Mean Field Games and Systemic Risk

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Joint work with René Carmona and Li-Hsien Sun

Mathematics for New Economic Thinking

INET Workshop at the Fields Institute

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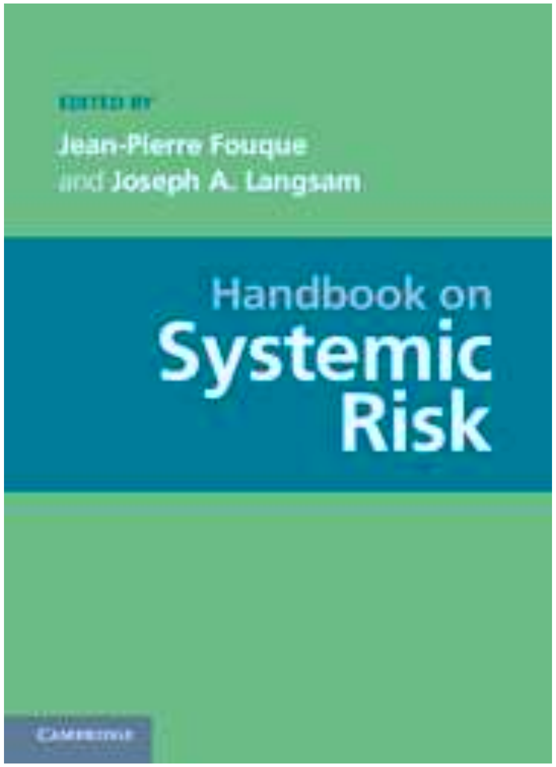
Editors: J.-P. Fouque and J. Langsam

Cambridge University Press (2013)



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- 2012: Director nominated: **Richard Berner** is forming a **Financial Research Advisory Committee** (first meeting on December 5, 2012 in Washington DC)

Correlated Diffusions: Credit Risk

$X_t^{(i)}, i = 1, \ldots, N$ denote log-values

$$dX_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N.$$

Three ingredients:

- Drifts $b_t^{(i)}$
- Volatilities $\sigma_t^{(i)}$
- Brownian motions $W_t^{(i)}$

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Correlation can also be created through stochastic volatilities $\sigma_t^{(i)}$

(Fouque-Wignall-Zhou 2008)

Coupled Diffusions: Systemic Risk

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Model **borrowing and lending** through the drifts:

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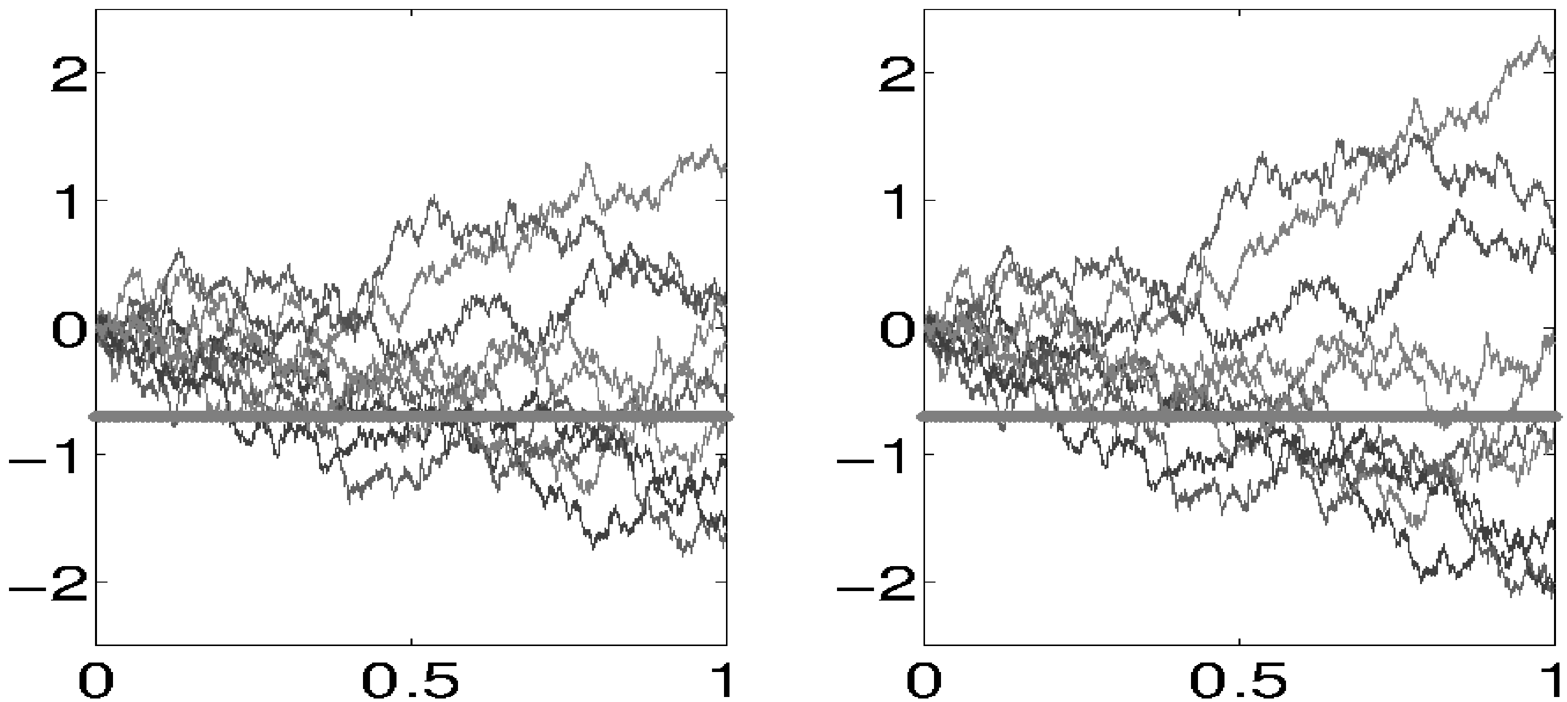
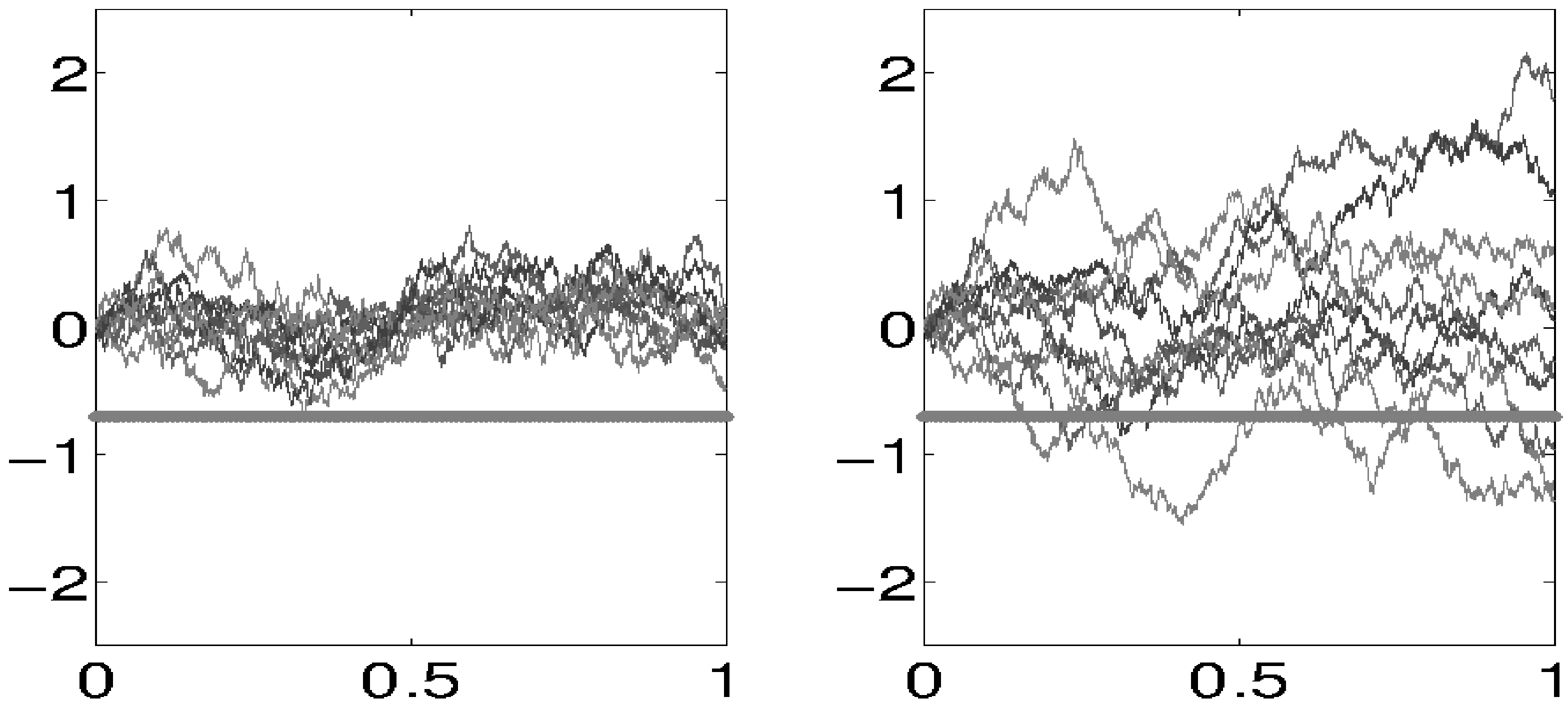
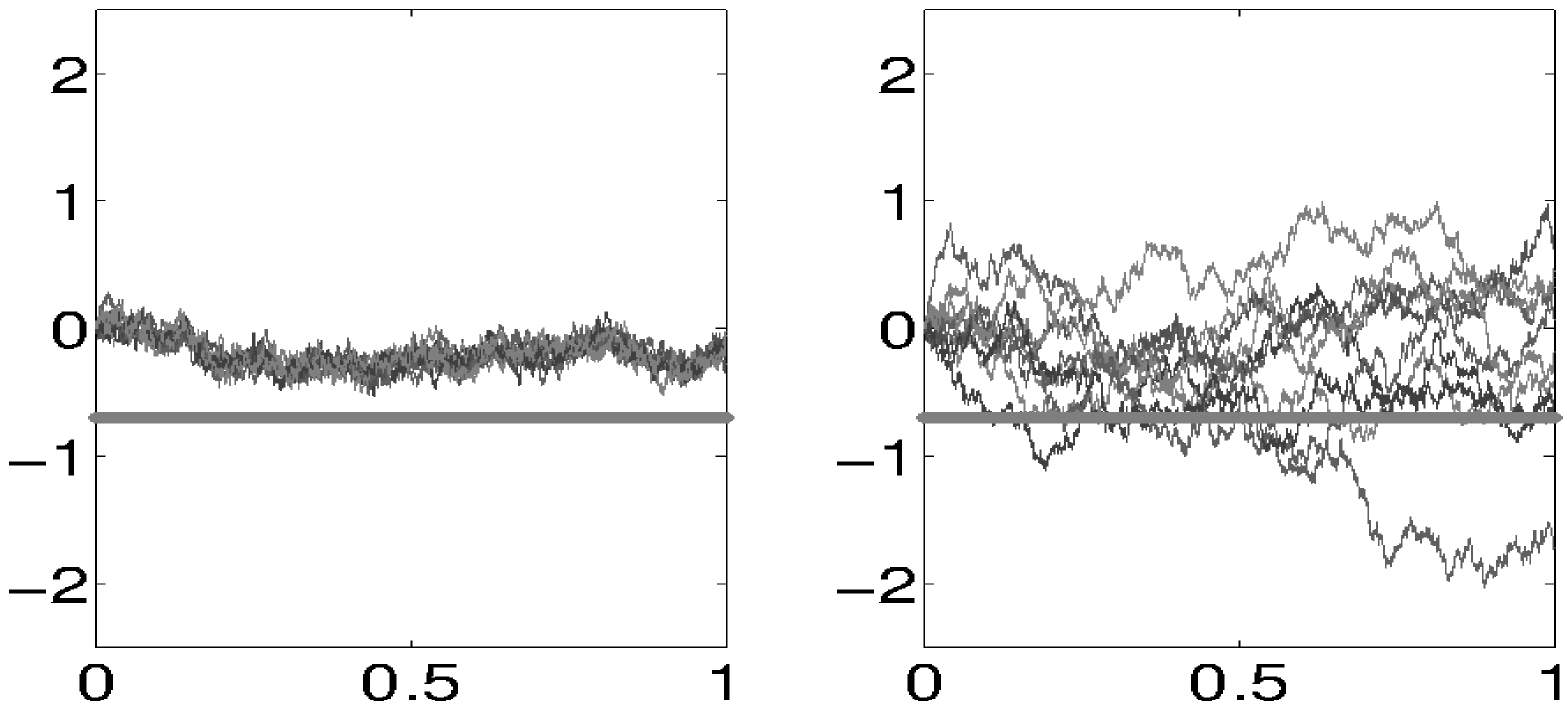
The overall **rate of borrowing and lending** a/N has been normalized by the number of banks and we assume $a > 0$
Denote the **default level** by $D < 0$ and simulate the system for various values of **a**: $\mathbf{0}, \mathbf{1}, \mathbf{10}, \mathbf{100}$ with fixed $\sigma = 1$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 1** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 10** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 100** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

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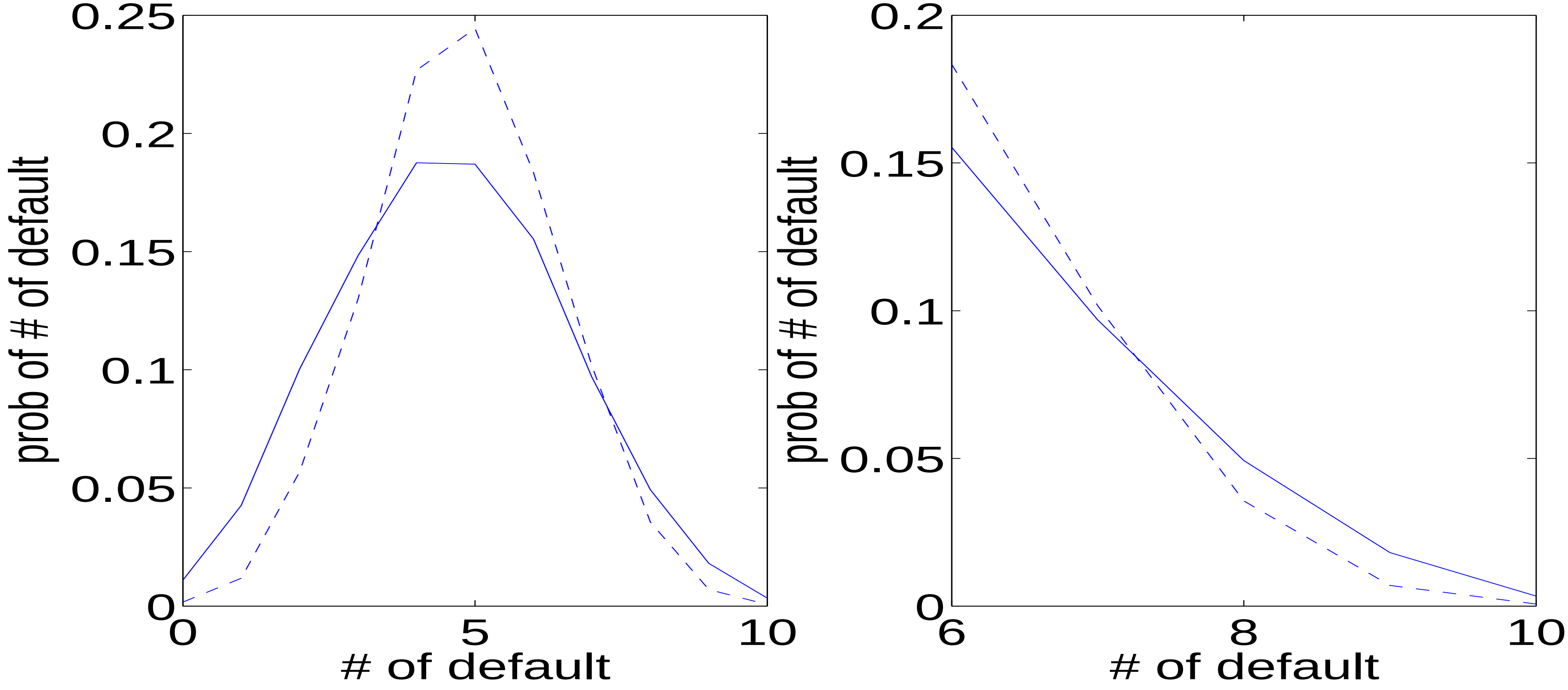
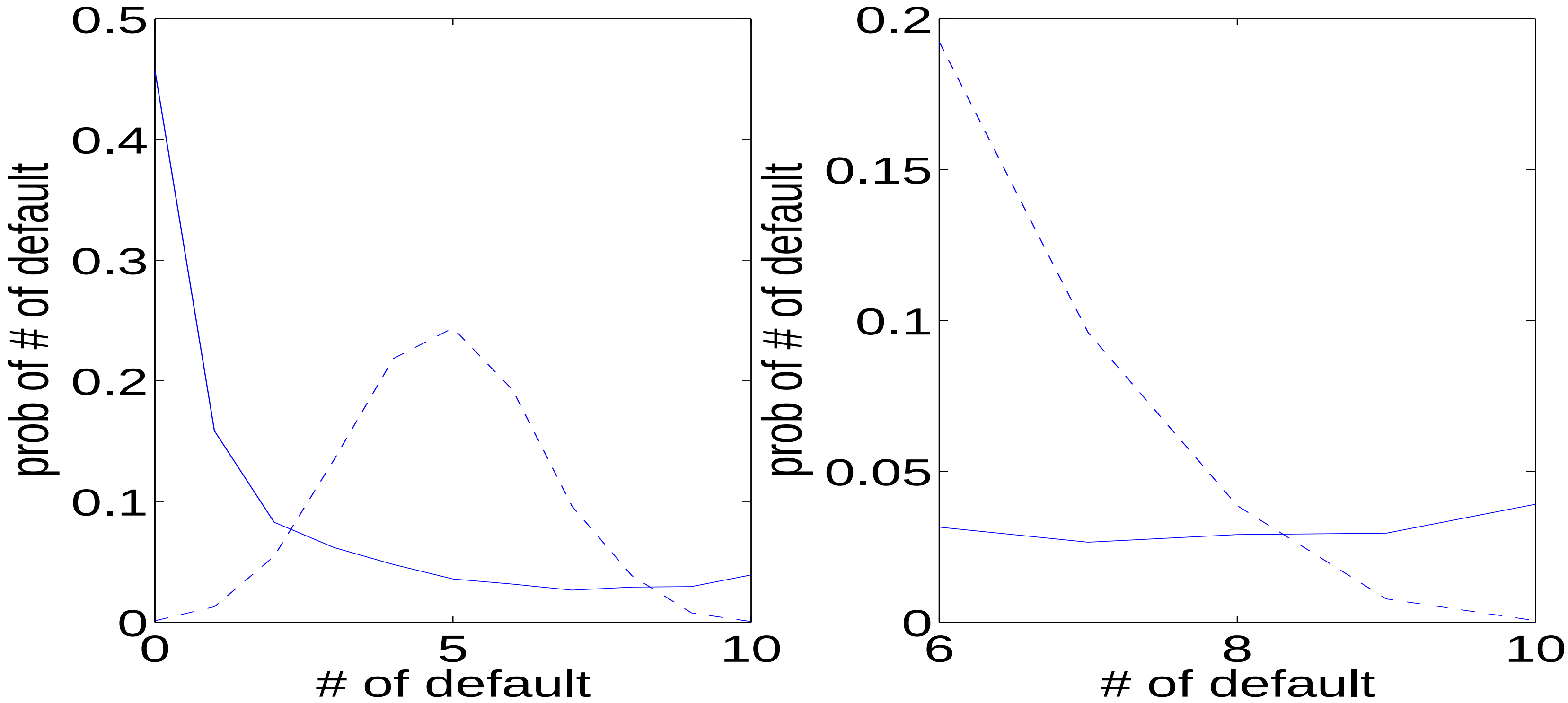
In the independent case, the loss distribution is Binomial(N, p) with parameter p given by

$$\begin{aligned} p &= \mathbb{P}\left(\min_{0\le t\le T}(\sigma W_t) \le D\right) \\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right), \end{aligned}$$

where Φ denotes the $\mathcal{N}(0,1)$-cdf, and we used the distribution of the minimum of a Brownian motion (see Karatzas-Shreve 2000 for instance). With our choice of parameters, we have $p \approx 0.5$



On the left, we show plots of the loss distribution for the coupled diffusions with $\mathbf{a = 1}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.



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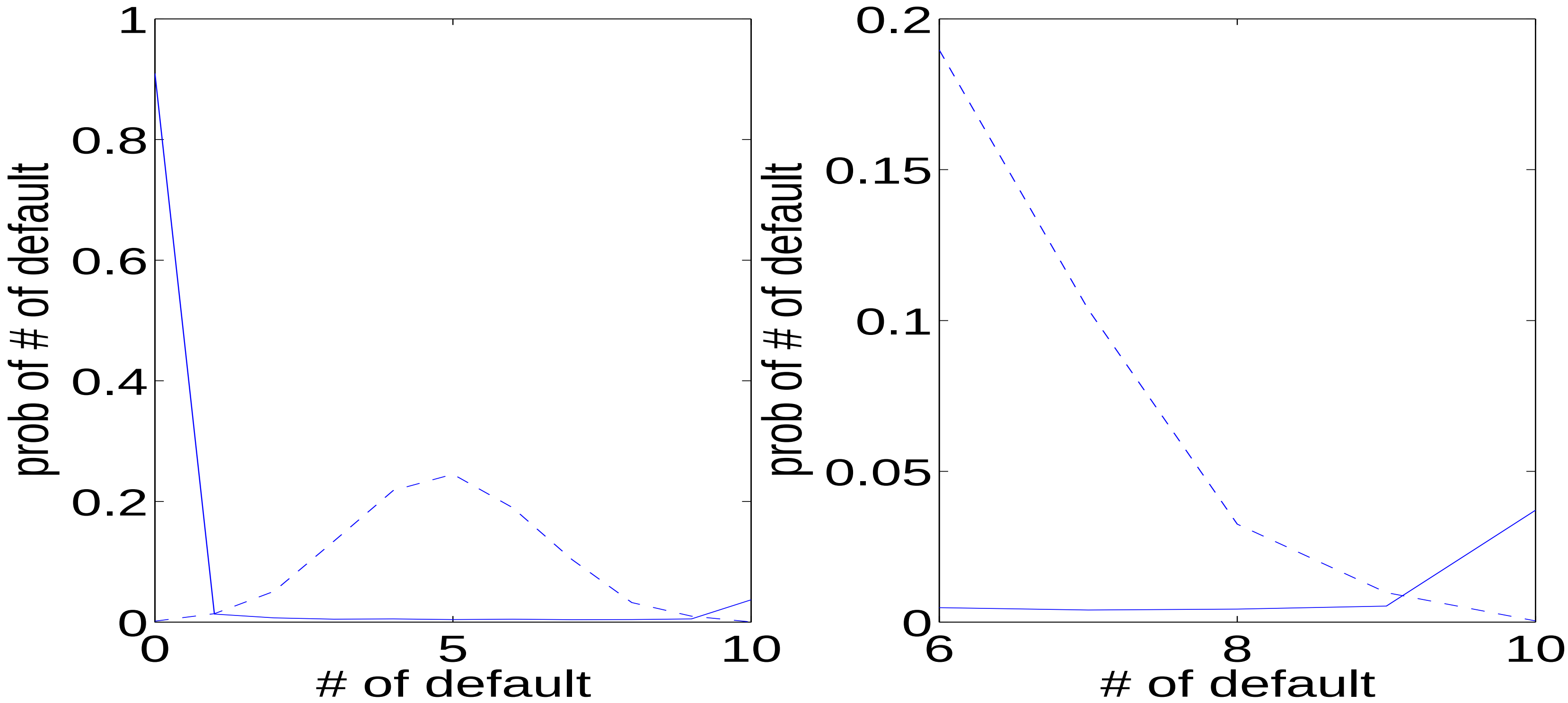

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Mean-field Limit

Rewrite the dynamics as:

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\begin{aligned}
dX_t^{(i)} &= \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\, dt + \sigma dW_t^{(i)} \\
&= a\left[\left(\frac{1}{N}\sum_{j=1}^{N} X_t^{(j)}\right) - X_t^{(i)}\right] dt + \sigma dW_t^{(i)}.
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\end{aligned}
$$

The processes $X^{(i)}$'s are "OUs" **mean-reverting** to the **ensemble average** which satisfies

$$
d\left(\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right).
$$

Assuming for instance that $x_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

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In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^{(j)} \to 0 \quad a.s. ,$$

and therefore, the processes $X^{(i)}$'s converge to independent OU processes with long-run mean zero.

In fact, $X_t^{(i)}$ is given explicitly by

$$X_t^{(i)} = \frac{\sigma}{N} \sum_{j=1}^{N} W_t^{(j)} + \sigma e^{-at} \int_0^t e^{as} dW_s^{(i)} - \frac{\sigma}{N} \sum_{j=1}^{N} \left(e^{-at} \int_0^t e^{as} dW_s^{(j)} \right) ,$$

and therefore, $X_t^{(i)}$ converges to $\sigma e^{-at} \int_0^t e^{as} dW_s^{(i)}$ which are independent OU processes.

This is a simple example of a **mean-field limit** and propagation of chaos studied in general by Sznitman (1991).

Large Deviation

We focus on the event where the ensemble average reaches the default level. The probability of this event is small (as N becomes large), and is given by the theory of **Large Deviation**.

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In our simple example, this probability can be computed explicitly as follows:

$$\begin{aligned} I\!P\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N}W_t^{(i)}\right)\le D\right) &= I\!P\left(\min_{0\le t\le T}\widetilde{W}_t\le\frac{D\sqrt{N}}{\sigma}\right) \\ &= 2\Phi\left(\frac{D\sqrt{N}}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $\widetilde{W}$ is a standard Brownian motion.

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbb{P}\left(\min_{0\le t\le T} \left(\frac{\sigma}{N} \sum_{i=1}^{N} W_t^{(i)} \right) \le D \right) = \frac{D^2}{2\sigma^2 T} \, .$$

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$$\text{Since}\quad \frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{we identify}$$

$$\left\{ \min_{\mathbf{0\le t\le T}} \left(\frac{\sigma}{\mathbf{N}} \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{X}_{\mathbf{t}}^{(\mathbf{i})} \right) \le \mathbf{D} \right\} \quad \text{as a } \textbf{systemic event}$$

Observe that this event does not depend on $a > 0$

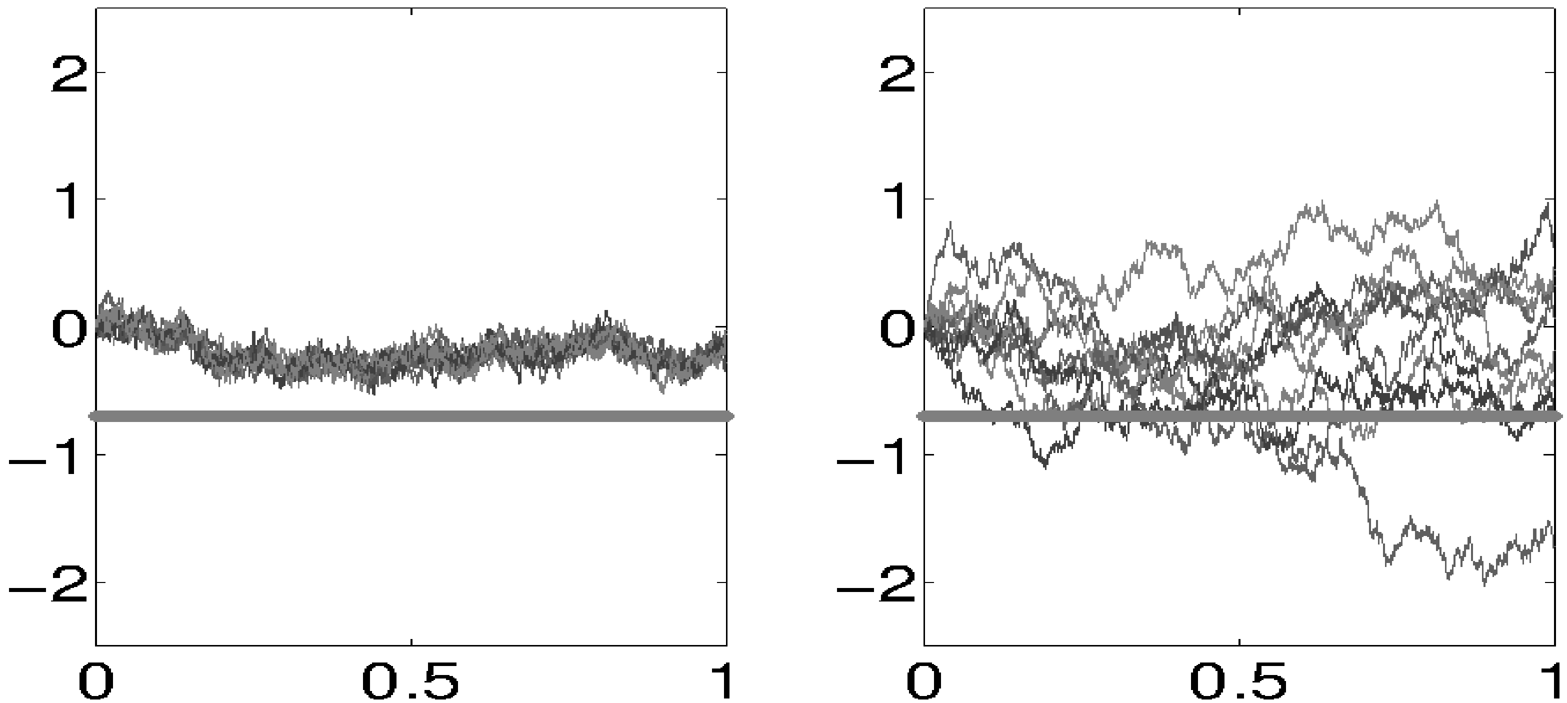
The probability

$$\exp\left(-\frac{D^2 N}{2\sigma^2 T}\right)$$

of a systemic event does not depend on $a > 0$, in other words:

“ Increasing stability by increasing the rate of borrowing and lending does not prevent a systemic event where a large number of banks default”

In fact, once in this event, increasing α creates even more defaults by **“flocking to default”**. This is illustrated in the simulation with $a = 100$ where the probability of systemic risk is roughly 3%.



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The probability of a systemic event is roughly 3%

Systemic Risk and Common Noise

$$dX_t^i = a\left(\frac{1}{N}\sum_{j=1}^{N} X_t^j - X_t^i\right) dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\, dW_t^i\right), \quad i = 1, \cdots, N,$$

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The ensemble average:

$$\frac{1}{N}\sum_{i=1}^{N} X_t^i = \frac{\sigma}{N}\sum_{i=1}^{N} \widetilde{W}_t^i \quad = \quad \sigma\left(\rho W_t^0 + \frac{\sqrt{1-\rho^2}}{N}\sum_{i=1}^{N} W_t^i\right)$$

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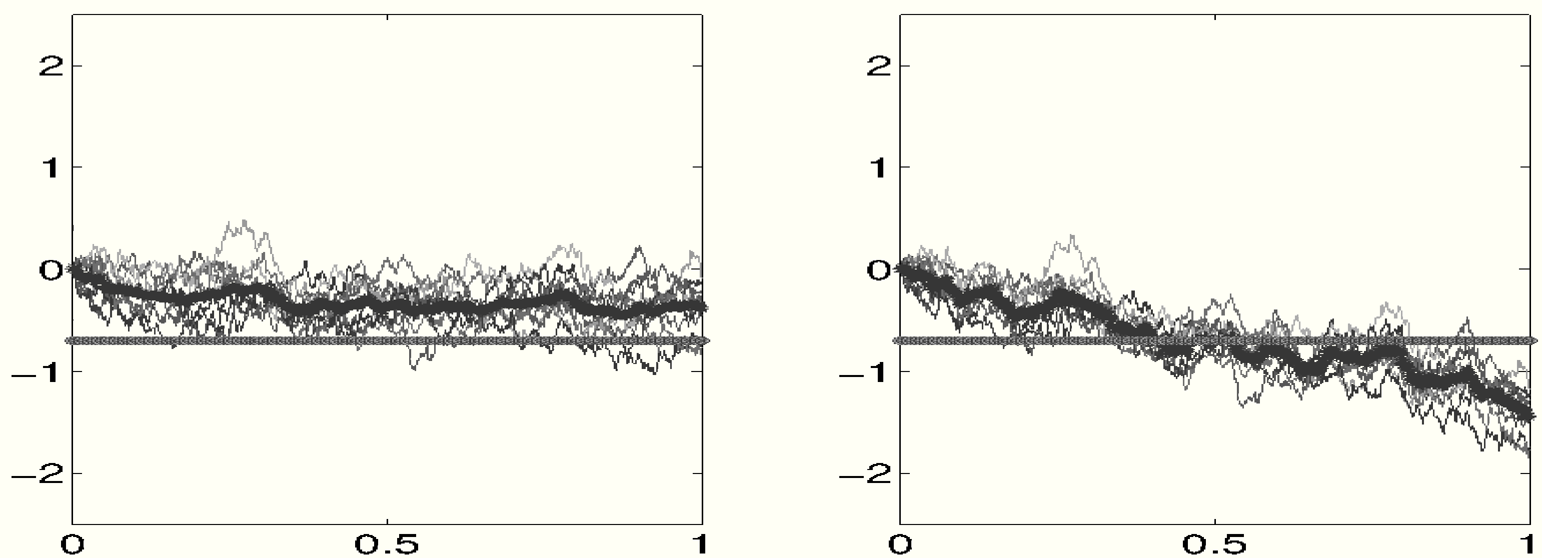
$$= (\text{in } \mathcal{D}) \quad \sigma\sqrt{\rho^2 + \frac{(1-\rho^2)}{N}}\,B_t,$$

The **probability of the systemic** event becomes

$$I\!P\left(\min_{0\le s\le T}\frac{1}{N}\sum_{i=1}^{N} X_s^i < D\right) = 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\sqrt{\frac{N}{N\rho^2 + (1-\rho^2)}}\right) \to 2\Phi\left(\frac{D}{\sigma|\rho|\sqrt{T}}\right)$$

$\rho = 0$ $\qquad\qquad$ $\rho = 0.5$



So far:

We proposed a simple toy model of coupled diffusions to represent **lending and borrowing** between banks. We show that, as expected, this activity **stabilizes** the system in the sense that it decreases the number of defaults. Indeed, and naively, banks in difficulty can be "saved" by borrowing from others. In fact, the model illustrates the fact that stability increases as the rate of borrowing and lending increases.

However, there is a small probability, computed explicitly in our model, that the average of the ensemble reaches the default level. Combined with the "flocking" behavior **"everybody follows everybody"**, this leads to a **systemic event** where almost all default, in particular when the rate of borrowing and lending is large.

Related Papers

- *Diversification in Financial Networks may Increase Systemic Risk*

  by J. Garnier, G. Papanicolaou, and T.-W. Yang

  Handbook of Systemic Risk (2013)

- *Stability in a model of inter-bank lending*

  by J.-P. Fouque and T. Ichiba (To appear in SIAM Journal on Financial Mathematics).

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- Can we find an equilibrium in which the previous analysis can still be performed?
- Can we find and characterize a Nash equilibrium?

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- Can we find and characterize a Nash equilibrium?

What follows is from
Mean Field Games and Systemic Risk
by R. Carmona, J.-P. Fouque and L.-H. Sun (2013)

Mean Field Game

Denoting $\overline{X}_t = \frac{1}{N}\sum_i^N X_t^i$, the dynamics is

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

where α^i is the control of bank i, and they **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T^i)\right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \quad q^2 \le \epsilon,$$

and **terminal cost** $\quad g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2.$

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This is an example of **Mean Field Game (MFG)** studied extensively by P.L. Lions and collaborators (see also the recent work of R. Carmona and F. Delarue).

Nash Equilibria (FBSDE Approach)

The Hamiltonian:

$$H^i(x, y^{i,1}, \cdots, y^{i,N}, \alpha^1(t,x), \cdots, \alpha^i_t, \cdots, \alpha^N(t,x))$$
$$= \sum_{k \neq i} \left[a(\overline{x} - x^k) + \alpha^k(t,x)\right] y^{i,k} + \left[a(\overline{x} - x^i) + \alpha^i\right] y^{i,i}$$
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Minimizing H^i over α^i gives the choices:

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&= \sum_{k \neq i} \left[a(\overline{x} - x^k) + \alpha^k(t,x)\right] y^{i,k} + \left[a(\overline{x} - x^i) + \alpha^i\right] y^{i,i} \\
&+ \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2,
\end{aligned}$$

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Ansatz:

$$Y_t^{i,j} = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\overline{X}_t - X_t^i),$$

where η_t is a deterministic function satisfying the terminal condition $\eta_T = c$.

Forward-Backward Equations

Forward Equation:

$$\begin{aligned} dX_t^i &= \partial_{y^{i,i}} H^i dt + \sigma dW_t^i \\ &= \left[a + q + (1 - \frac{1}{N})\eta_t \right] (\overline{X}_t - X_t^i) dt + \sigma dW_t^i, \end{aligned}$$

with initial conditions $X_0^i = x_0^i$.

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&= \left[a + q + (1 - \frac{1}{N}) \eta_t \right] (\overline{X}_t - X_t^i) dt + \sigma dW_t^i,
\end{aligned}
$$

with initial conditions $X_0^i = x_0^i$.

Backward Equation:

$$
\begin{aligned}
dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k \\
&= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i) \left[(a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon \right] dt \\
&\quad + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k, \qquad Y_T^{i,j} = c(\frac{1}{N} - \delta_{i,j})(\overline{X}_T - X_T^i).
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Solution to the Forward-Backward Equations

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and η_t **must satisfy the Riccati equation**

$$
\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\epsilon - q^2),
$$

with the terminal condition $\eta_T = c$.

Solution to the Riccati Equation

$$\eta_t = \frac{-(\epsilon - q^2)\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right) - c\left(\delta^+ e^{(\delta^+-\delta^-)(T-t)} - \delta^-\right)}{\left(\delta^- e^{(\delta^+-\delta^-)(T-t)} - \delta^+\right) - c(1 - \frac{1}{N^2})\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right)},$$

with the notations

$$\begin{aligned} \delta^\pm &= -(a+q) \pm \sqrt{R}, \\ R &= (a+q)^2 + \left(1 - \frac{1}{N^2}\right)(\epsilon - q^2) > 0. \end{aligned}$$

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Observe that η_t is well defined for any $t \leq T$ since the denominator can be written as

$$-\left(e^{(\delta^+ - \delta^-)(T-t)} + 1\right)\sqrt{R} - \left(a + q + c\left(1 - \frac{1}{N^2}\right)\right)\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right),$$

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which stays negative because $\delta^+ - \delta^- = 2\sqrt{R} > 0$.
In fact, using $q^2 \leq \epsilon$, we see that η_t is positive with $\eta_T = c$.

Financial Implications

1. Once the function η_t has been obtained, bank i implements its strategy by using its control

$$\hat{\alpha}_t^i = -Y_t^{i,i} + q(\overline{X}_t - X_t^i) = \left[q + (1 - \frac{1}{N})\eta_t \right] (\overline{X}_t - X_t^i),$$

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However, the control affects the rate of borrowing and lending by adding the time-varying component $q + (1 - \frac{1}{N})\eta_t$ to the uncontrolled rate a.

Financial Implications

2. In fact, the controlled dynamics can be rewritten

$$dX_t^i = \left(a + q + (1 - \frac{1}{N})\eta_t\right) \frac{1}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) dt + \sigma dW_t^i.$$

The effect of the banks using their optimal strategies corresponds to inter-bank borrowing and lending at the increased **effective rate**

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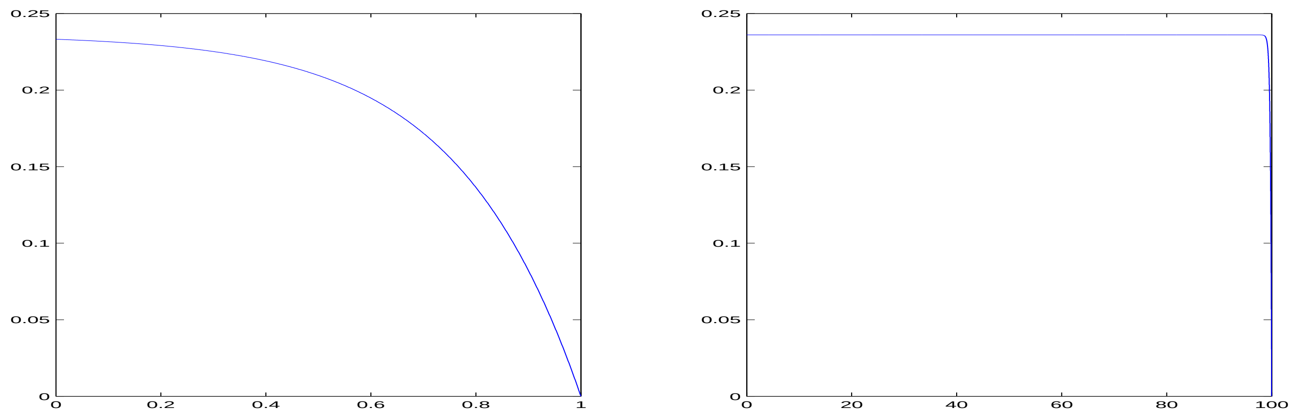
Under this equilibrium, the system is operating as if banks were borrowing from and lending to each other at the rate A_t, and the net effect is **additional liquidity** quantified by the rate of lending/borrowing.

Financial Implications

3. For T large (most of the time $T - t$ large), η_t is mainly constant. For instance, with $c = 0$, $\lim_{T\to\infty} \eta_t = \frac{\epsilon - q^2}{-\delta^-} := \overline{\eta}$.



Plots of η_t with $c = 0$, $a = 1$, $q = 1$, $\epsilon = 2$ and $T = 1$ on the left, $T = 100$ on the right with $\overline{\eta} \sim 0.24$ (here we used $1/N \equiv 0$).

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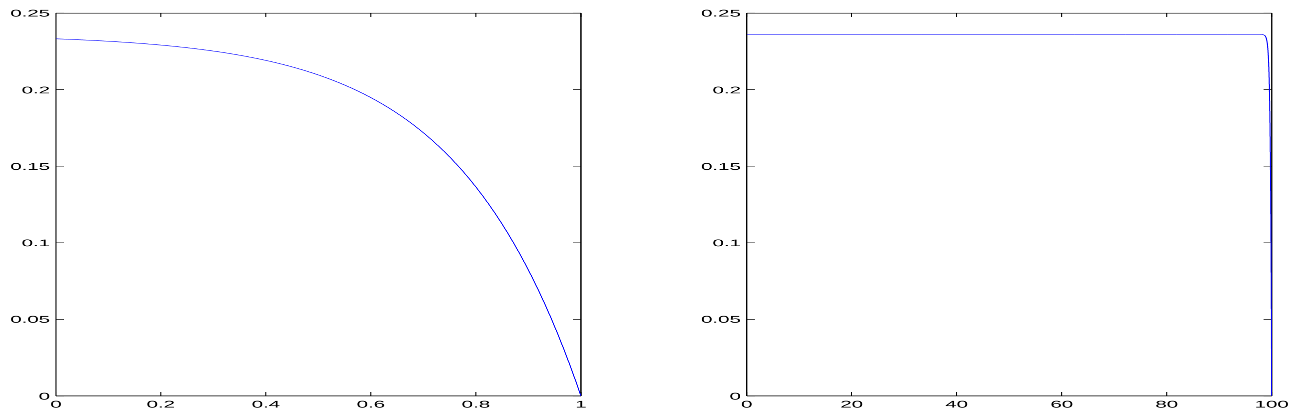

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Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := a+q+(1-\frac{1}{N})\overline{\eta}.$$

Mean Field Game ($N \to \infty$) with Common Noise

1. Fix $(m_t)_{t\geq 0}$ (the limit of $\overline{X}_t$ as $N \to \infty$ which depends on W^0)

Mean Field Game ($N \to \infty$)

- Fix $(m_t)_{t\geq 0}$ (the limit of $\overline{X}_t$ as $N \to \infty$ which depends on W^0)
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subject to:

$$dX_t = [a(m_t - X_t) + \alpha_t]\, dt + \sigma \left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t \right)$$

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Hamiltonian:

$$H(t,x,y,\alpha) = [a(m_t - x) + \alpha]\, y + \frac{1}{2}\alpha^2 - q\alpha(m_t - x) + \frac{\epsilon}{2}(m_t - x)^2$$

$$\frac{\partial H}{\partial \alpha} \longrightarrow \hat{\alpha} = q(m_t - x) - y$$

Adjoint Equations

$$
\begin{aligned}
dX_t &= [(a+q)(m_t - X_t) - Y_t]\,dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right), \quad X_0 = \xi \\
dY_t &= -\frac{\partial H}{\partial x}dt + Z_t^0 dW_t^0 + Z_t dW_t\,, \qquad Y_T = c(X_T - m_T) \\
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Taking conditional expectation given $(W_s^0)_{s\le t}$ in the second equation and using $m_t = m_t^X$ for all $t \le T$ and consequently $m_T^Y = c(m_T^X - m_T) = 0$, gives:

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Then, taking conditional expectations in the first equation gives:

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Adjoint Equations (continued)

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From $m_t^Y = -\int_t^T e^{(a+q)(s-t)} Z_s^0 dW_s^0$, we obtain $m_t^Y = 0$.

Mean Field Game Solution

Equating the drifts in the two Itô decompositions, we get

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which is the same Riccati equation as before but with "$N = \infty$".

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Once a solution to the MFG is found, on can use it to construct approximate Nash equilibria for the finitely many players games. Here, if one assumes that each player is given the information $\overline{X}_t$, and if player i uses the strategy $\alpha_t^i = (q+\eta_t)(\overline{X}_t - X_t^i)$, which is the limit as $N \to \infty$ of the strategy used in the finite players game, one sees how solving the limiting MFG problem can provide approximate Nash equilibria for which the financial implications are identical as the ones given for the exact Nash equilibria.

THANKS FOR YOUR ATTENTION