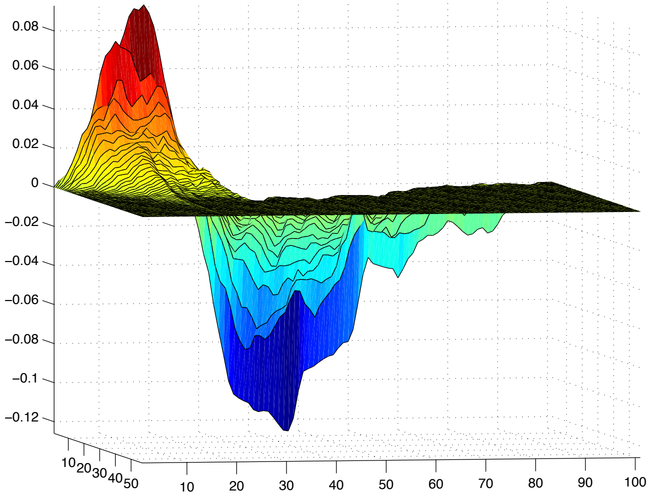

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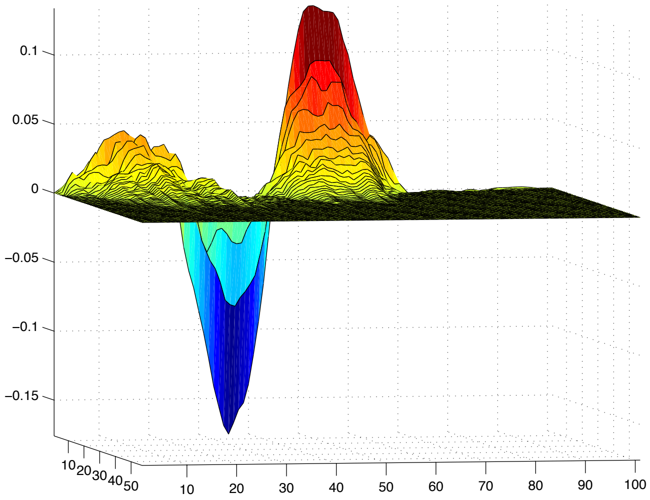

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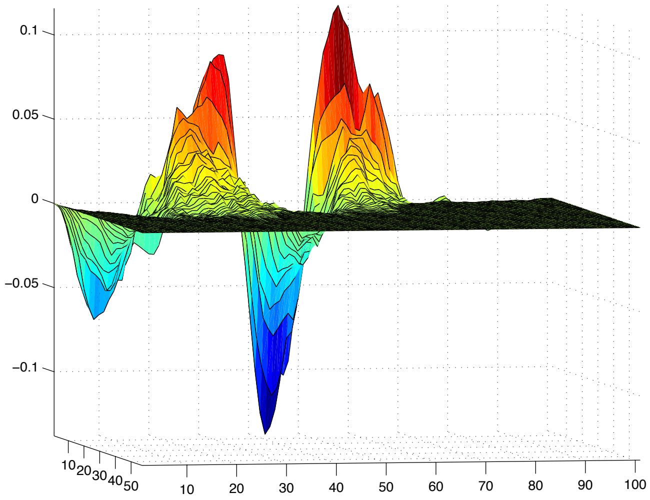

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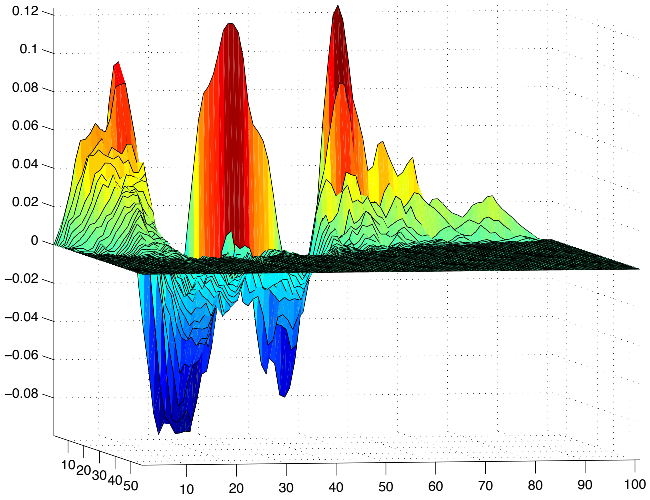

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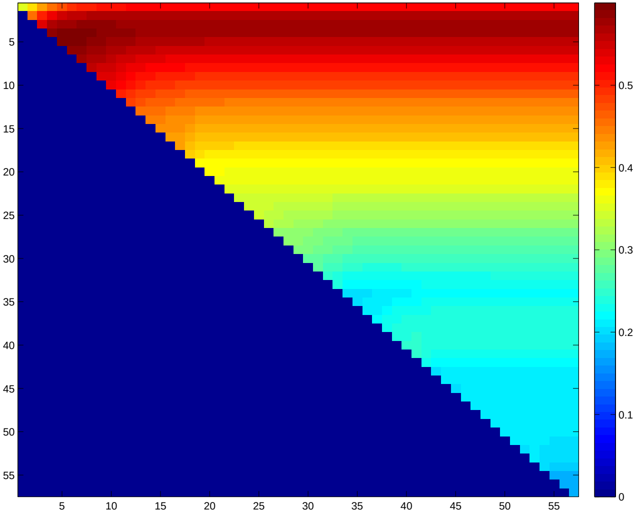
Values of r using statistics derived from persistence landscape:

landscapes used	1-norm	first princ comp
$\lambda_1, \ldots, \lambda_{57}$	0.5077	0.5216
$\lambda_2, \ldots, \lambda_{57}$	0.5214	0.5666
$\lambda_5, \ldots, \lambda_5$	0.5582	0.6000

Correlation of age with PCA1 on weighted norms



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- Topology promising tool for analyzing data
- Persistence landscapes easy to combine with standard statistical techniques
- Looking for collaborators

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Thank you!