Iterative rounding approximation algorithms for degree-bounded node-connectivity network design

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Survivable network design (SND)

Problem

Input:

- an undirected or directed graph $G = (V, E)$
- edge-cost $c : E \to \mathbb{Q}_+$
- terminal set $T \subseteq V$
- connectivity requirements $r : T \times T \to \mathbb{N}$

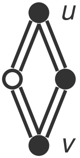
Solution: a minimum cost subgraph of G

Constraints: $\forall u, v \in T$: (connectivity between u and v) $\geq r(u, v)$

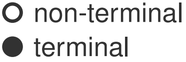
Connectivity

- **edge-connectivity** λ: max # of **edge-disjoint** paths
- **element-connectivity** λ_T:

  max # of paths **disjoint in edges and non-terminals**
- **node-connectivity** κ: max # of paths **disjoint in inner-nodes**



$\lambda(u, v) = 4$
$\lambda_T(u, v) = 3$
$\kappa(u, v) = 2$



Many special cases are defined according to r
(e.g., uniform req., rooted req., subset req.)

Degree-bounded SND

Degree bounds

- Undirected graphs: Given $B \subseteq V$ and $b : B \to \mathbb{N}$,

$$\text{degree of } \forall v \in B \; \leq b(v)$$

- Digraphs: Given $B^-, B^+ \subseteq V$, $b^- : B^- \to \mathbb{N}$ and $b^+ : B^+ \to \mathbb{N}$,

$$\text{in-degree of } \forall v \in B^- \leq b^-(v)$$

$$\text{out-degree of } \forall v \in B^+ \leq b^+(v)$$

Feasible solutions of Degree-bounded SND are Hamiltonian paths

- connectivity requirements: an undirected connected graph
- degree bounds: $B = V$, and $b(v) = 2$ for $\forall v$

➜ **NP-hard to find a feasible solution**

Multi-criteria approximation

Approximation for undirected degree-bounded SND

- $\alpha \in \mathbb{Q}$
- $\beta : \mathbb{N} \to \mathbb{N}$

An algorithm achieves (α, β)**-approximation** if it outputs $F \subseteq E$ such that

- $c(F) \leq \alpha \mathrm{OPT}$ (edge-cost approx)
- degree of $v \leq \beta(b(v))$ for $\forall v \in B$ (degree bounds approx)

for each instance that has a feasible solution.

Key idea: iterative rounding

Iterative rounding is a powerful tool

- Jain '01: 2-approx algorithm for edge-connectivity SND
- Fleischer, Jain, Williamson '01: Extended [Jain '01] to element-connectivity SND and node-connectivity SND w/ $k \leq 2$ ($k := \max_{u,v} r(u, v)$)
- Breakthrough around '07: Applied to degree-bounded spanning tree and degree-bounded SND w/ edge-connectivity req.

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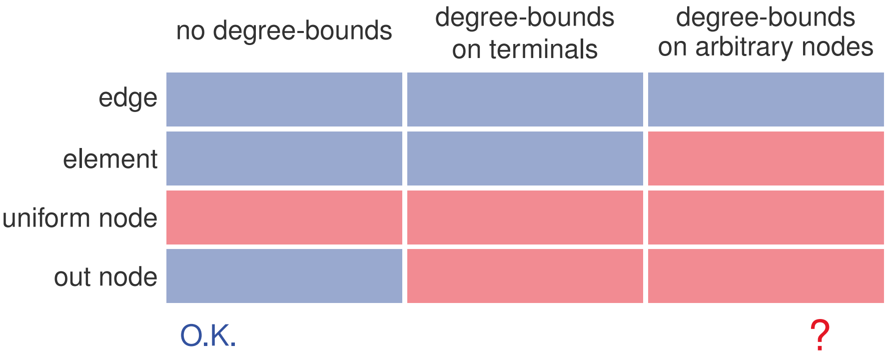
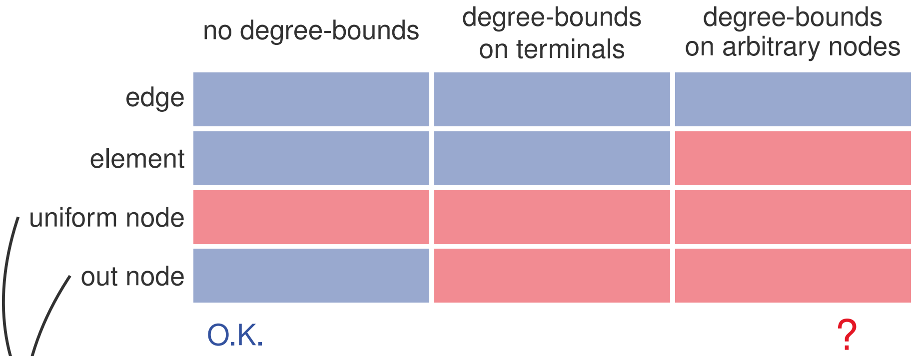
But, **iterative rounding did NOT work well** for

- element-connectivity SND w/ degree-bounds on arbitrary nodes
- node-connectivity SND even w/o degree-bounds if $k \geq 3$

Situation


no degree-bounds
degree-bounds on terminals
degree-bounds on arbitrary nodes
edge
element
uniform node
out node
O.K.
?

Situation

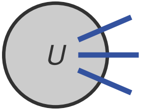


- uniform node-conn. req.: undirected graph, $r(u, v) = k, \forall u, v \in V$
- out node-conne. req.: directed graph, root $s \in V$,

$$r(u, v) = \begin{cases} k & \text{if } u = s, \\ 0 & \text{otherwise.} \end{cases}$$

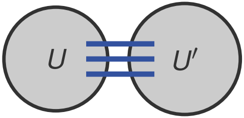
Why were they difficult?

- edge-connectivity SND ➜ covering set functions by edges



$$\geq R(U) := \max_{u \in U, v \notin U} r(u, v)$$

- node-connectivity SND ➜ covering set-pair functions by edges



$$\geq R(U, U') := \max_{u \in U, v \in U'} r(u, v)$$

There was no good analysis of iterative rounding for covering set-pair functions except a few restricted cases.

What did we do?

- We gave two definitions of laminarity for set-pairs
  - Laminarity of set-pairs
  - Strongly laminarity of set-pairs

What did we do?

- We gave two definitions of laminarity for set-pairs
  - Laminarity of set-pairs
  - Strongly laminarity of set-pairs

- We characterized structure on tight set-pair families of element-connectivity and node-connectivity SND
  - Iterative rounding was known to work

    ➜ strongly laminar family (undirected graphs)

    or laminar family, one direction (directed graphs)
  - Iterative rounding was NOT known to work

    ➜ laminar family (undirected graphs)

    or laminar family, both directions (directed graphs)

What did we do?

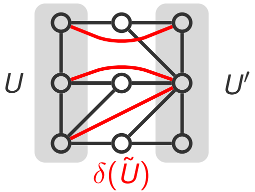
- We gave a new analysis for
  - laminar families (both in undirected and in directed graphs)
  - strongly laminar families w/ degree-bounds
  - no edge-cost case

Our ideas

1. New token counting method for laminar family of set-pairs
2. Using two different counting methods according to # of tight set-pairs v.s. # of tight degree nodes

Set-pair

- set-pair (= biset): ordered pair $\tilde{U} = (U, U')$ of disjoint node sets
- $U :=$ tail, $U' :=$ head
- $\delta(\tilde{U}) := \{uv \in E : u \in U, v \in U'\}$
- $\Gamma(\tilde{U}) := V \setminus (U \cup U')$ (boundary)



LP relaxation

- $R(\tilde{U}) := \max_{u \in U, v \in U'} r(u, v) - |\Gamma(\tilde{U})|$
- $R(\tilde{U}) > 0 \Rightarrow |\Gamma(\tilde{U})| < k$
- $\mathcal{F}$: a family of set-pairs defined depending on the connectivity

Set-pair relaxation for undirected graphs

$$\begin{aligned} \min \quad & c^T x & \\ \text{s.t.} \quad & x(\delta(\tilde{U})) \geq R(\tilde{U}) & \forall \tilde{U} \in \mathcal{F} \\ & x(\delta(v)) \leq b(v) & \forall v \in B \\ & 0 \leq x(e) \leq 1 & \forall e \in E \end{aligned}$$

Laminarity of set-pairs

Laminar family of set-pairs

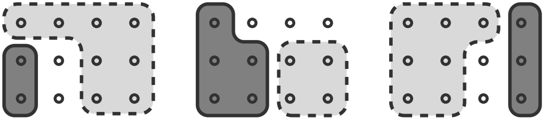
$\mathcal{L}$ is a laminar family of set-pairs if

- $\{U : (U, U') \in \mathcal{L}\}$ is a laminar set family,
- $\forall (U, U'), (W, W') \in \mathcal{L} : U \subseteq W \Rightarrow W' \subseteq U'$.

Strongly laminar family of set-pairs

$\mathcal{L}$ is a strongly laminar family of set-pairs if

- $\mathcal{L}$ is a laminar family of set-pairs,
- $\forall \tilde{U} = (U, U'), \tilde{W} = (W, W') \in \mathcal{L}$:
  $U \cap W = \emptyset \Rightarrow U \cap \Gamma(\tilde{W}) = \emptyset, \Gamma(\tilde{U}) \cap W = \emptyset$.



Laminar
NOT strongly laminar

Results via structure of tight constraints

	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary nodes
element			
uniform node			
out node			

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	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary nodes
element	← strongly	laminar →	laminar
uniform node	←	laminar, if $n > 3k - 3$	→
out node	laminar, only entering arcs	← laminar	→

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1

1. Laminar, undirected ➜ $(O(k), O(k) \cdot b(v))$-approx

Results via structure of tight constraints

	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary nodes	
element	⟵ strongly	laminar ⟶	laminar	
uniform node	⟵	laminar, if $n > 3k-3$	⟶	
out node	laminar, only entering arcs	⟵	laminar ⟶	2

1. Laminar, undirected ➜ $(O(k), O(k) \cdot b(v))$-approx
2. Laminar, directed ➜ $(2, k, 2b^{+}(v) + O(k))$-approx

Results via structure of tight constraints

	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary nodes	
element	⟵ strongly laminar ⟶		laminar	3
uniform node	⟵ laminar, if $n > 3k - 3$ ⟶			
out node	laminar, only entering arcs	⟵ laminar ⟶		

1. Laminar, undirected ➜ $(O(k), O(k) \cdot b(v))$-approx
2. Laminar, directed ➜ $(2, k, 2b^{+}(v) + O(k))$-approx
3. Strongly laminar w/ degree-bounds, undirected ➜ $(4, 4b(v) + O(k))$-approx

Approximation factors: SND w/o degree bounds

node-connectivity

$k \leq 2$	2-approx	[Fleischer et al. 06]	**iterative rounding**
general	$O(k^3 \log n)$-approx	[Chuzhoy, Khanna 09]	**decomposition**
rooted	$O(k \log k)$-approx	[Nutov 09]	**decomposition**
subset	$O(k^2)$-approx	[Nutov 09]	**decomposition**
uniform	$O(\log^2 k)$-approx	[Fakcharoenphol, Laekhanukit 08]	
		[Nutov 09]	**decomposition**
	$O(\sqrt{n/\epsilon})$-approx	[Cheriyan et al. 06]	**iterative rounding**
	$\Omega(\sqrt{k})$-fractionality	[Aazami et al. 10]	**iterative rounding**
uniform ($n > 3k-3$)	**$O(k)$-approx**	**This talk**	**iterative rounding**

Approximation factors: Edge- and element-connectivity SND w/ degree-bounds

edge-connectivity

	edge-cost	degree	
spanning tree	1	$b(v) + 1$	[Singh, Lau 07]
general	2	$b(v) + O(k)$	[Lau et al. 07]

element-connectivity

edge-cost	deg terminals	deg non-terminals	
2	$b(v) + O(k)$	$+\infty$	[Lau et al. 07]
$O(\log k)$	$O(\log k \cdot b(v) + k)$	$O(2^k) \cdot b(v)$	[Nutov 12]
4	$\mathbf{4b(v) + O(k)}$	$\mathbf{4b(v) + O(k)}$	**This talk**

Approximation factors: Node-connectivity SND w/ degree-bounds

node-connectivity, undirected graphs

	edge-cost	degree	
general	$O(k^3 \log k \log \vert T \vert)$	$O(2^k k^3 \log \vert T \vert) \cdot b(v)$	[Nutov 12]
	$\boldsymbol{O(k^3 \log \vert T \vert)}$	$\boldsymbol{O(k^3 \log \vert T \vert) \cdot b(v)}$	**This talk**
rooted	$O(k^2 \log k \log \vert T \vert)$	$O(2^k k^2 \log \vert T \vert) \cdot b(v)$	[Nutov 12]
	$\boldsymbol{O(k \log k)}$	$\boldsymbol{O(k \log k) \cdot b(v)}$	**This talk**
subset	$O(k^2 \log k \log \vert T \vert)$	$O(2^k k^2 \log \vert T \vert) \cdot b(v)$	[Nutov 12]
$\vert T \vert = O(k)$	$O(k^2)$	$O(k^2)$	trivial
$\boldsymbol{\vert T \vert = \omega(k)}$	$\boldsymbol{O(k \log k)}$	$\boldsymbol{O(k \log k) \cdot b(v)}$	**This talk**

Note: $(+\infty, 2^{\log^{1-\epsilon} n} b(v))$-approx hardness is known for subset node-connectivity SND when k is large [Lau et al. 09]

Approximation factors: Degree-bounded SND for digraphs

node-connectivity, digraphs

	edge-cost	in-degree	out-degree	
out-conn	$O(\log k)$	$+\infty$	$O(2^k) \cdot b^+(v)$	[Nutov 12]
	2	$\boldsymbol{k}$	$\boldsymbol{2b^+(v) + O(k)}$	**This talk**
uniform	$O(k)$	$+\infty$	$O(2^k) \cdot b^+(v)$	[Nutov 12]
	$\boldsymbol{O(k)}$	$\boldsymbol{O(k\sqrt{k})}$	$\boldsymbol{2b^+(v) + O(k\sqrt{k})}$	**This talk**

implications for undirected graphs

	edge-cost	degree	
out-conn	$O(\log k)$	$O(2^k) \cdot b(v)$	[Nutov 12]
	4	$\boldsymbol{2b(v) + O(k)}$	**This talk**
uniform	$O(k)$	$O(2^k) \cdot b(v)$	[Nutov 12]
	$\boldsymbol{O(k)}$	$\boldsymbol{2b(v) + O(k\sqrt{k})}$	**This talk**

Result 1

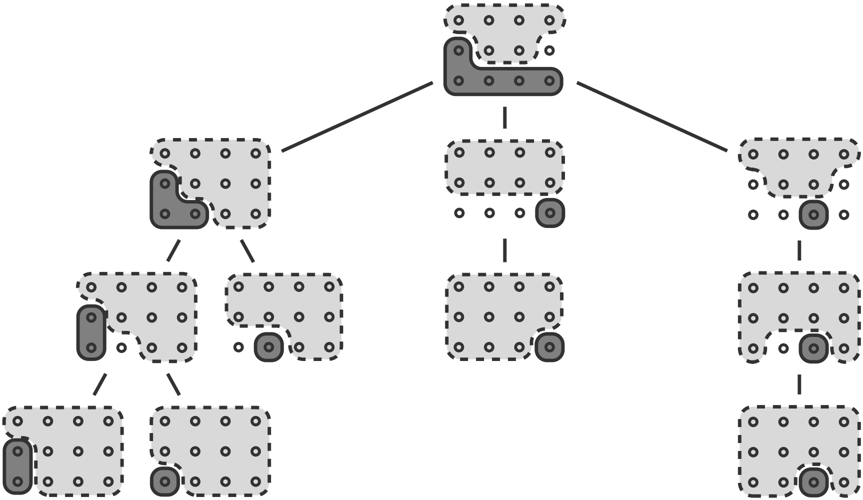
	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary terminals
element	← strongly laminar →		laminar
uniform node	← laminar, if $n > 3k - 3$ →		
out node	laminar, only entering arcs	← laminar →	

1

1. Laminar, undirected ➜ $(O(k), O(k) \cdot b(v))$-approx

Laminar family of set-pairs defines a forest

(U, U') is the parent of (W, W') if $W \subset U$ or if $W = U$ and $U' \subset W'$.



Token distribution

We prove

Theorem

If x^* is uniquely defined from the laminar family of tight set-pairs, one of the following holds:

- $\exists e \in E : x^*(e) = 0$
- $\exists e \in E : x^*(e) \geq 1/(4k-1)$
- $\exists v \in B : |\delta(v)| \leq 4k-1$

Assume $0 < x^*(e) < 1/(4k-1)$ for $\forall e \in E$, and $|\delta(v)| \geq 4k$ for $\forall v \in B$.

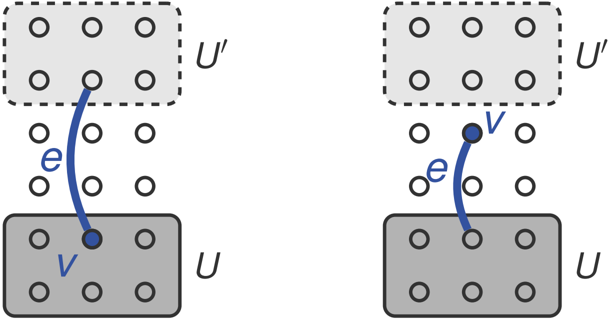
1. We make each edge distributes at most 2 tokens to set-pairs.
2. We show each set-pair receives ≥ 2 tokens, and the root receives ≥ 4 tokens.

Initial distribution

Token distribution rule

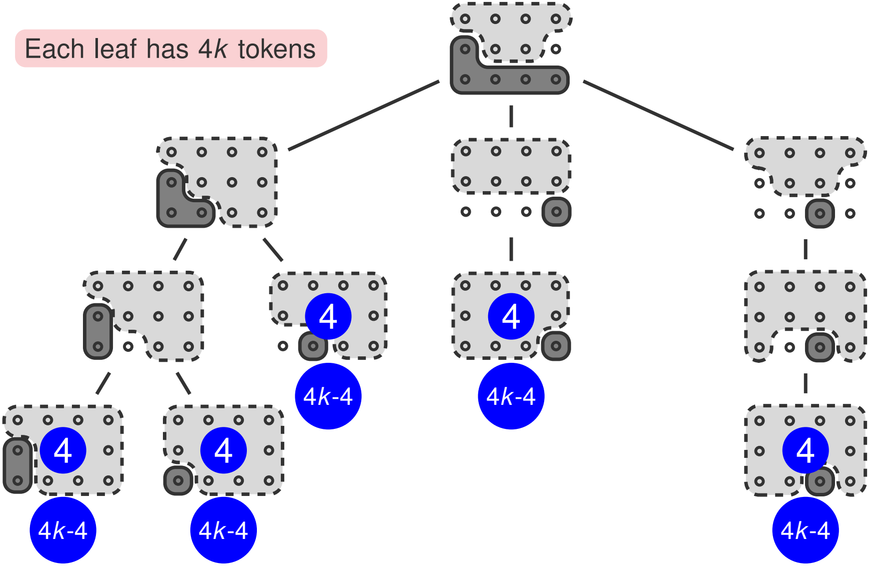
For each $e = uv$ and its end-node v, e gives a token to

1. minimal (U, U') s.t. $e \in \delta(U, U')$ and $v \in U$ if it exists,
2. minimal (U, U') s.t. $e \notin \delta(U, U')$ and $u \in U$ otherwise.



Inductive distribution

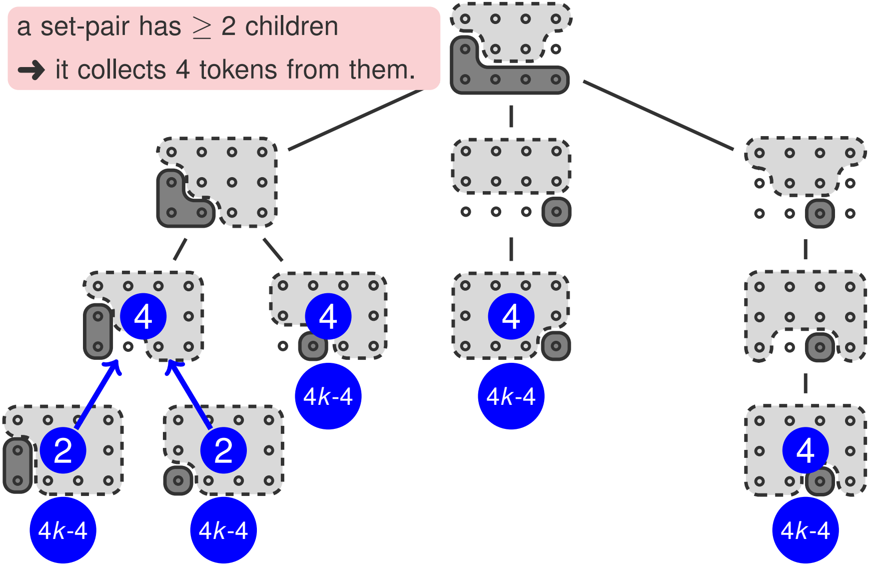
Each leaf has $4k$ tokens



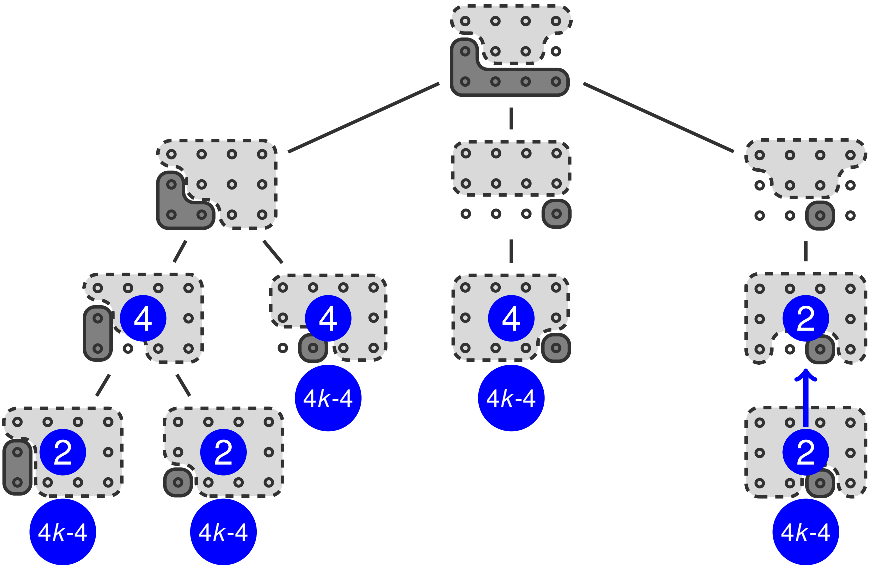
Inductive distribution

a set-pair has $\geq$ 2 children

➜ it collects 4 tokens from them.

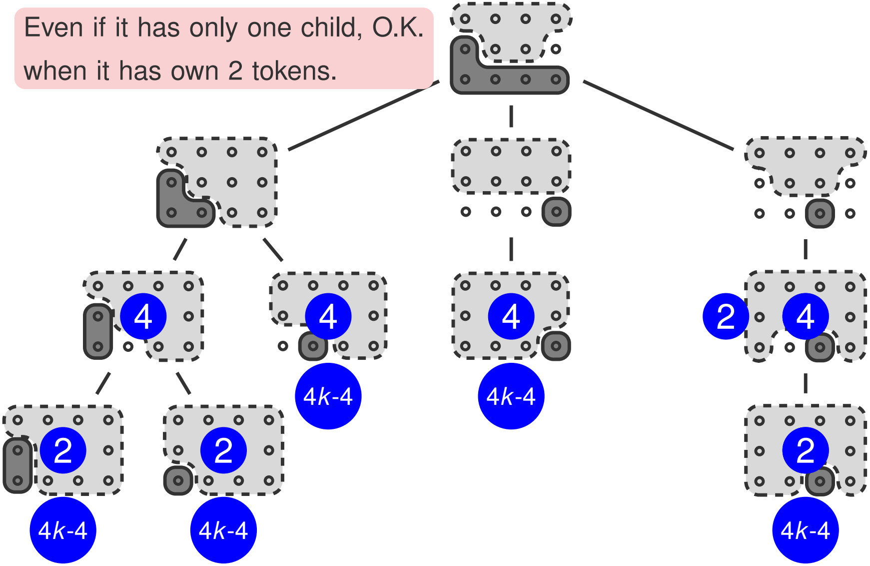


Inductive distribution



Inductive distribution

Even if it has only one child, O.K. when it has own 2 tokens.



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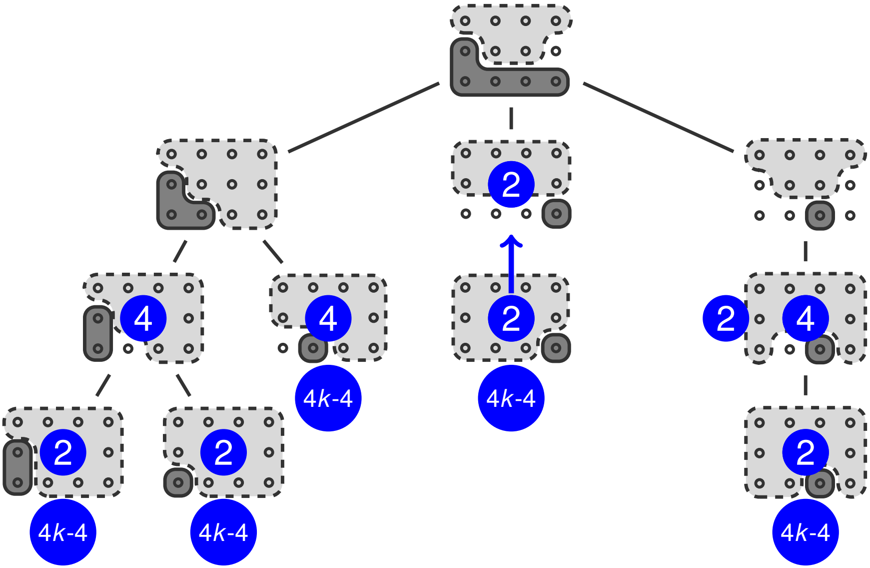

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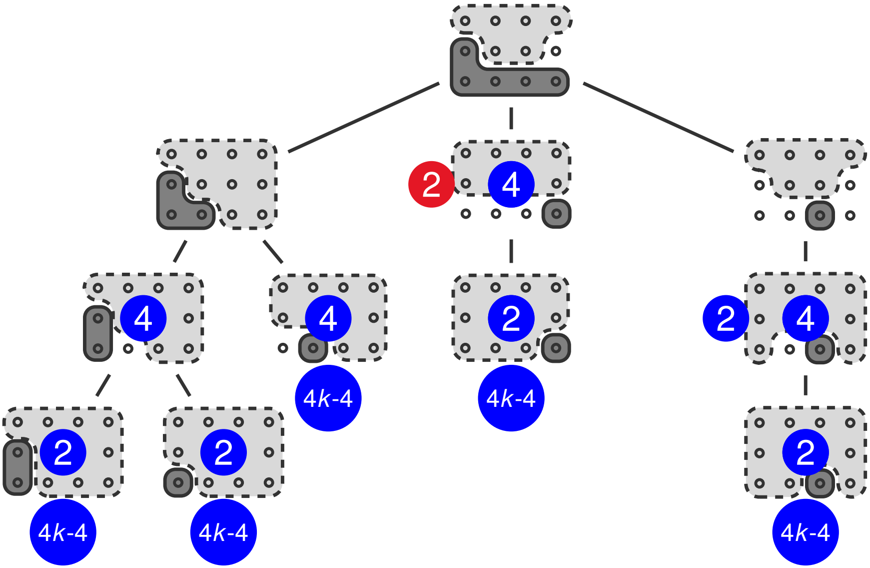

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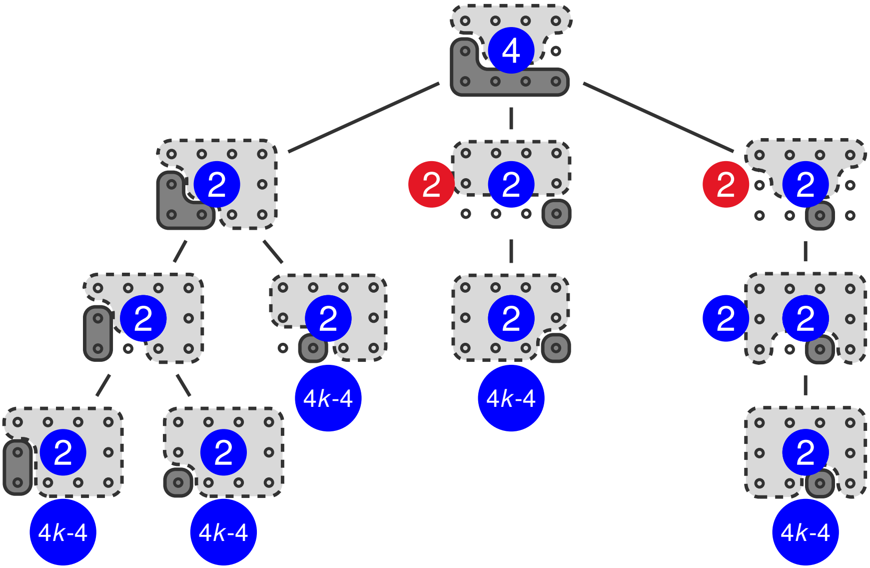
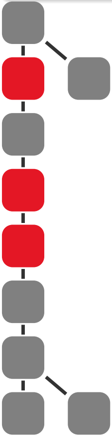


How many red tokens?

Theorem

# of red tokens in the forest $< 4(k-1)\times$ (# of leaves)



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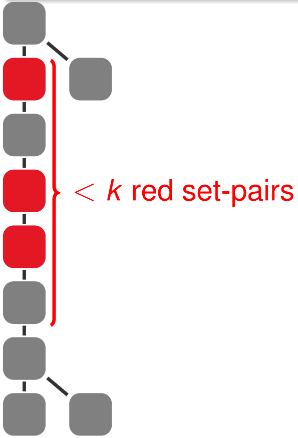

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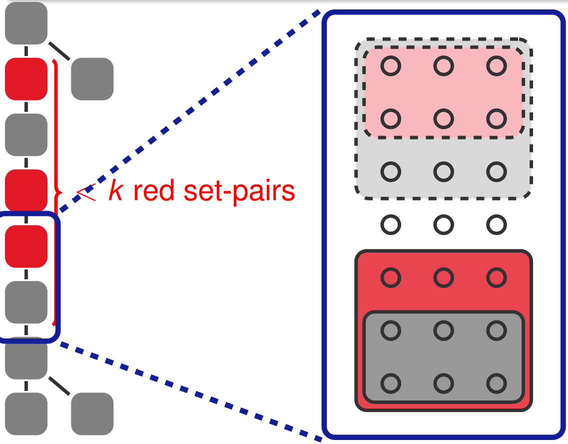

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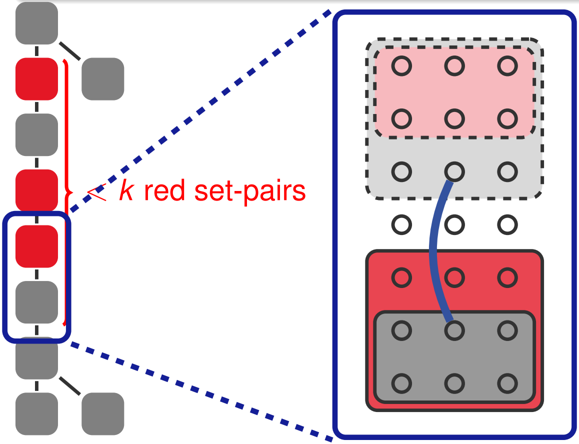

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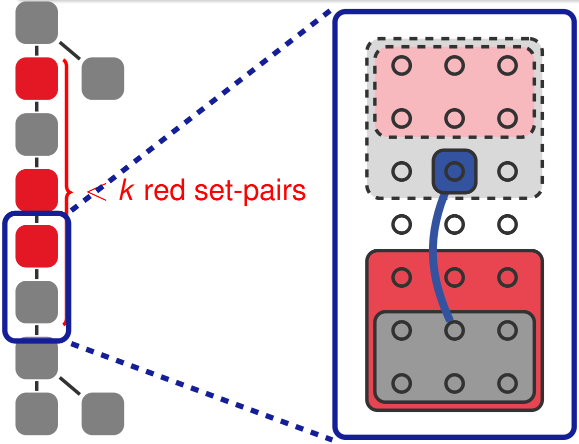


blue set remains in the boundary of the last set-pair on the path

↓

#red set-pairs on the path $\leq k-1$

↓

#red set-pairs in the forest $< 2(k-1) \times$ #leaves

Result 3

	no degree-bounds	degree-bounds on terminals	degree-bounds on arbitrary terminals
element	⟵ strongly laminar ⟶		laminar
uniform node	⟵ laminar, if $n > 3k - 3$ ⟶		
out node	laminar, only entering arcs	⟵ laminar ⟶	

3

3. Strongly laminar w/ degree-bounds, undirected ➜ $(4, 4b(v) + O(k))$-approx

Strongly laminar family of set-pairs with degree-bounds

Theorem

If x^* is defined from a strongly laminar family of tight set-pairs and tight degree-bounds, then one of the following holds:

- $\exists e \in E : x^*(e) = 0$
- $\exists e \in E : x^*(e) \geq 1/4$
- $\exists v \in B : |\delta(v)| < 2.5k + 6.25$

Idea 2: Using two different counting methods

$\mathcal{L}$:= strongly laminar family of set-pairs
C := set of tight degree-bounded nodes

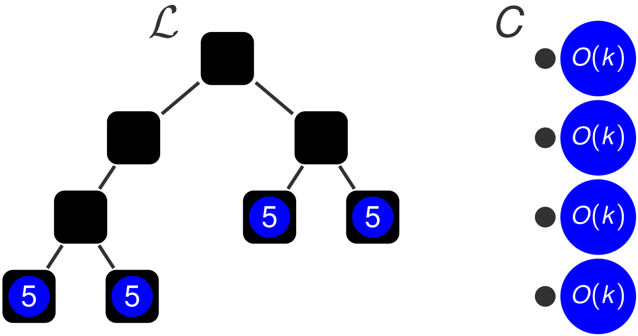
Case (i): C is small (i.e. $2|C| \leq$ # leaves)

A leaf gives tokens to nodes in C, and follow the 2-approx proof without C

Case (ii): C is large (i.e. $2|C| >$ # leaves in $\mathcal{L}$)

Nodes in C give tokens to leaves, and follow the proof for the laminar set-pair family in undirected graphs

Token distribution



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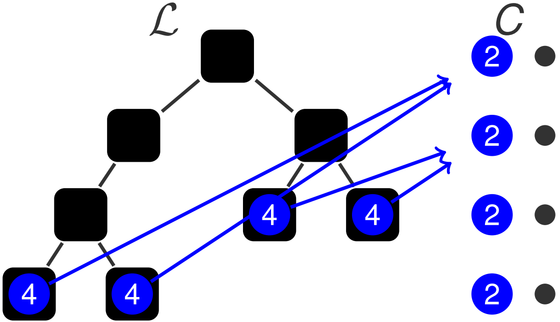
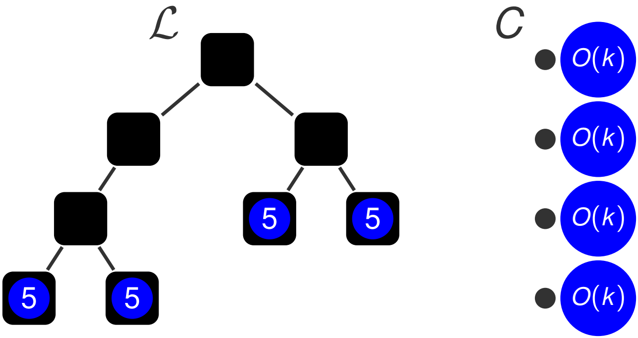


When $2|C| \leq$ # leaves of $\mathcal{L}$

- each leaf gives 2 tokens to a node in C
- nodes in C release their tokens

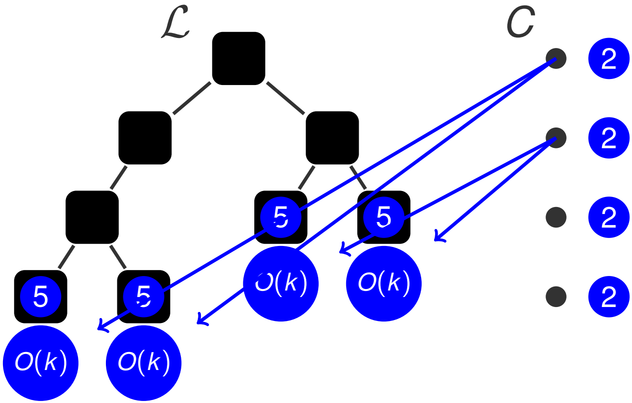
Token distribution



When $2|C| \geq$ # leaves of $\mathcal{L}$

- each node in C keeps 2 tokens
- nodes in C give the other tokens to leaves of $\mathcal{L}$

Token distribution



When $2|C| \geq$ # leaves of $\mathcal{L}$

- each node in C keeps 2 tokens
- nodes in C give the other tokens to leaves of $\mathcal{L}$

Conclusion

- Laminar, undirected ➜ $(O(k), O(k) \cdot b(v))$-approx
- Laminar, directed ➜ $(2, k, 2b^{+}(v) + O(k))$-approx
- Strongly laminar w/ degree-bounds, undirected ➜ $(4, 4b(v) + O(k))$-approx
- Laminar, undirected ➜ $(+\infty, 6b(v) + O(k^2))$-approx
- Strongly laminar w/ degree-bounds, undirected ➜ $(+\infty, 2b(v) + O(k^2))$-approx

Future works

- Narrow the gap between $O(k)$ and $\Omega(\sqrt{k})$ for uniform node-connectivity req. by iterative rounding
- Iterative rounding for other cases of node-connectivity