Nonuniform Models for Robust Network Design

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Joint work with Sebastian Stiller and Rico Zenklusen

July 31, 2013

Some Keywords

- Combinatorial Problems (Shortest Path, Spanning Tree...)
- Adversarial Failure Model ("Robust" Optimization)
- Covering Problems
    - Nonuniform failure scenarios ? , ?
- Complexity/Algorithms (exact, approximation)

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Nonuniform Faults in Networks

How do failures patterns look like? depends...

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What is the cause for failures? depends...

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What does the network represent?

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Nonuniform Faults in Networks (Example 1)

Distributed Computer Systems

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Distributed Computer Systems

- Many processes running on various physical machines, using various resources
- Faults: downtime of resources.

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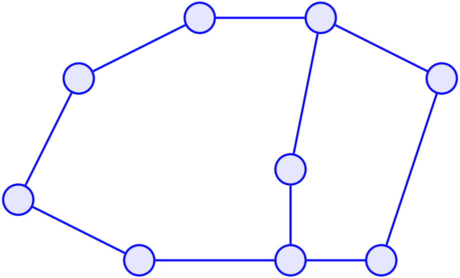

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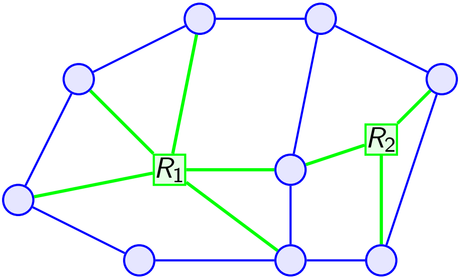

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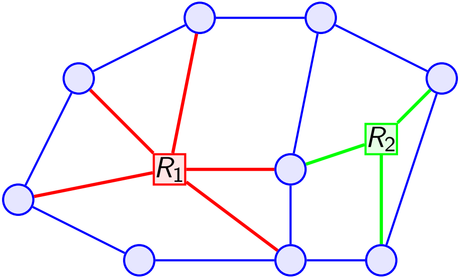

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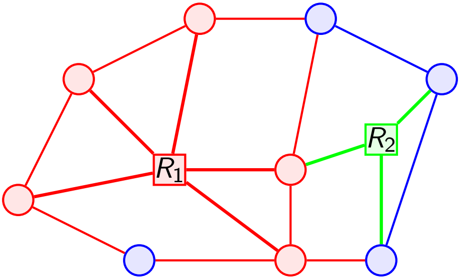

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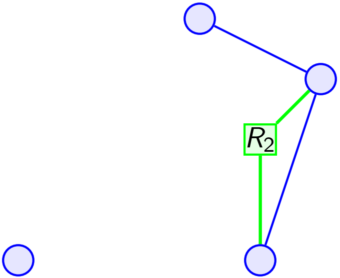

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Similar characteristics in networks representing

- Health care facilities
- Digitally controlled infrastructures
- Hierarchical organizations
- ...

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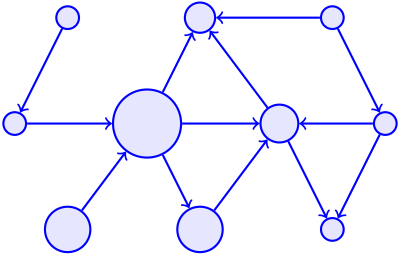

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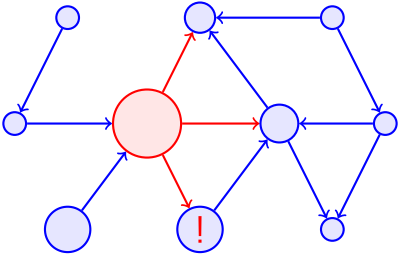

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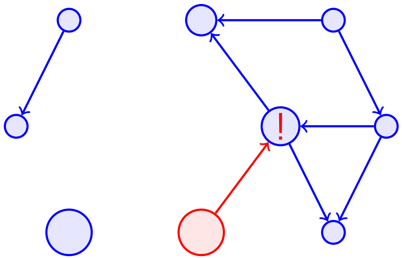

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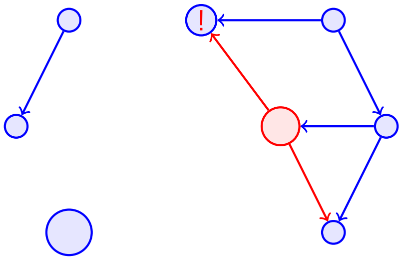

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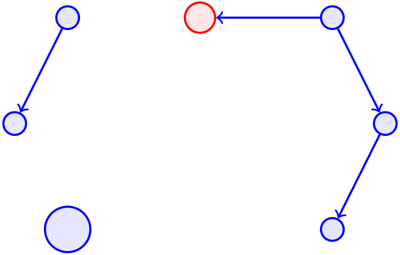

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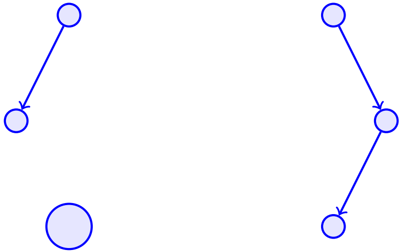

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Similar characteristics in networks representing

- Electricity networks
- Insurance networks
- Social networks
- ...

Nonuniform Faults in Networks

Scenario Sets $\Omega \subset 2^N$ that feature

- Simultaneous failure of variable-size parts of the network
- Failure of a single resource that causes failure of multiple network components
- Propagation effects

Certainly,

$$\Omega \neq \{A \subset N : w(A) \leq k\} = \text{"Uniform"}$$

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Nonuniform Models - Bulk Robustness

Combinatorial (covering) problem P:

$$\min\{c(X) : X \in \mathcal{S}\} \qquad (\mathcal{S} \subset 2^N \textit{ feasible set, } c : N \to \mathbb{Z}_+)$$

Scenario set:

$$\Omega = \{F_1, \cdots, F_m\} \qquad (F_1, \cdots, F_m \subset N \textit{ scenarios})$$

Bulk-Robust counterpart $\mathrm{Bulk}(\mathrm{P})$:

$$\min\{c(X) : X \setminus F_i \in \mathcal{S} \quad \forall F_i \in \Omega\}$$

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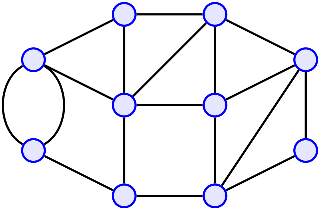

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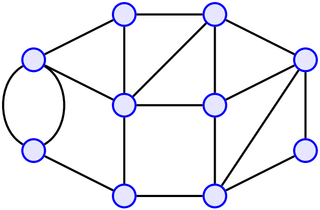

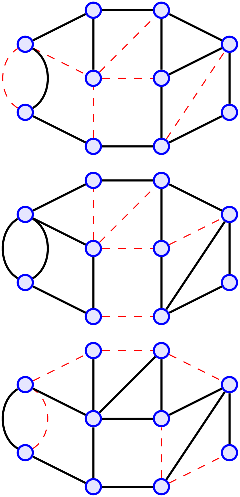

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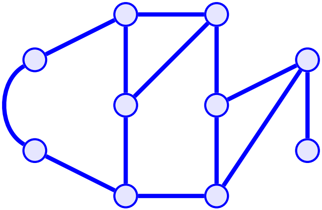

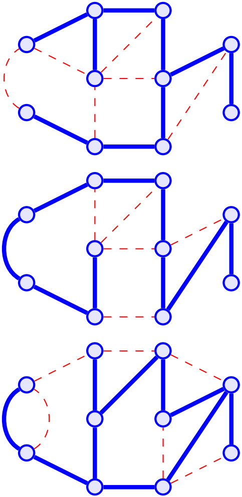

Properties of Bulk(P)

- Bulk(P) can model arbitrary failure types
  - $\Rightarrow$ e.g. edge-sets failures, vertex-sets failures...
- Bulk(P) instance feasible iff $N \setminus F \in \mathcal{S}$ for all $F \in \Omega$.
- β-approximation for P implies a $|\Omega|\beta$-approximation for Bulk(P):
  - Compute β-approximate solution X_F for relaxation

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Bulk(P) and Set Cover

Let $N = \{1, \cdots, n\}$ and P^U : $\min\{c(X) : X \subset N,\ |X| \geq 1\}$

$$\Rightarrow \text{Bulk}(\mathrm{P}^U) \equiv \text{Set Cover}$$

$\Rightarrow$ Bulk(Spanning Tree), Bulk(Shortest Path), etc. unlikely to admit polynomial constant-factor approximation algorithms

(not better than $\ln |\Omega|$)

At the same time, e.g., Bulk(Set Cover) $\approx$ Set Cover

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Bulk(Spanning Tree)

Theorem. There is a polynomial $(\log|\Omega| + \log r)$-approximation algorithm for Bulk(Minimum Matroid Basis).

$\Rightarrow$ $(\log|\Omega| + \log|V|)$-approximation for Bulk(Spanning Tree).

Proof sketch:

Let $r(\cdot)$ denote the rank function of the matroid $\mathcal{M}$.

For $F \in \Omega$ define $r^F(X) = r(X \setminus F)$. $\Rightarrow$ Submodular

Define:

$$f(X) = \sum_{F \in \Omega} r^F(X) \qquad \Rightarrow \text{Submodular}$$

Note: $S \subset N$ feasible iff $f(S) = f(N) = r(\mathcal{M})|\Omega|$

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Theorem. There is a polynomial $(\log|\Omega| + \log r)$-approximation algorithm for Bulk(Minimum Matroid Basis).

$\Rightarrow$ $(\log|\Omega| + \log|V|)$-approximation for Bulk(Spanning Tree).

Proof sketch:

Let $r(\cdot)$ denote the rank function of the matroid $\mathcal{M}$.

For $F \in \Omega$ define $r^F(X) = r(X \setminus F)$. $\Rightarrow$ Submodular

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Try to maximize f without paying more than OPT.

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$\Rightarrow$ Submodular Function maximization with knapsack constraint

NP-hard, but constant-factor approximations exist
e.g. $1 - \frac{1}{e} - \epsilon$ [Kulik, Shachnai, Tamir 10']

$\Rightarrow$ Obtain a set Y_1 with

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Bulk(Shortest Path)

Bulk(Shortest Path) can be even harder:

Steiner Forest can be modeled as Bulk(Shortest Path)

$\Rightarrow$ No $O(2^{\log^{1-\epsilon}|V|})$-approximations for directed graphs.

However: when $k = \max_{F \in \Omega} |F|$ bounded

Theorem. Bulk(Shortest Path) is

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APX-complete when $k = 2$.

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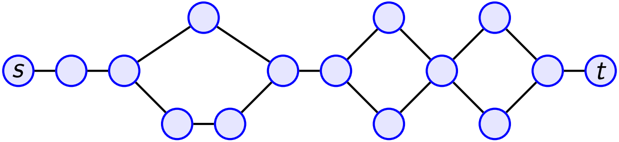

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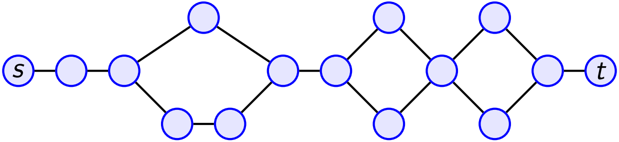

Step 1: Solve for $\Omega_1 = \{\{e\} \subset E : \exists F \in \Omega \, : \, e \in F\}$

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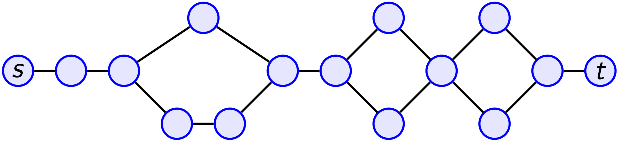

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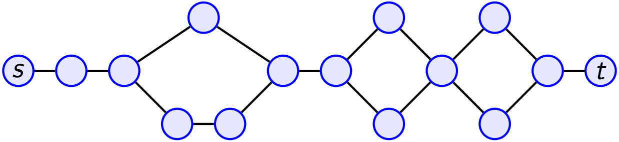

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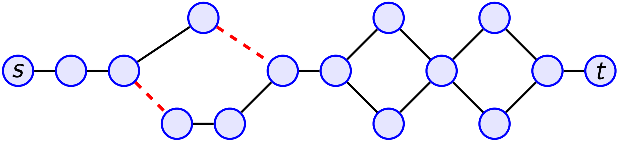

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Bulk(Shortest Path)

Proof sketch (cont.)

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Lemma 2. Minimal X such that $X \cup S_1$ is feasible is a forest.

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Proof sketch (cont.)

Bulk(Shortest Path)

Proof sketch (cont.)

$\Rightarrow$ Want a $O(1)$-apx for that Set Cover problem

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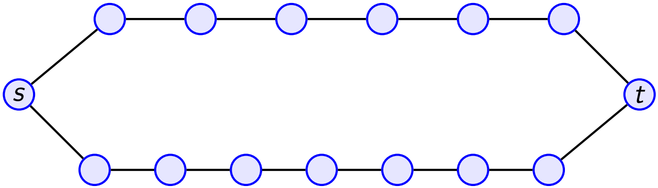

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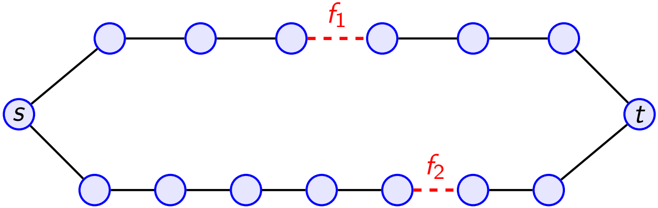

$$F = \{f_1, f_2\}$$

Proof sketch (cont.)

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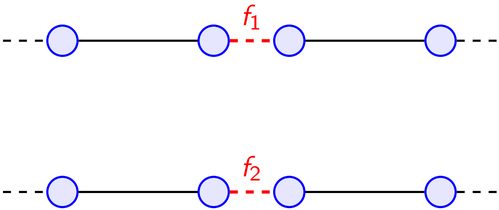

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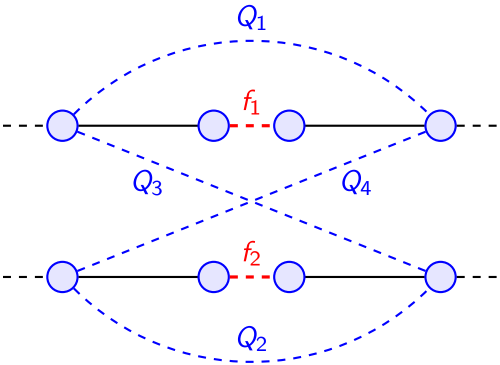

Bulk(Shortest Path)

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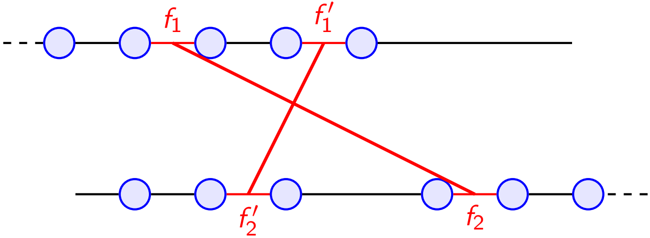

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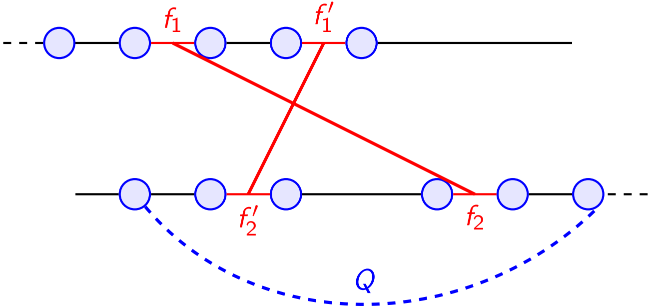

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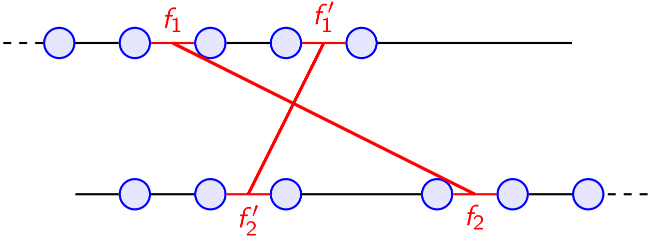

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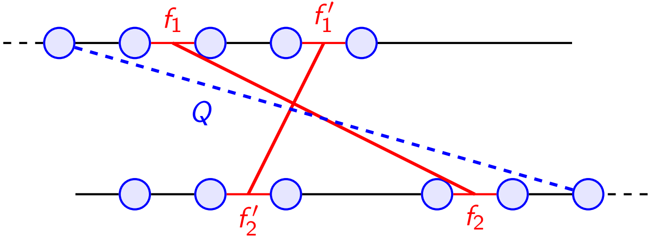

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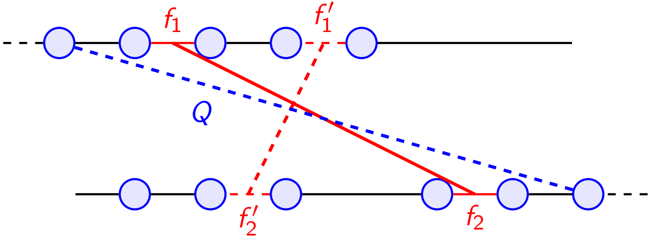

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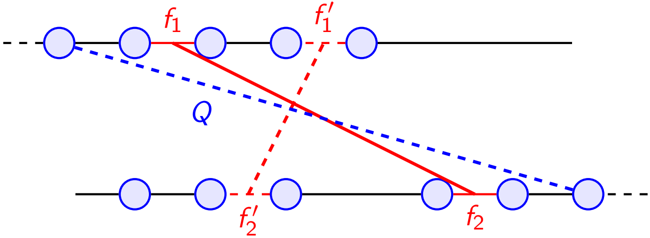

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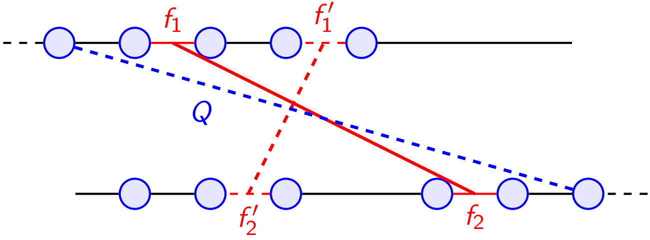

Which scenarios F to cover with which paths? Ask the LP!

Proof sketch (cont.)

Bulk(Shortest Path)

Proof sketch (cont.)

Step 4: Solve

$$\min \left\{ \sum_{u,v \in V} c(Q_{u,v}) x_{u,v} \;:\; \sum_{F \times (u,v)} x_{u,v} \geq 1 \; \forall F \in \Omega \right\}$$

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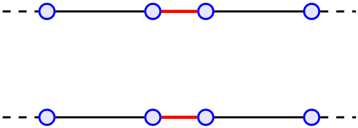

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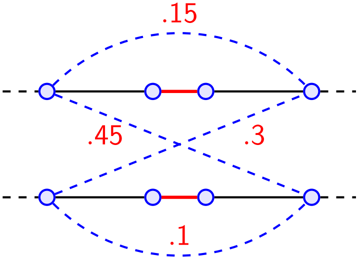

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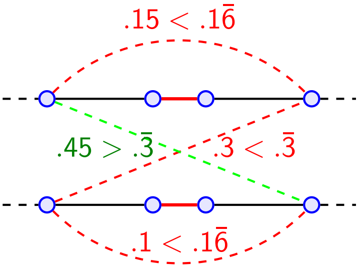

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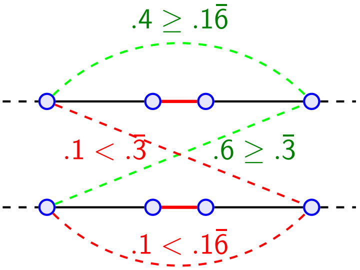

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Partitioning of x^* plus integrality of Interval Cover $\square$

Other nonuniform models

A more realistic model: (*multi*-Bulk Robustness)

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$$\Omega^k = \left\{ Y = \bigcup_{F \in \Omega'} F \ : \ \Omega' \subset \Omega, \ |\Omega'| \leq k \right\}$$

Perhaps more importantly: reliability analysis (interdiction)

- Find $\Omega' \subset \Omega^k$ such that $G - \bigcup_{F \in \Omega'} F \ldots$
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