An LP-based $\frac{3}{2}$-approximation algorithm for the graphic s-t path TSP

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Graphic s-t path Traveling Salesman Problem (TSP)

Graphic s-t path TSP

Let G be a connected graph with unit cost.

s-t **path TSP:** Find a minimum-size **connected spanning** subgraph of $2G$ with $\{s, t\}$ as the **odd-degree vertex set**.

Literature

Metric s-t path TSP		
Due to	Year	Approx. Factor
J. A. Hoogeveen	1991	$\frac{5}{3} \approx 1.666$
H-C An et al (AKS)	2012	$\frac{1+\sqrt{5}}{2} \approx 1.618$
A. Sebő	2012	$\frac{8}{5} = 1.6$

Graphic s-t path TSP		
Due to	Year	Approx. Factor
H-C An & D. Shmoys	2011	$(\frac{5}{3} - \epsilon)$
T. Mömke & O.Svensson	2011	1.586
M. Mucha	2012	$\frac{19}{12} + \epsilon \approx 1.58333 + \epsilon$
A. Sebő & J. Vygen	2012	1.5

Scheme of Algorithm

1. Find a spanning tree J of G.

   —— Build Connectivity

2. Find min-size D-join F for the spanning tree J.

   D : wrong degree vertex set of J

   D-join: subgraph with D as odd-degree vertex set

   —— Correct Degree

3. Output $J \dot{\cup} F$.

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Question

- How to bound the min-size of D-join ?
- How to find a 'good' spanning tree for the algorithm?

LP relaxations

s-t Path TSP (on $2G$): x^* optimal solution

$$\begin{array}{lrcll} \text{minimize :} & \sum_{e \in E} x_e & & & \\ \text{subject to :} & x(\delta(\mathcal{W})) & \geq & |\mathcal{W}| - 1 & \forall \text{ partition } \mathcal{W} \text{ of } V \\ & x(\delta(S)) & \geq & 2 & \forall \emptyset \subsetneq S \subsetneq V, |S \cap \{s,t\}| \text{ even} \\ & 2 \geq & x_e & \geq 0 & \forall e \in E \end{array}$$

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- Bad Cuts S: $x^*(S) < 2$, $D - odd$ and aslo s, t-odd.
- If no bad cuts, $\frac{1}{2}x^*$ fixes our spanning tree.

Idea: No Bad Cuts

Lemma

For a spanning J with wrong degree vertex set D, if a cut S is both $D - odd$ and s, t-odd, then $|J \cap \delta(S)|$ is **EVEN**.

Idea: Construct a spanning tree that has **ODD** number of edges at the potential Bad Cuts.

Narrow Cuts: Potential Bad Cuts

- Bad Cuts S: $x^*(S) < 2$, $D - odd$ and s, t-odd.
- Narrow cut S: $x^*(S) < 2$ and s, t-odd cut.
- Bad cut must be a narrow cut.
- Construct a spanning tree that has **ODD** number of edges at the narrow cuts.

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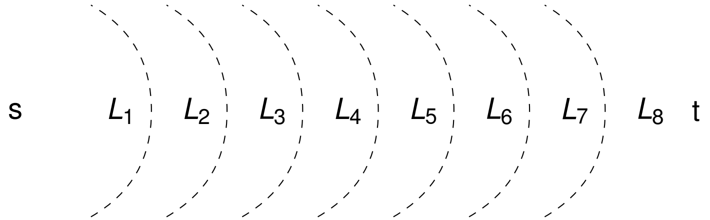
AKS's Lemma on Narrow Cuts

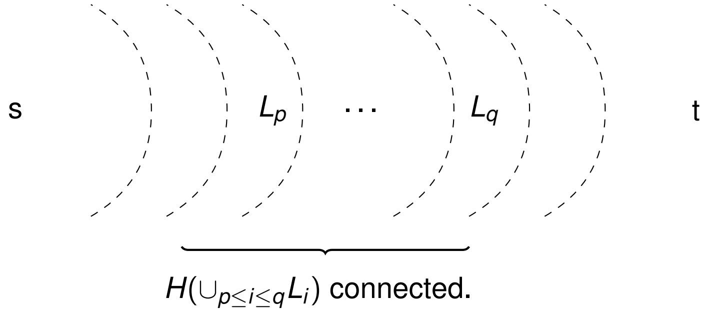
Narrow cuts is a nested family.



Key Lemma

Lemma

Let H be the support graph of an optimal solution x^* of s-t path TSP LP. Then, $H(\cup_{p\leq i\leq q} L_i)$ is connected.



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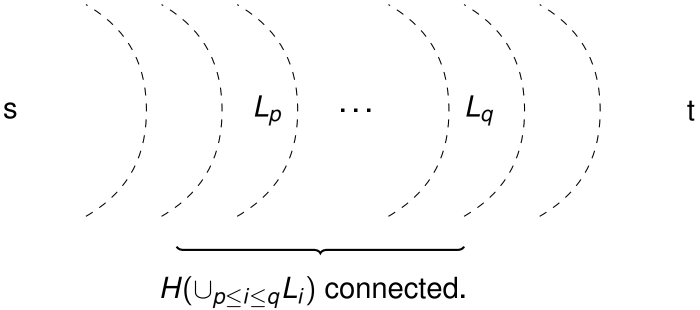
- $H(L_i)$ is connected, and $\exists$ edge connecting L_i and L_{i+1} in H.



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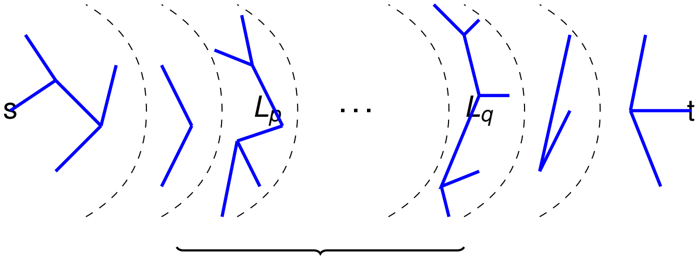

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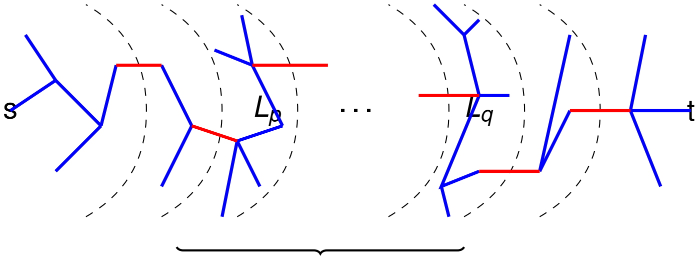

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LP-based Algorithm

1. Find the support graph H of x^*.
2. Find all narrow cuts and corresponding partition $L_1, L_2, \ldots, L_{k+1}$.
3. For each $1 \leq i \leq k+1$, find a local spanning tree J_i on $H(L_i)$.
4. For each $1 \leq i \leq k$, take an edge e_i connecting each two consecutive L_i and L_{i+1}
5. Let the spanning J be the union of the edge sets of previous twp steps.
6. Let D be the wrong degree vertex set of J. Find the minimum size D-join F.
7. Output $J \dot{\cup} F$.

Analysis of the LP-based Algorithm

- D-join: $|F| \leq \frac{1}{2} \sum_{e \in E} x_e^*$.
- Spanning tree: $|J| = n - 1 \leq \sum_{e \in E} x_e^*$.

Theorem

The LP-based algorithm is a $\frac{3}{2}$-approximation for the graphic s-t path TSP.

Thank you!