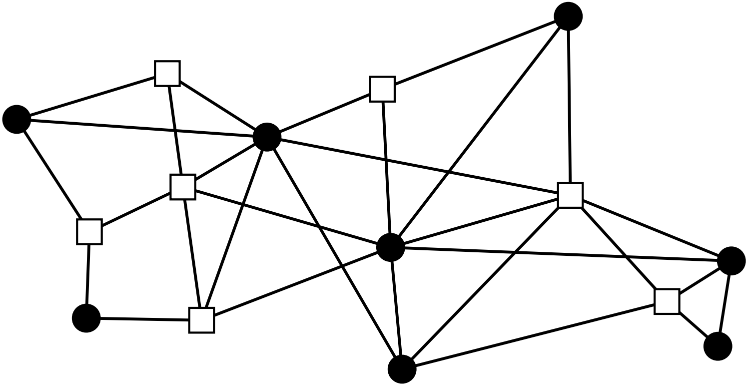
Integrality Gaps of Bidirected Cut and Directed Component Relaxations for Steiner Trees

Andreas Emil Feldmann   Jochen Könemann   Laura Sanità

University of Waterloo
Combinatorics & Optimization

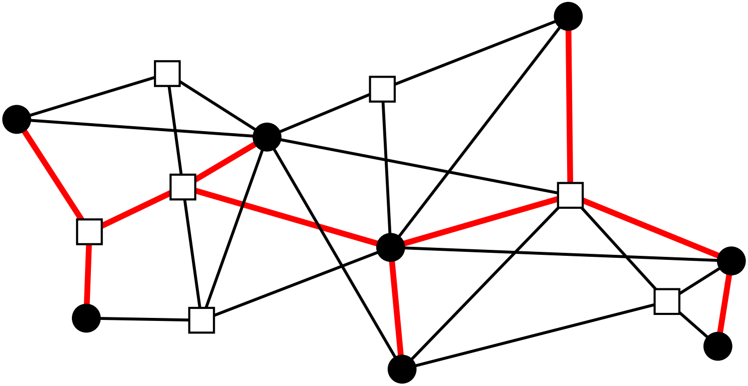
The Steiner Tree Problem



● Terminals

□ Steiner vertices

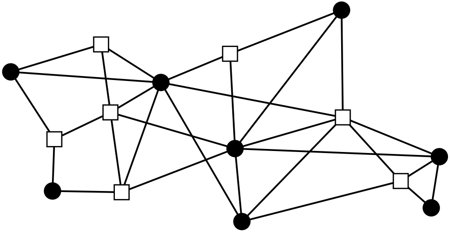
The Steiner Tree Problem



● Terminals

□ Steiner vertices

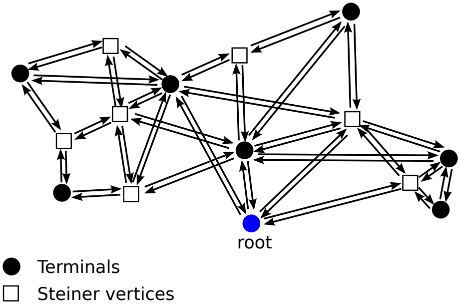
Bidirected Cut Relaxation (BCR)



● Terminals
□ Steiner vertices

[Edmonds 1967]

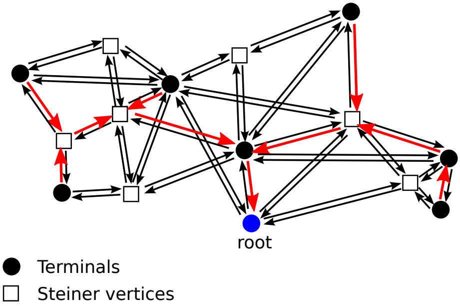
Bidirected Cut Relaxation (BCR)



- add root
- bi-direct edges

[Edmonds 1967]

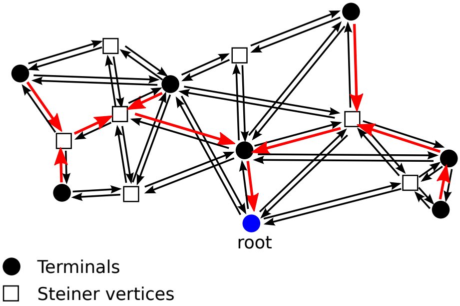
Bidirected Cut Relaxation (BCR)



- add root
- bi-direct edges
- integral flow = Steiner tree

[Edmonds 1967]

Bidirected Cut Relaxation (BCR)



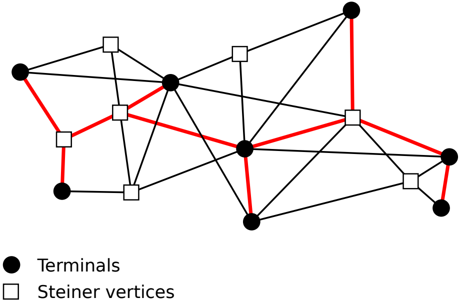
[Edmonds 1967]

- add root
- bi-direct edges
- integral flow = Steiner tree

Relaxation:

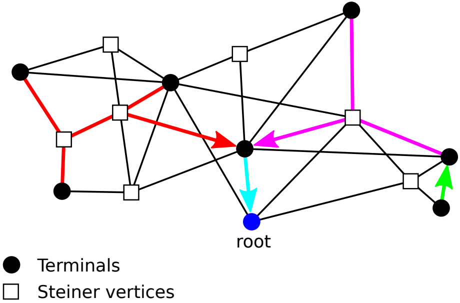
- find capacities x_a for flow demand 1 to root
- cost: $\sum_{a \in A} c_a x_a$

Directed Component Relaxation (DCR)

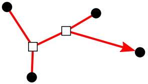


[Polzin, Daneshmand 2003]

Directed Component Relaxation (DCR)



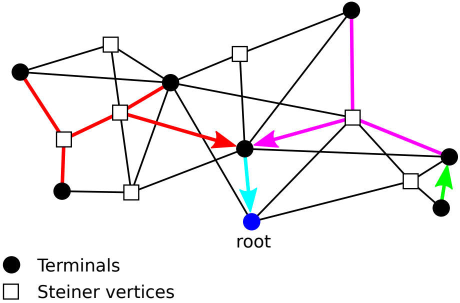
- add root
- integral flow through *full components*



= Steiner tree

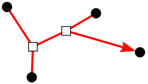
[Polzin, Daneshmand 2003]

Directed Component Relaxation (DCR)



[Polzin, Daneshmand 2003]

- add root
- integral flow through *full components*



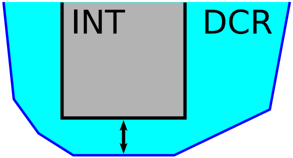
$=$ Steiner tree

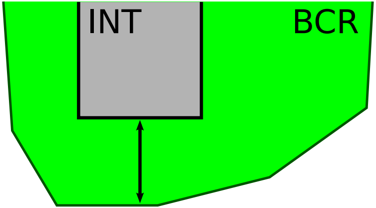
Relaxation:

- find capacities x_K of *full components* for flow demand 1 to root
- cost: $\sum_{K \in \mathcal{K}} c_K x_K$

  $(c_K = \sum_{e \in K} c_e)$

BCR v.s. DCR Gaps



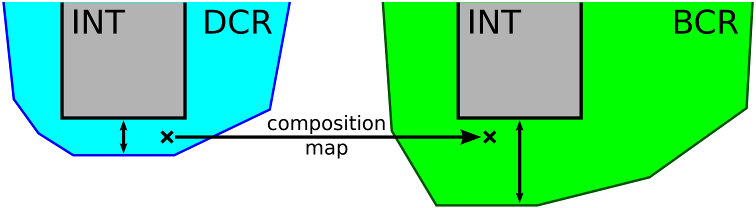


DCR gap $\leq \ln(4) \approx 1.39$

[Goemans et al. 2012]

BCR gap ≤ 2

BCR v.s. DCR Gaps



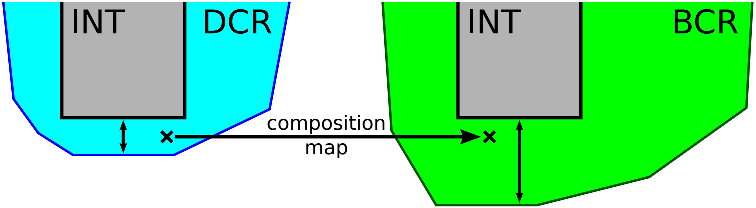
DCR gap $\leq \ln(4) \approx 1.39$

[Goemans et al. 2012]

BCR gap ≤ 2

DCR gap $\leq$ BCR gap

BCR v.s. DCR Gaps



DCR gap $\leq \ln(4) \approx 1.39$

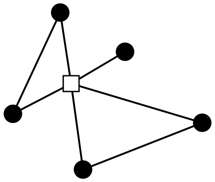
[Goemans et al. 2012]

BCR gap ≤ 2

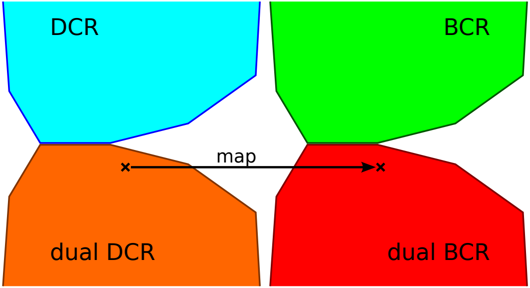
DCR gap $\leq$ BCR gap

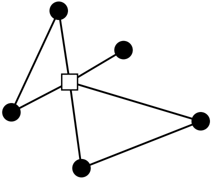
DCR gap $\stackrel{?}{=}$ BCR gap
NP-hard

Quasi-Bipartite Graphs



Quasi-Bipartite Graphs





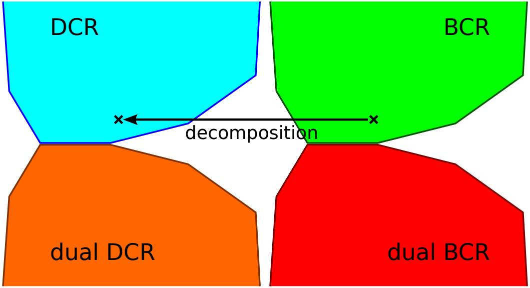
Non-constructive:

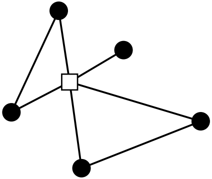
$$\begin{aligned} \text{DCR opt} &= \\ \text{dual DCR opt} &\leq \text{dual BCR opt} \\ &= \text{BCR opt} \end{aligned}$$

[Chakrabarty et al. 2011]

$$\boxed{\text{DCR gap} = \text{BCR gap}}$$

Quasi-Bipartite Graphs





Non-constructive:

$$\text{DCR opt} = \text{dual DCR opt} \leq \text{dual BCR opt} = \text{BCR opt}$$

[Chakrabarty et al. 2011]

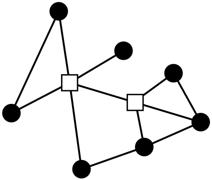
Constructive:

$$\text{BCR opt} \geq \text{DCR opt}$$

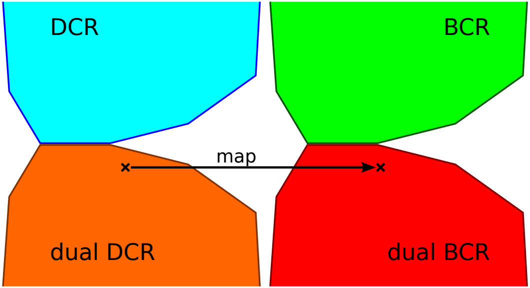
[Goemans et al. 2012]
[Fung et al. 2012]

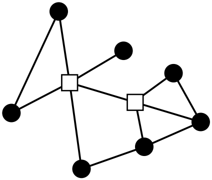
$$\boxed{\text{DCR gap} = \text{BCR gap}}$$

2-Quasi-Bipartite Graphs



2-Quasi-Bipartite Graphs



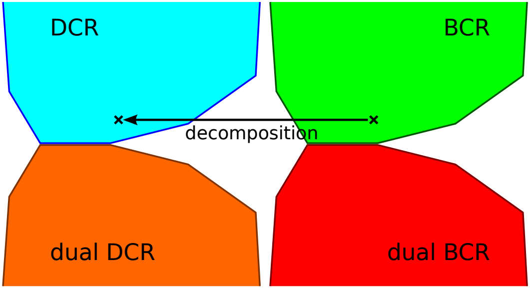


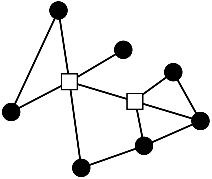
Non-constructive:

$$\begin{aligned} \text{DCR opt} &= \\ \text{dual DCR opt} &\leq \text{dual BCR opt} \\ &= \text{BCR opt} \end{aligned}$$

$$\boxed{\text{DCR gap} = \text{BCR gap}}$$

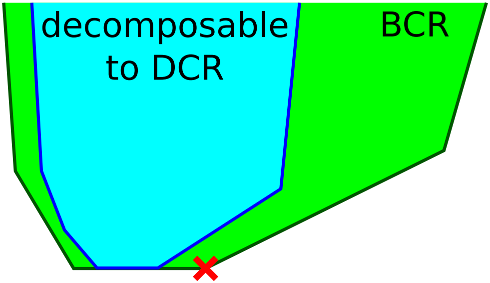
2-Quasi-Bipartite Graphs

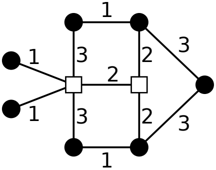




Constructive?

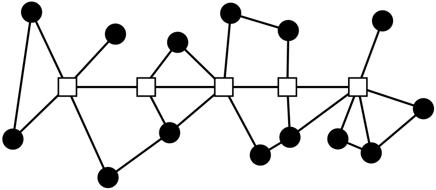
2-Quasi-Bipartite Graphs





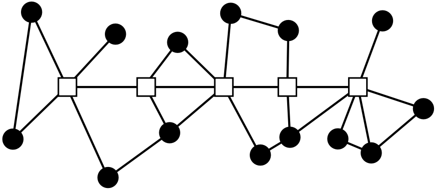
Constructive?

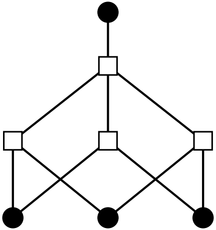
Further Generalizations



DCR gap v.s. BCR gap?

Further Generalizations



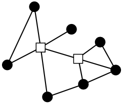


DCR gap v.s. BCR gap?

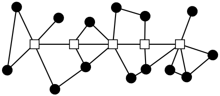
$$\frac{\text{DCR opt}}{\text{BCR opt}} = \frac{12}{11}$$

Open Problems

- Constructive proof for 2-quasi-bipartite graphs?



- Equal gaps for k-quasi-bipartite graphs?
  (Conjecture: yes.)



- Gap difference for more general graphs?