Lexicographic Labellings achieve fast algorithms for bump number, cocomp hamiltonicity and two-processor scheduling

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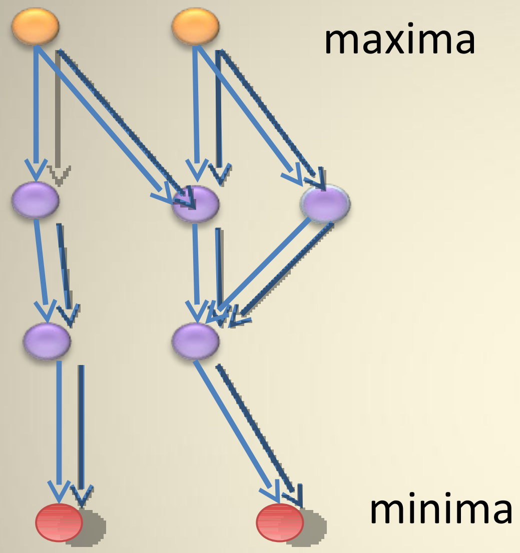
University of Toronto

Outline

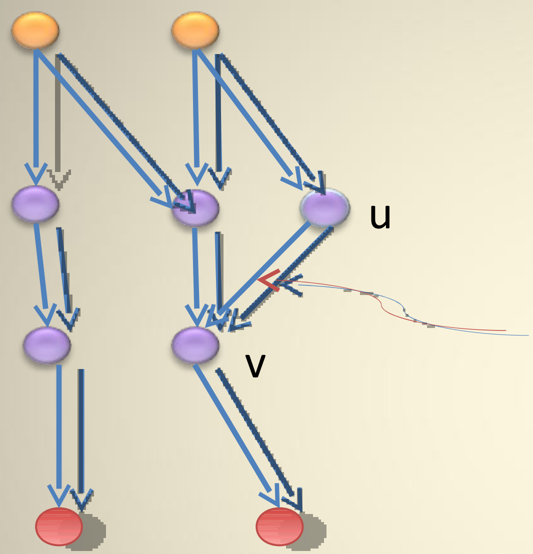
- Introduce Bump Number
- ~~Show relationship with 2-Proc Scheduling~~
- Show relationship with Min Path Cover in Cocomp Graphs
- Introduce Lexicographic Labelling
- Give Greedlex Algorithm
- Prove Greedlex is Correct
- Show how this fits into previous work
- Further work

Posets = partially ordered sets



Hasse Diagram

Posets = partially ordered sets



u *covers* v

u is an *upper cover* of v

v is a *lower cover* of u

u v

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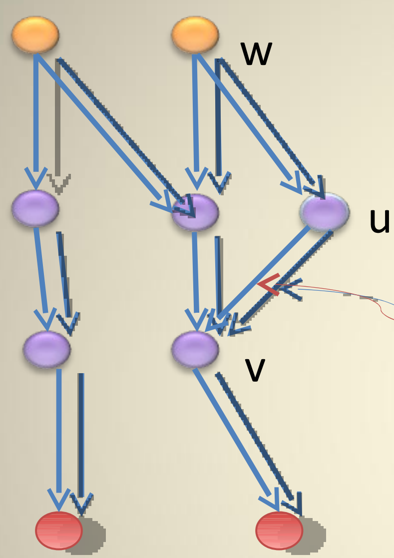

w > v
v < w
v and w are *transitively related*

u *covers* v
u is an *upper cover* of v
v is a *lower cover* of u
u ⋖ v

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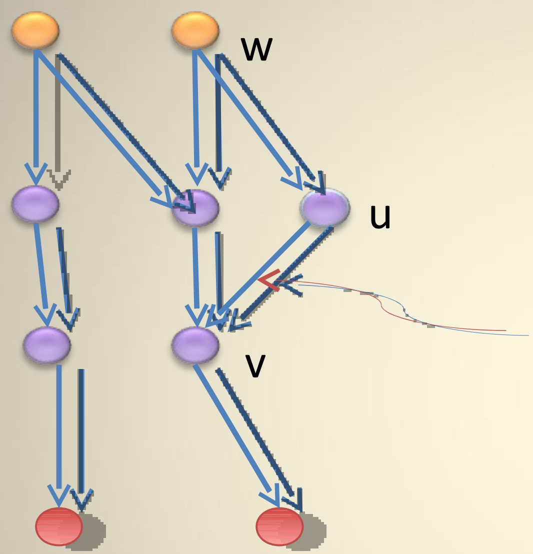

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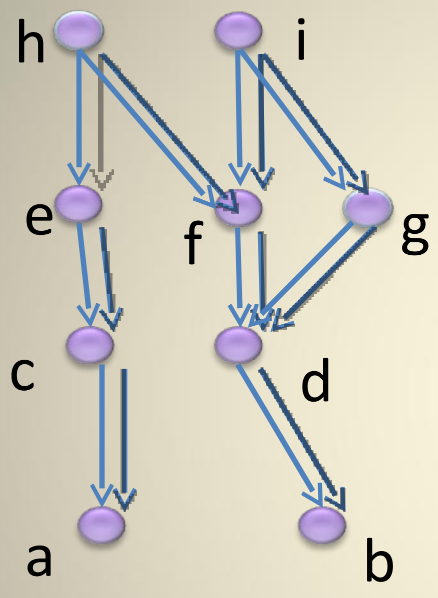
Hasse Diagram

-A compact representation of a set of relations

-i.e. can be $O(n)$ representation of $O(n^2)$ relations

Bumps in linear extensions



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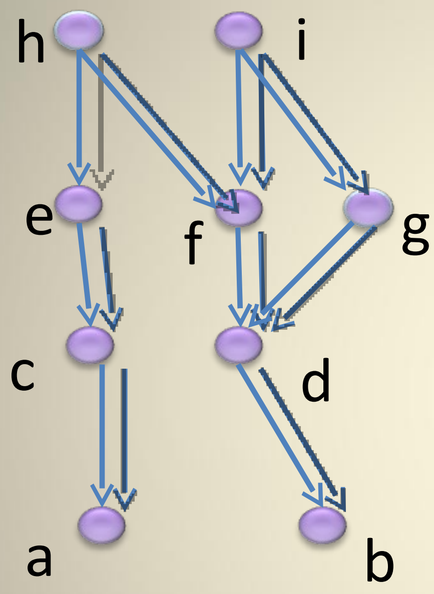
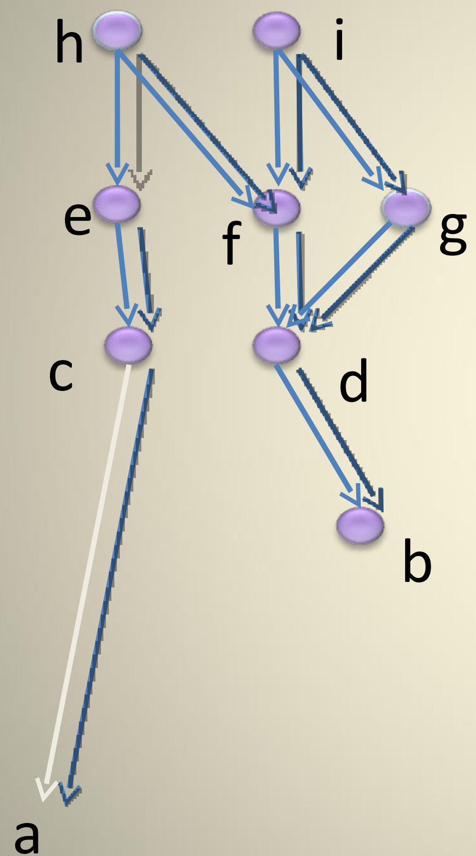


Linear extension (showing bumps)

Bumps in linear extensions



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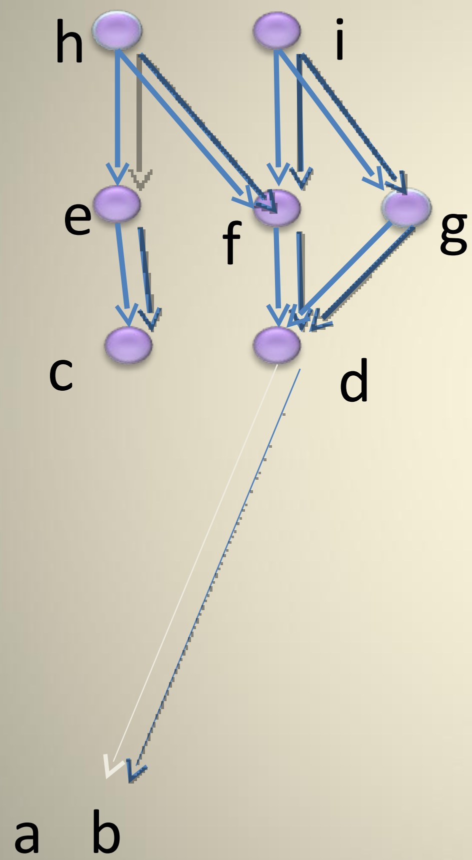

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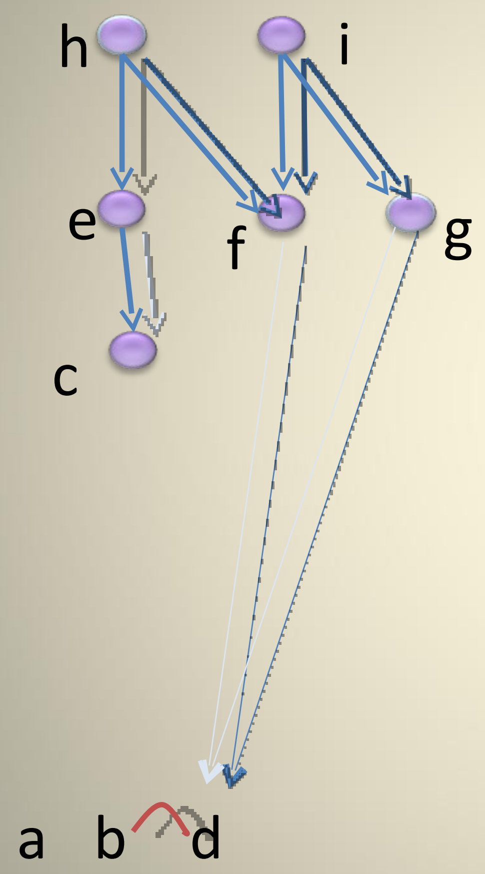

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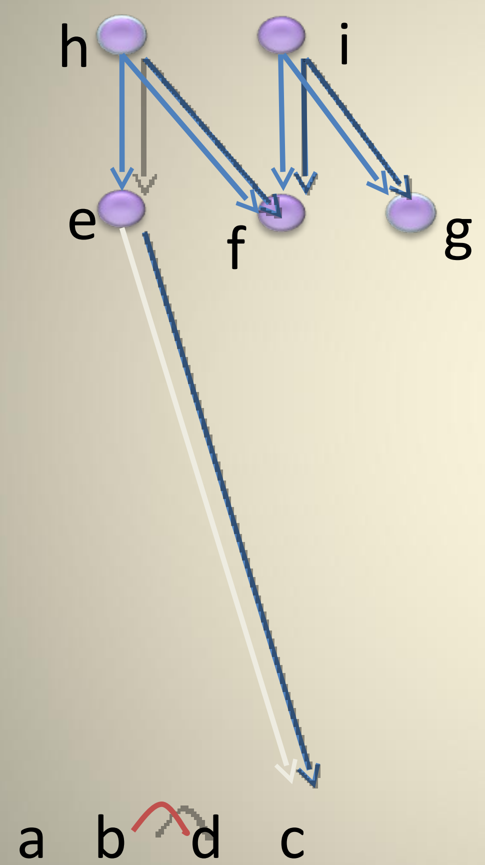

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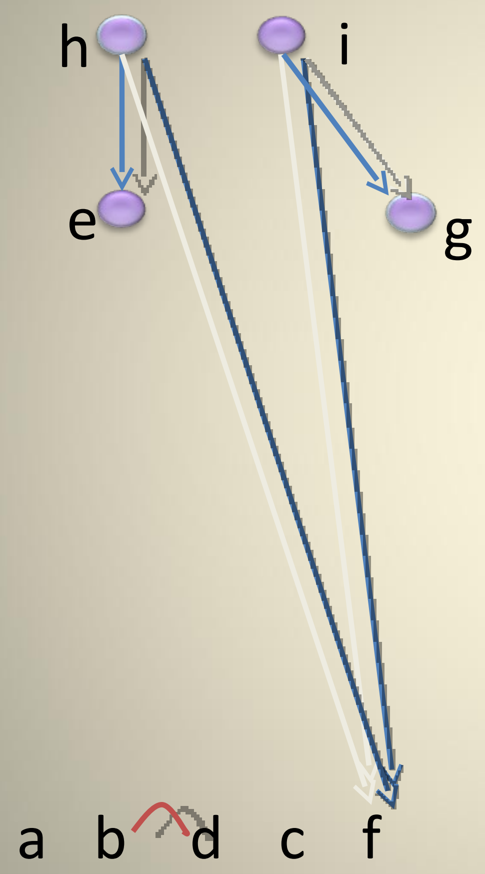

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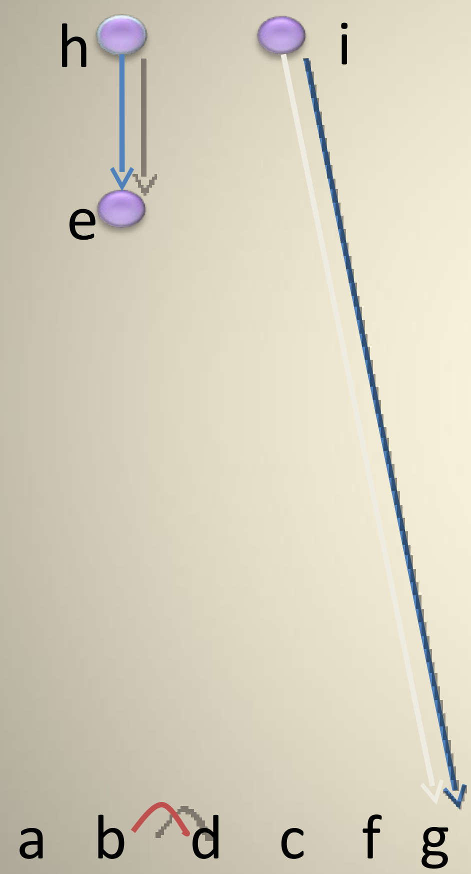

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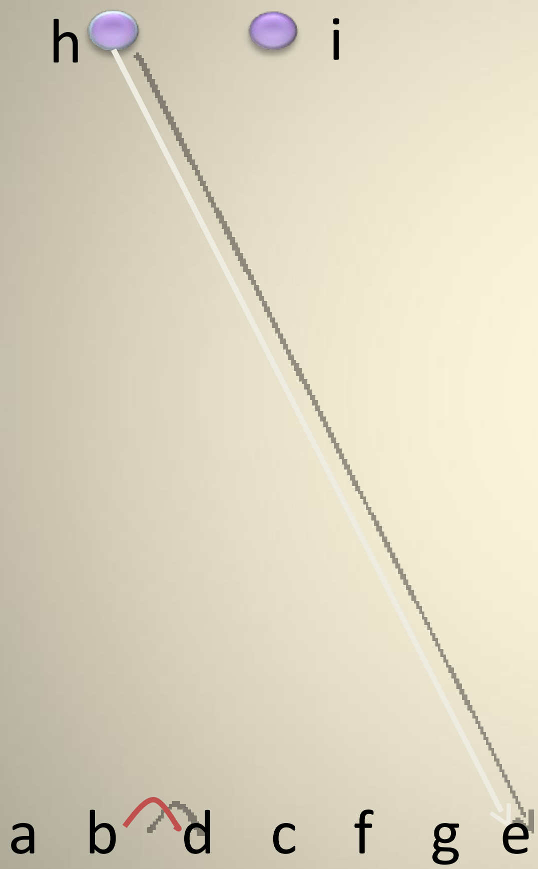


Linear extension (showing bumps)

Bumps in linear extensions

h

Linear extension (showing bumps)

a b d c f g e i

Bumps in linear extensions

Linear extension (showing bumps)

a b d c f g e i h

Bump Number Problem

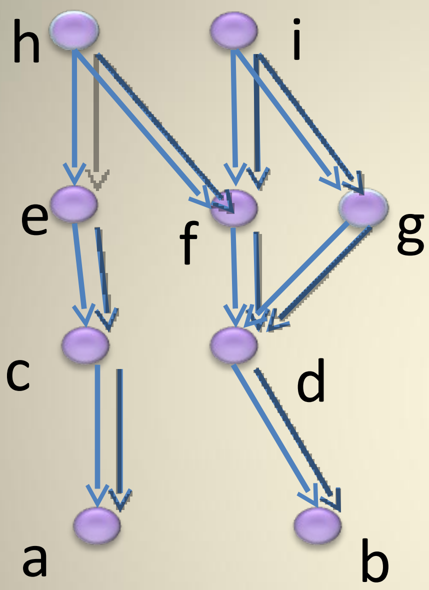
Given poset P, what is the least number of bumps realized by a linear extension of P?

b(P)= bump# of P

Find an algorithm to compute b(P) and construct a linear extension with fewest bumps

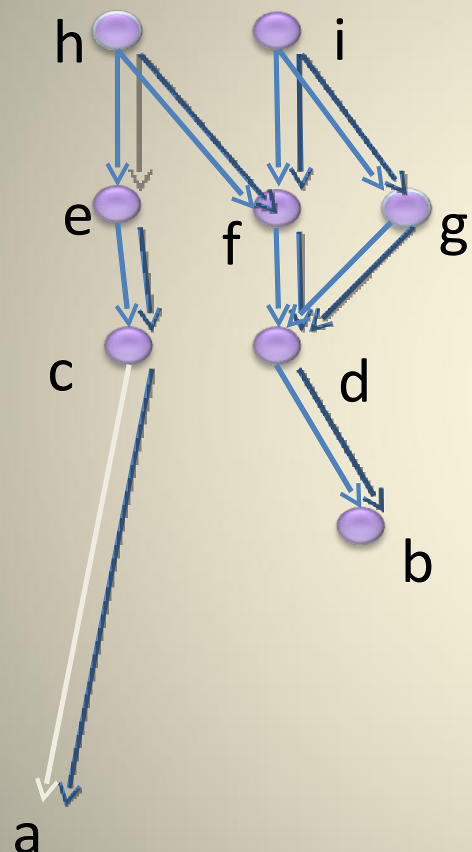
a b d c f g e i h

Greedily seeking min-bump l.e.



Linear extension (showing bumps)
Greedily selecting to avoid bumps

Greedily seeking min-bump l.e



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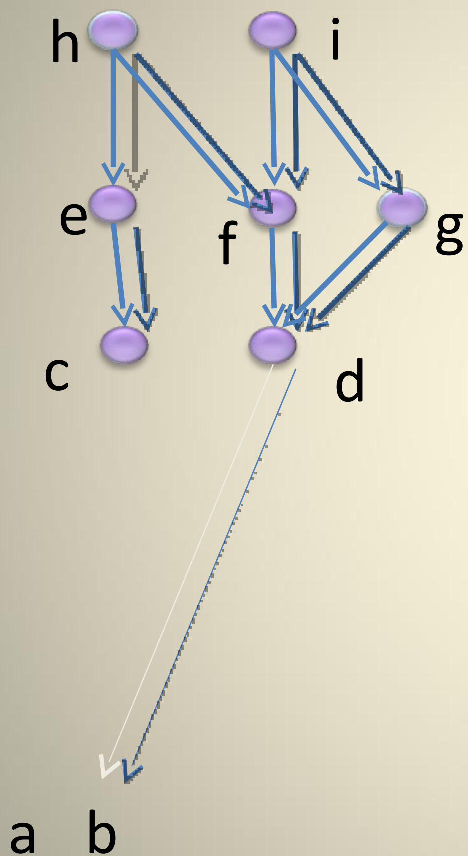

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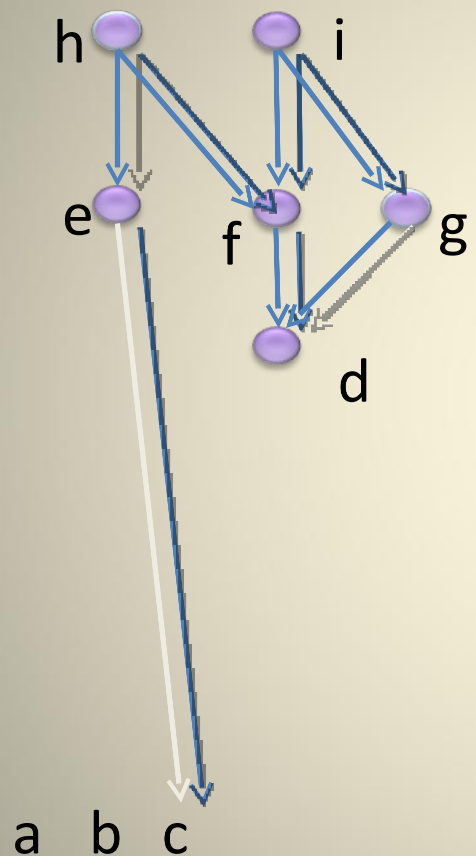

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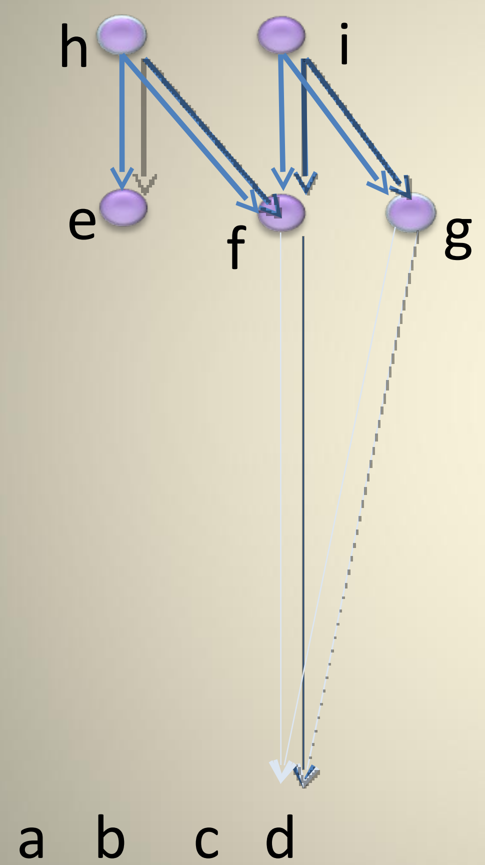

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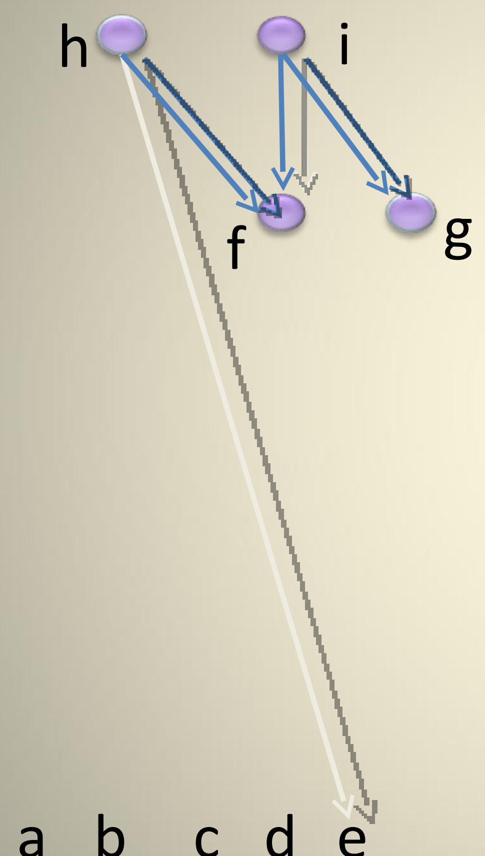

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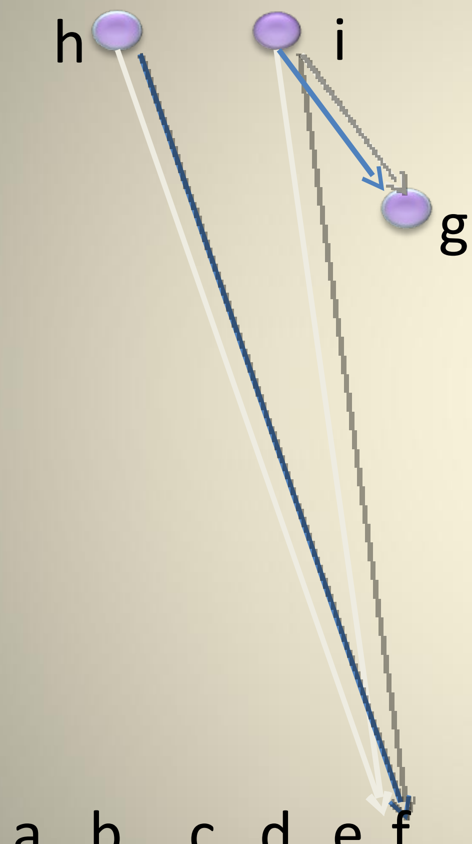

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Linear extension (showing bumps)
Greedily selecting to avoid bumps

a b c d e f g h i

Greedily seeking min-bump l.e

There is always some greedy l.e. that achieves minimum bump (Fishburn & Gehrlein, ‘86).

For which posets does greedy always work?

Greedily seeking min-bump l.e

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Greedy + ? works for all posets?

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For which posets does greedy always work? F&G’86

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For which posets does greedy always work? F&G’86

Greedy + ? works for all posets? This talk

Bump Number

- polynomial algorithms for interval order posets and for partial semiorder posets – both are based on the greedy shelling algorithms

  Fishburn and Gehrlein 1986

- polynomial algorithm for width=2 posets – not based on greedy shelling

  Zaguia 1987

-

- polynomial algorithm for any poset – not based on shelling

  Habib, Möhring, Steiner 1988

- linear time algorithm – based on Gabow's linear time 2-proc scheduling algorithm

  Schäffer & Simons 1988

Greedlex Algorithm does these quickly, simply

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•linear time algorithm – based on Gabow's linear time 2-proc scheduling algorithm Schäffer & Simons 1988

Linear Time Bump Number

relies on Gabow and Tarjan's special case Union-Find algorithm: union and find operations known in advance

O(n+m)

… relies on hybrid linked-list / array data structure … Switch to array representation of tree for subtrees that are small enough…

Algorithm, proof of correctness, and analysis

- Spread across several papers
- Proofs long and case-ridden
- Analysis complex

Question:

$\exists$ a simple algorithm

with a short proof

that can be made efficient (linear time) without recourse to Special Case of Union-Find?

Algorithm, proof of correctness, and analysis

- Spread across several papers
- Proofs long and case-ridden
- Analysis complex

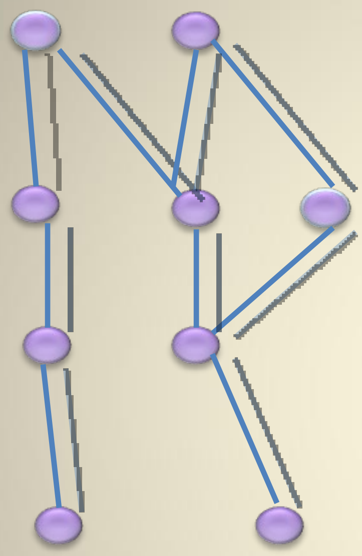
Question:

$\exists$ a simple algorithm YES

with a short proof YES

that can be made efficient (linear time) without recourse to Special Case of Union-Find? I think so.

Posets & comparability graphs



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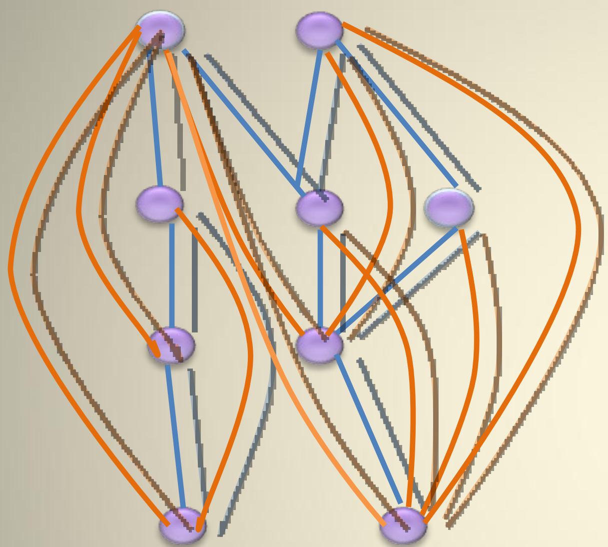
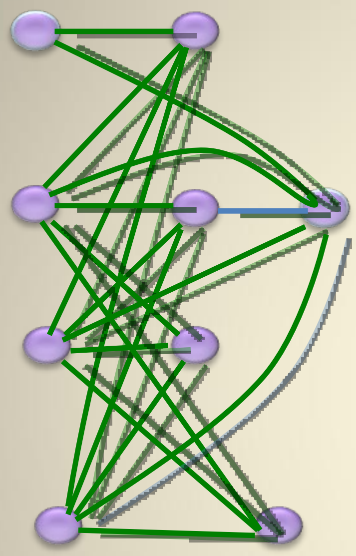


Posets & cocomparability graphs



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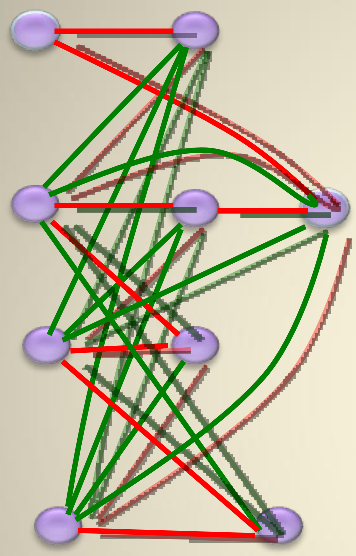
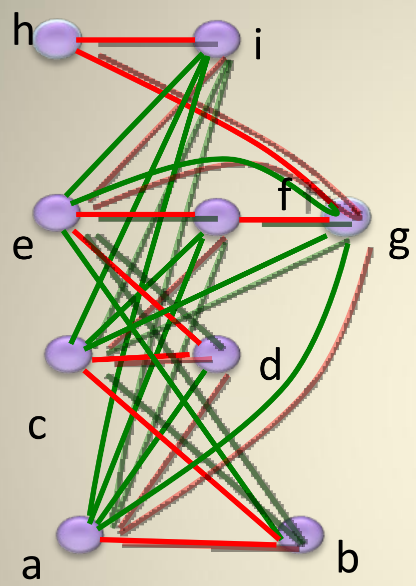


When is there a Hamilton Path in the cocomparability graph?

Posets & cocomparability graphs



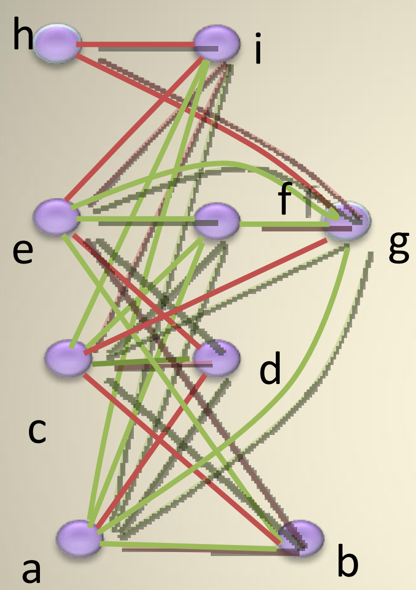
When is there a Hamilton Path in the cocomparability graph?

When there is an ordering of the vertices so that there is an edge between successive vertices

…i.e., so that there is a non-edge in the *comparability* graph

… i.e., so there is *no bump* between successive vertices in the linear extension (assuming your restrict to orderings that obey the partial order).

Posets & cocomparability graphs



When is there a Hamilton Path in the cocomparability graph?

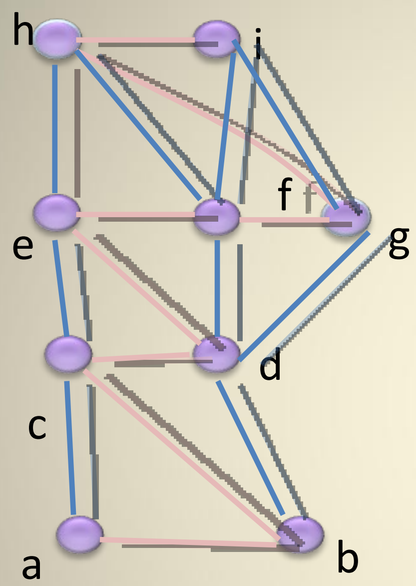
Of course, it is possible to trace the graph in ways that are not obedient to the partial order

a d e i h g c b

Exists Ham Path iff exists cocomp order that is a HamPath iff bump#=0

Exists k-path cover in cocomp graph iff bump# ≤ k

Posets & cocomparability graphs



When is there a k-path cover in the cocomparability graph?

Cocomp graph G ⟺ many posets

Cocomp graph G + cocomp ordering ⟺ one poset

Solve bump on the poset ⟺ Solve min-path-cover on cocomp graph

Solve MPC on cocomp graph using a cocomp order ⟺ Solve bump on the unique underlying poset

Hamiltonicity of Cocomp Graphs

Keil 1985

- Ham’n cycle in Interval graphs alg

Deogun Steiner 1990

- Poly-time Ham’n Cycle

Deogun Kratsch Steiner 1997

- 1-tough cocomp graphs are hamiltonian –

Damaschke Deogun Kratsch Steiner 1991

- Hamilton Path in cocomps using bump number algorithm

Corneil Dalton Habib 2013

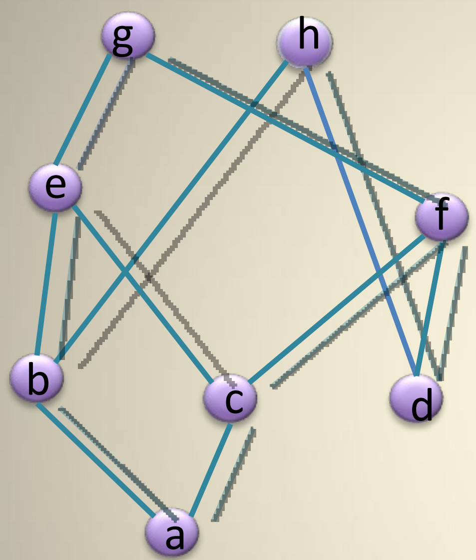
- Min Path Cover Alg (certified) in Cocomp Graphs

Recap:

- Definition of Bump Number
- Relationship (equivalency, up to data representation) to the Minimum Path Cover/Hamiltonicity of Cocomp Graphs
- Is related to Two-Processor Scheduling

- Introduce Lexicographic Labelling
- Give the Greedlex Algorithm solving Bump
- Prove Greedlex is correct
  - State the Lex-Yanking Lemma
  - Show that the Lex-Yanking Lemma implies Greedlex is Correct
  - Prove the Lex-Yanking Lemma
- How this work fits into previous results

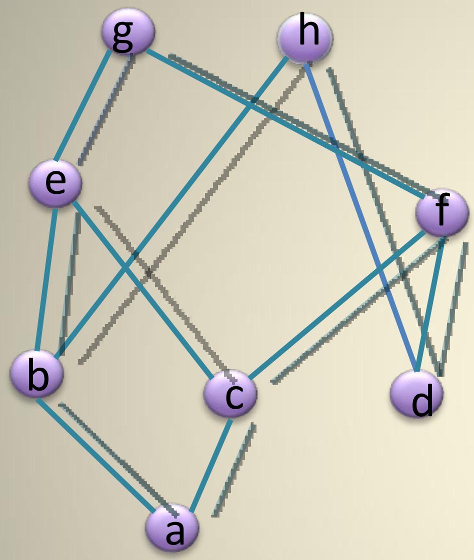
Greedy bump#



Greedy Approach

d a … oops

Greedy bump#



Greedy Approach

d a ... oops

a d b c h e f ...oops

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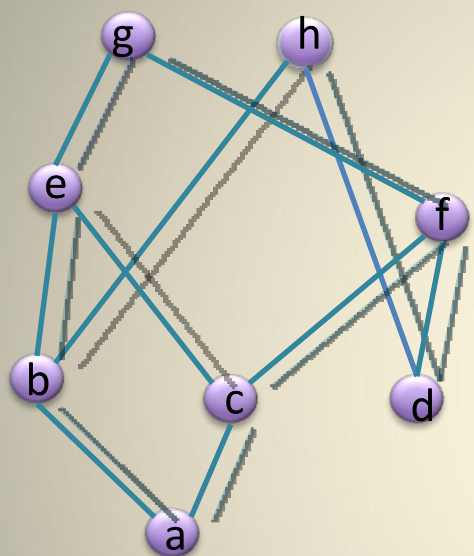


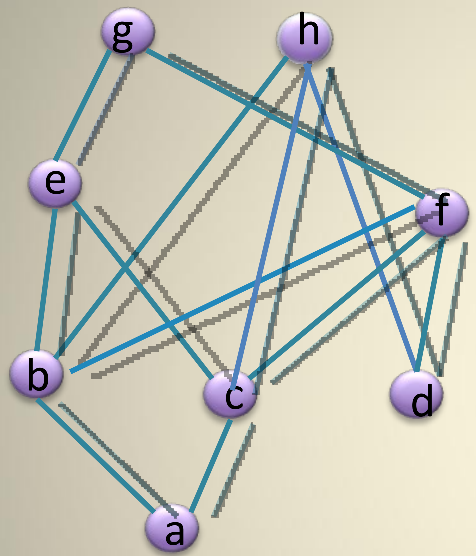
Greedy Approach

d a ... oops

a d b c h e f ...oops

a d c b f e h g

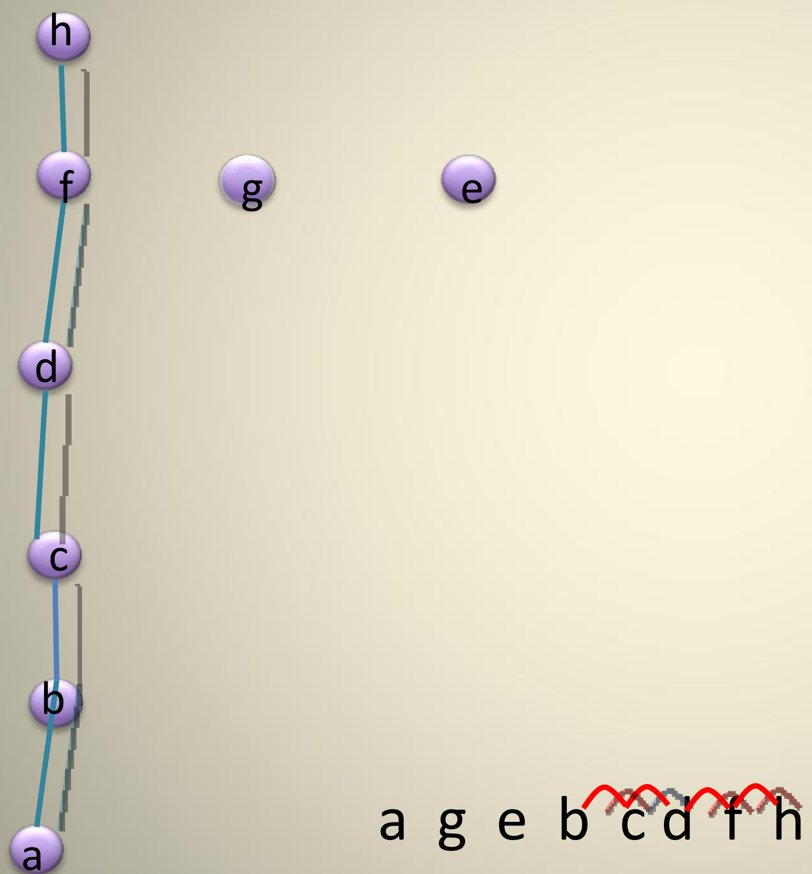
How can a bump be unavoidable



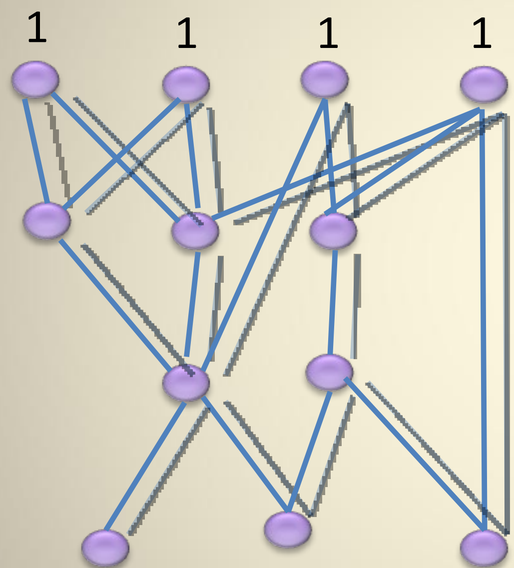
a d b c …

Now all minima e f g are upper covers of c

How can a bump be ~~un~~avoidable



Lexicographic Labelling



- Give minima arbitrary lex#

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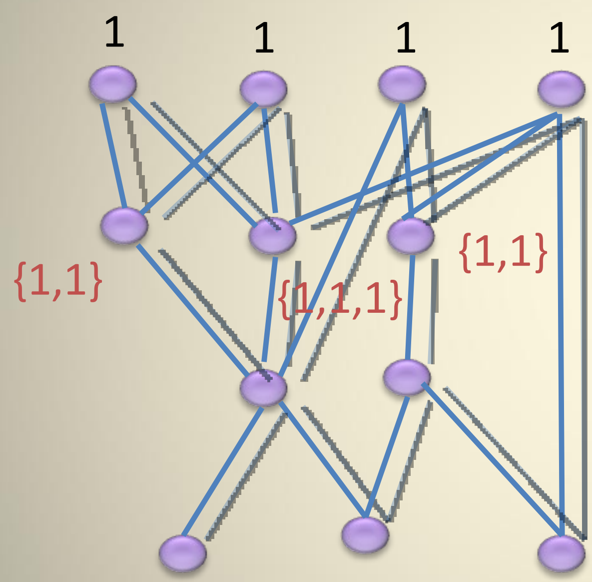

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- Assign lex# so that lex(u)<lex(v) whenever {lex(u'): u'covers u} $<_{lexico}$ {lex(v'): v' covers v}

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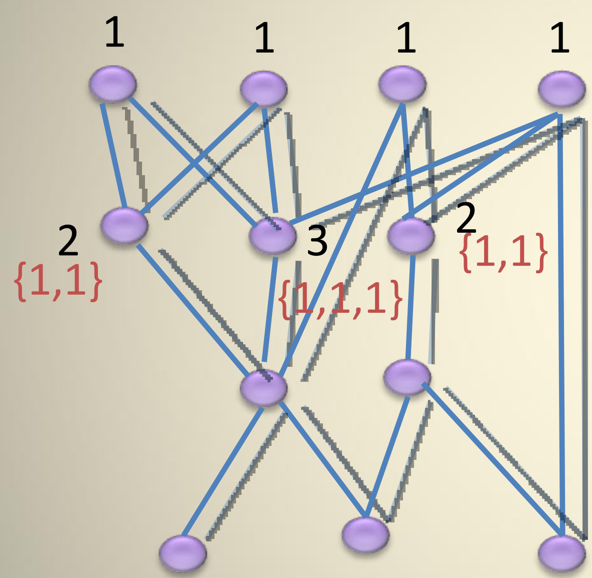

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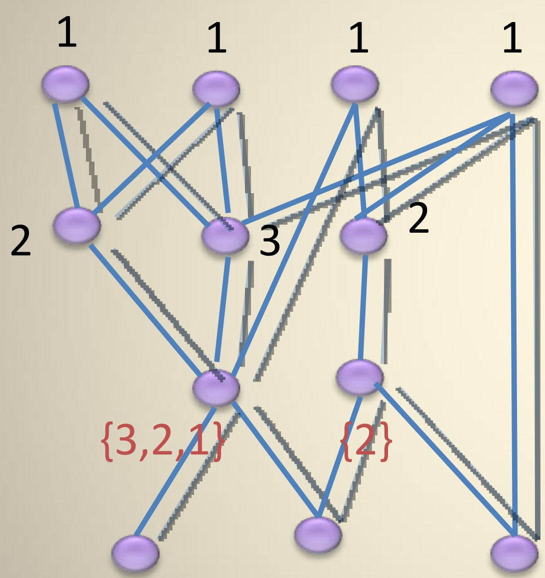

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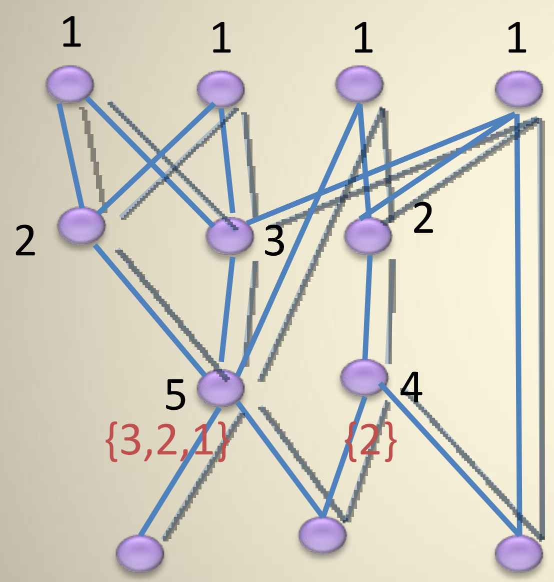

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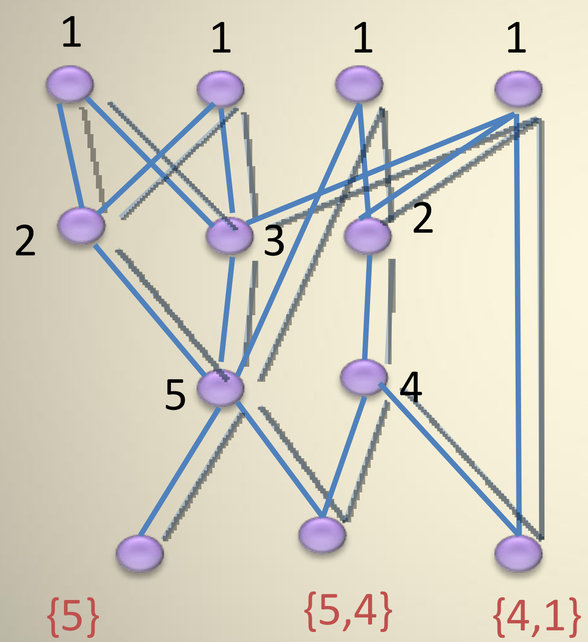

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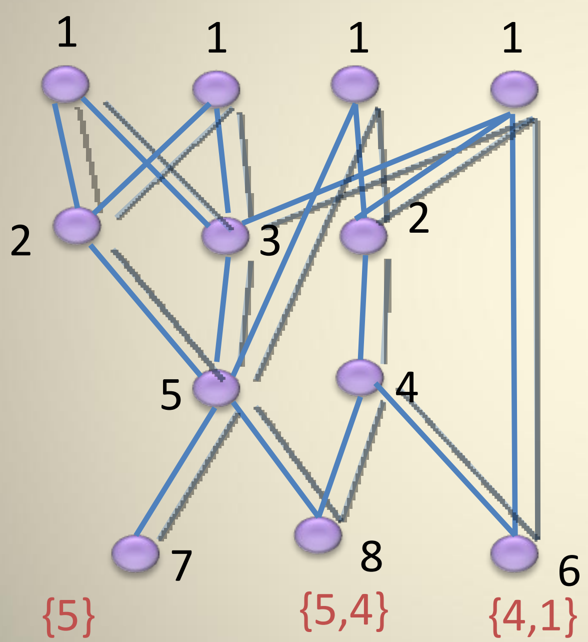

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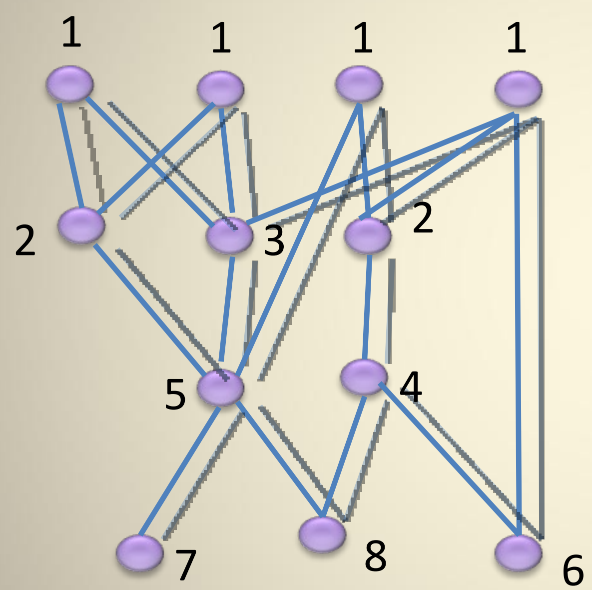

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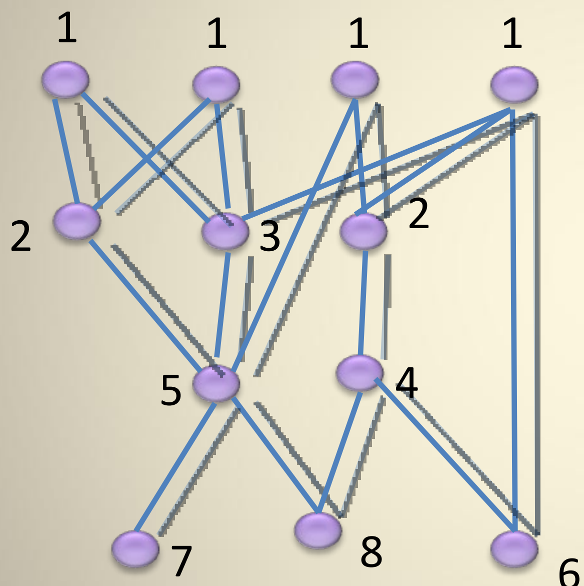
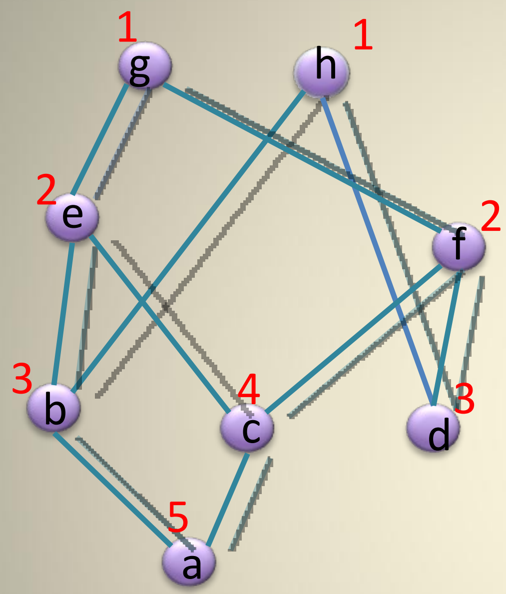


New: O(n+m) algorithm for lex-labelling

(Sethi 1976 algorithm also acheives linear time)

Greedy + Lex = Greedlex



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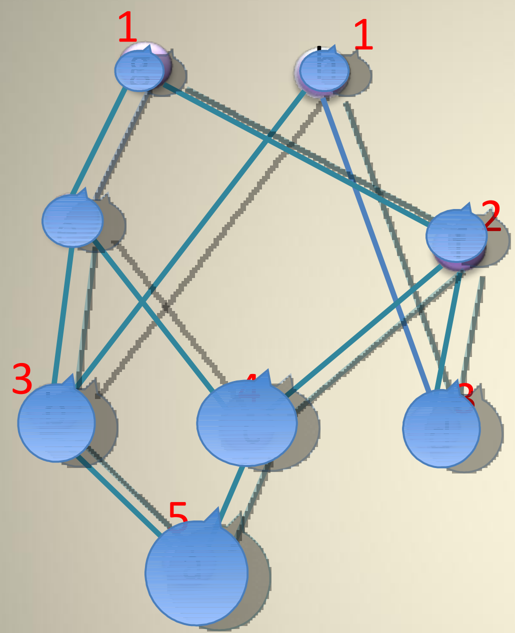

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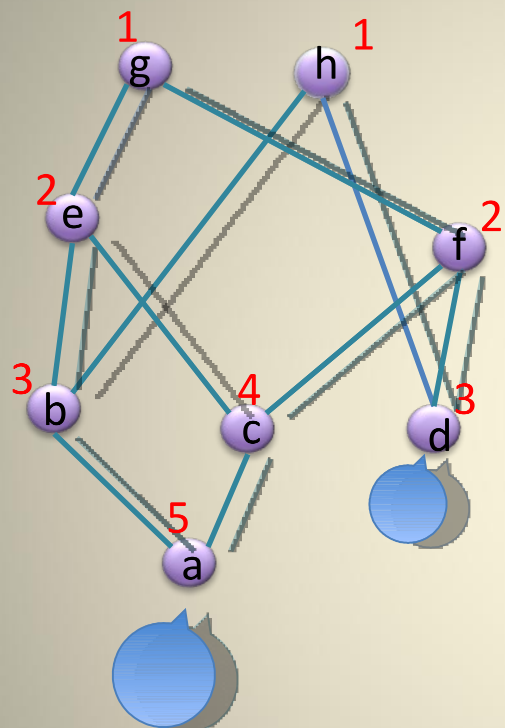

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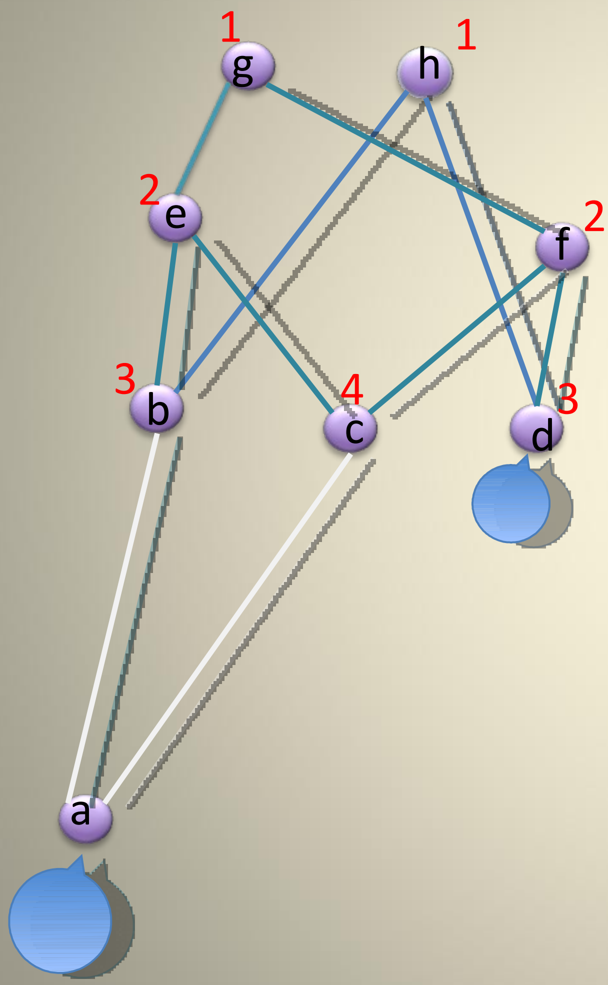

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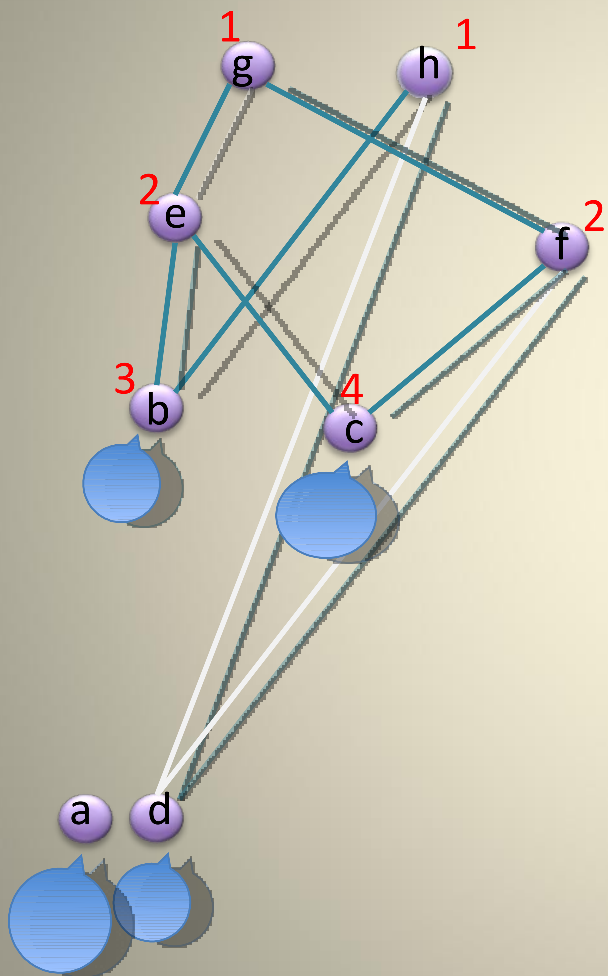

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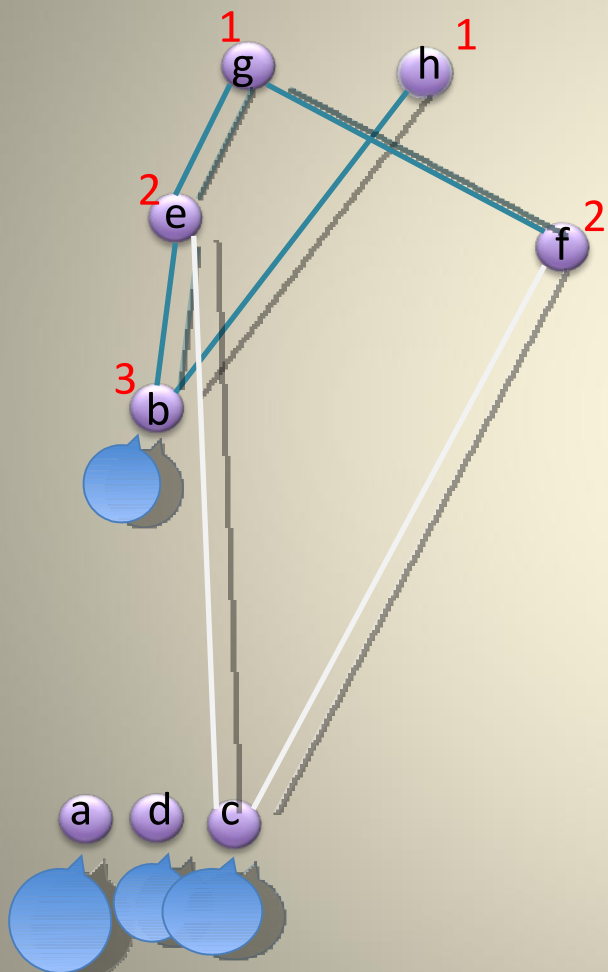

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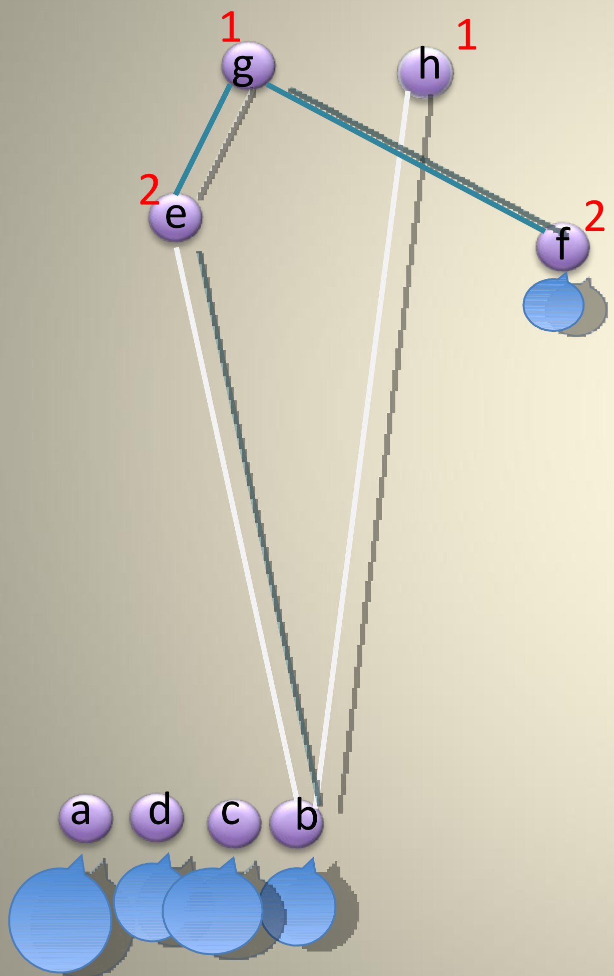

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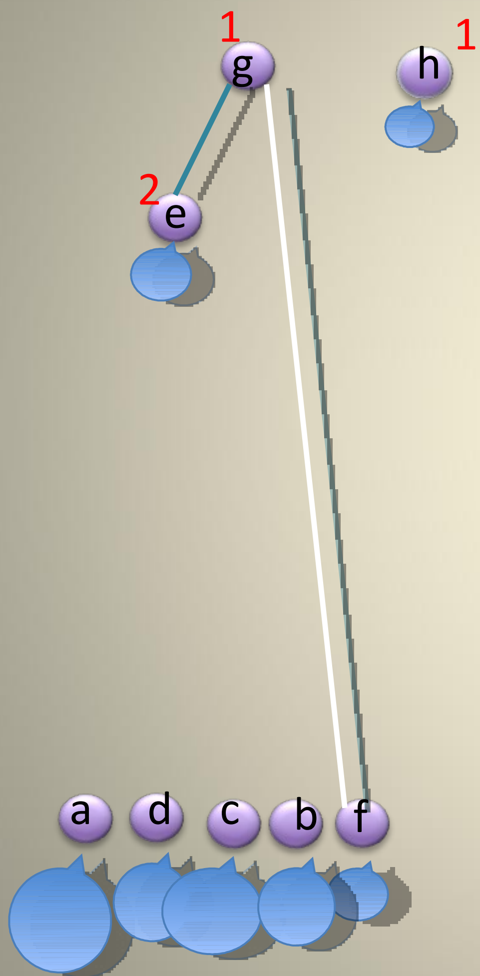
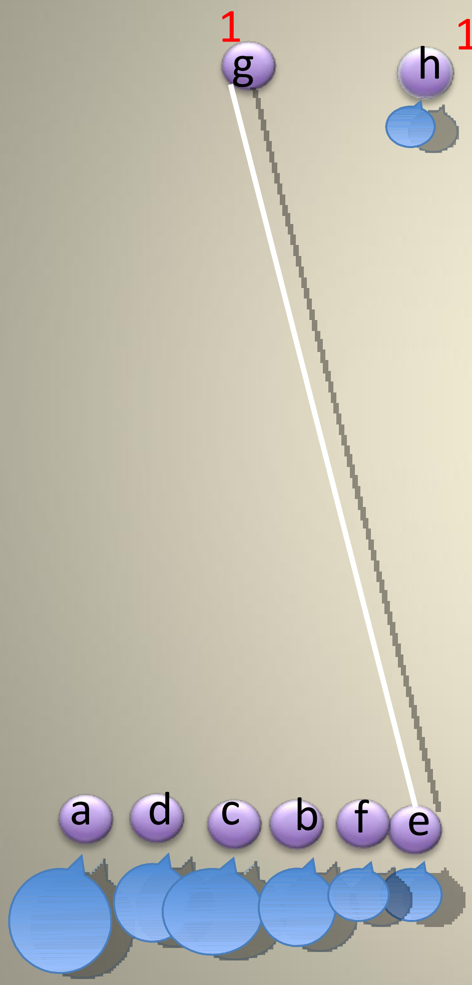


Greedy + Lex = Greedlex



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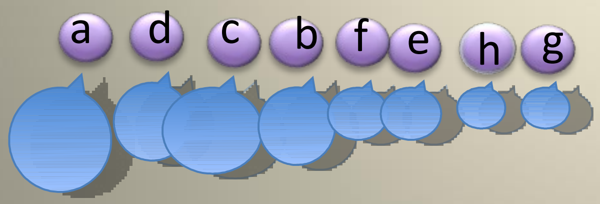


Greedlex Alg for bump#

1. Lex label all v in V(P)
2. Shell P, always removing

    (a) a non-cover of last-shelled u, if exists

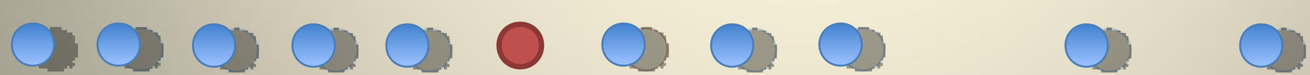
    (b) the highest lex-labelled v allowed by (a)

This always yields the min-bump l.e.!

Proof of Correctness

First, an observation:

When shelling to produce a low-bump l.e., if you make one bad selection, how many *added bumps* can that introduce?



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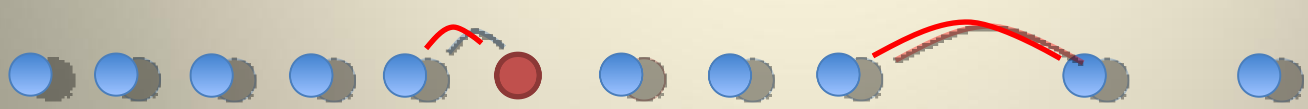

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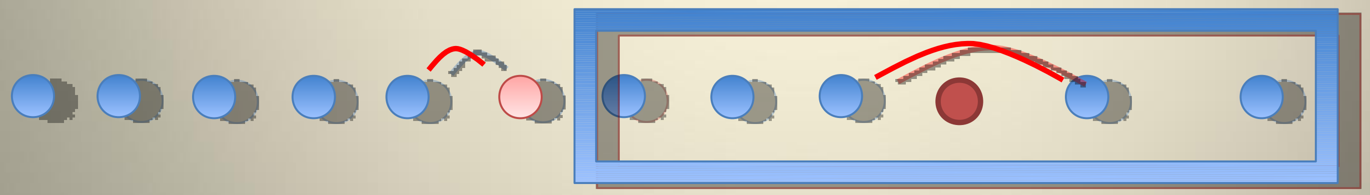

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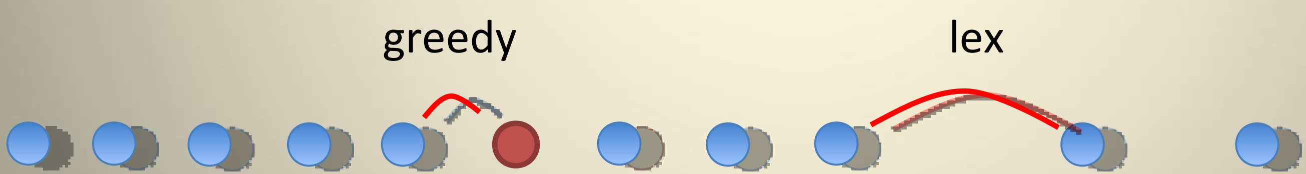


Lex-Yanking Lemma

b xxx..x a xx…x has k bumps and a is min

$\Rightarrow$ ∃ l.e. a x'x'x'…b…x' with k or fewer bumps

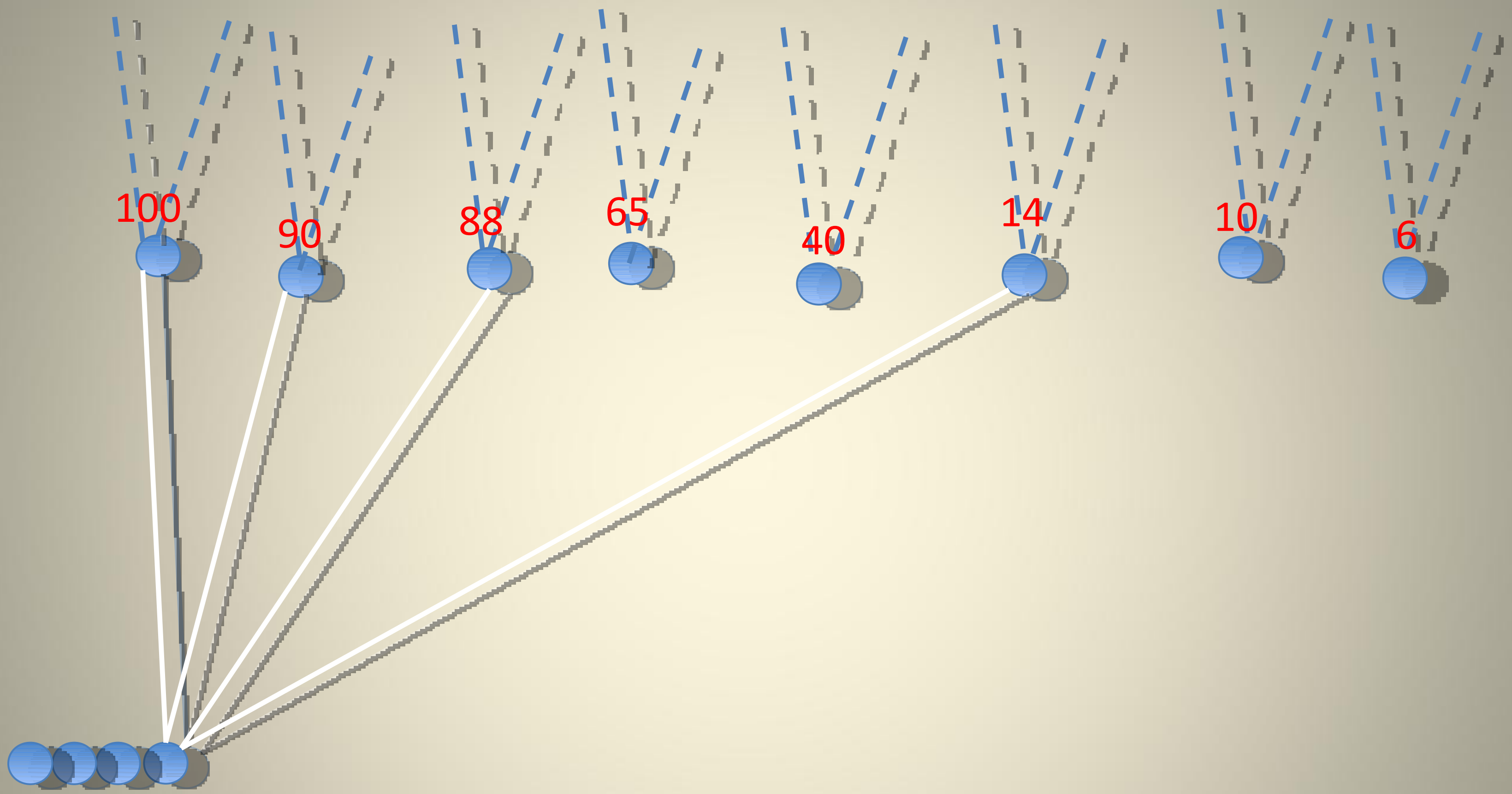
(Balloon size indicates relative Lex value)

Recap:

- Definition of Bump Number
- Relationship (equivalency, up to data representation) to the Minimum Path Cover/Hamiltonicity of Cocomp Graphs
- Is related to Two-Processor Scheduling

- Introduce Lexicographic Labelling
- Give the Greedlex Algorithm solving Bump
- Prove Greedlex is correct
  - State the Lex-Yanking Lemma
  - Show that the Lex-Yanking Lemma implies Greedlex is Correct
  - Prove the Lex-Yanking Lemma
- How this work fits into previous results

The LexYanking Lemma implies Greedlex works:



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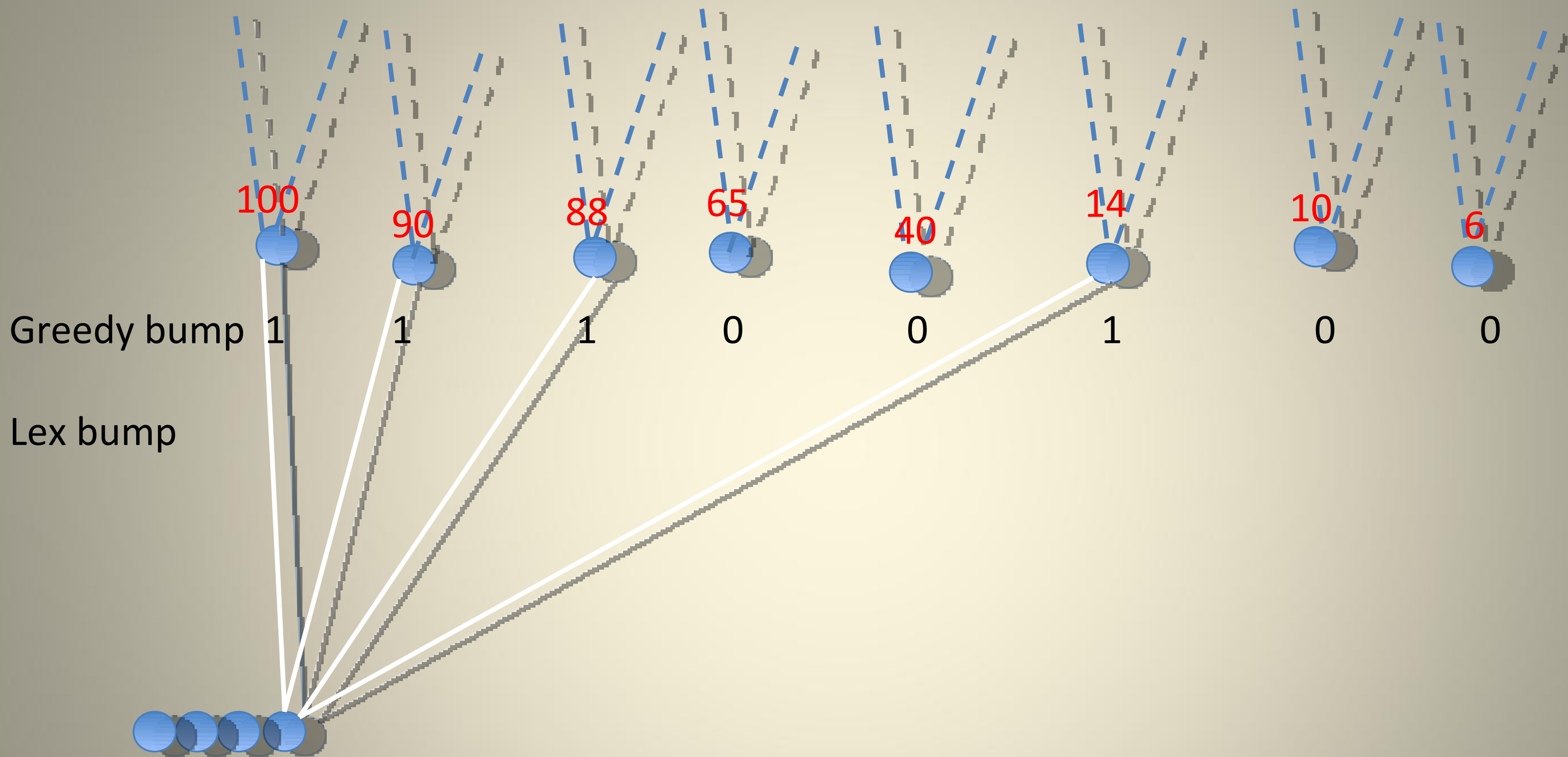

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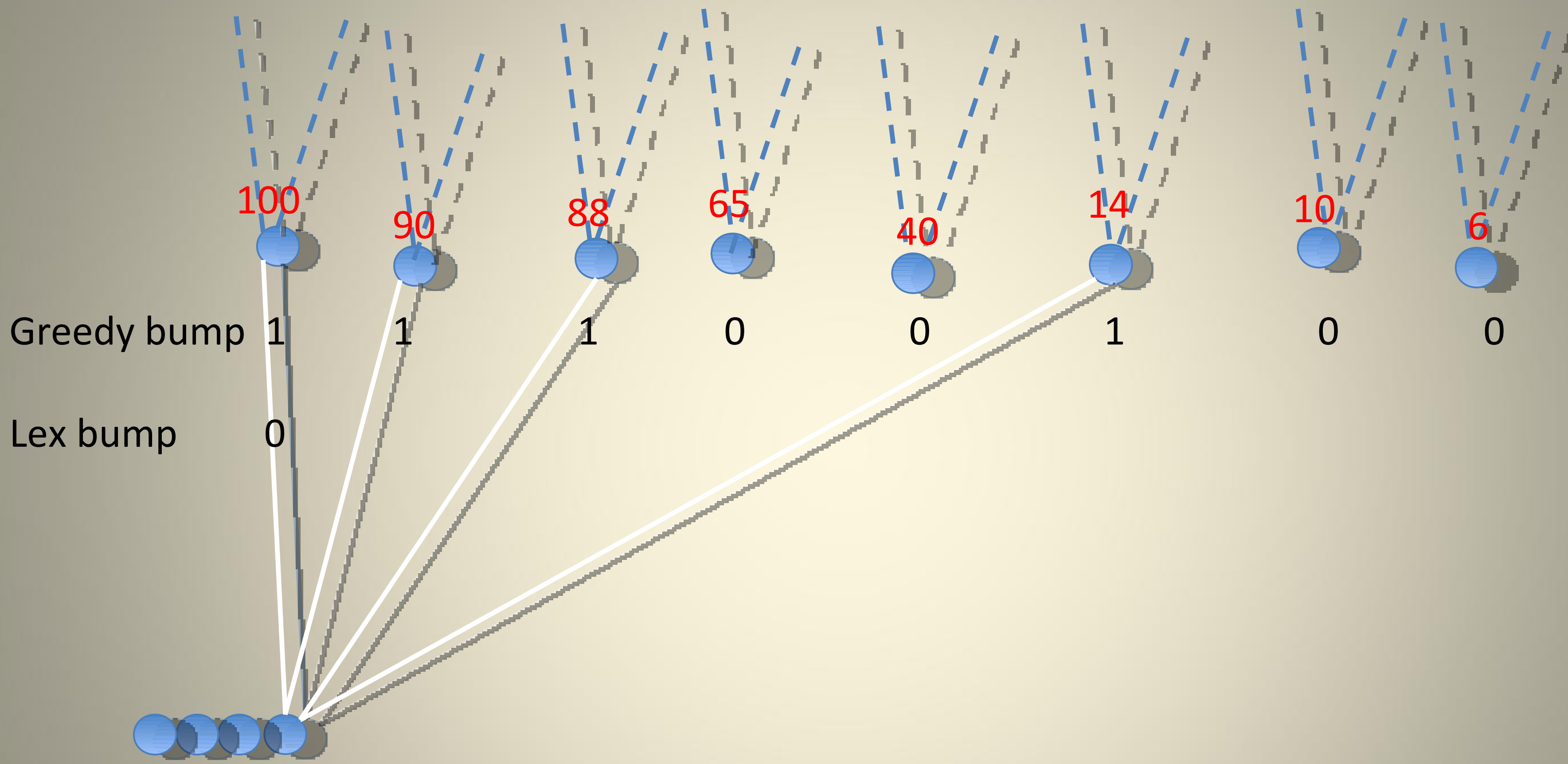

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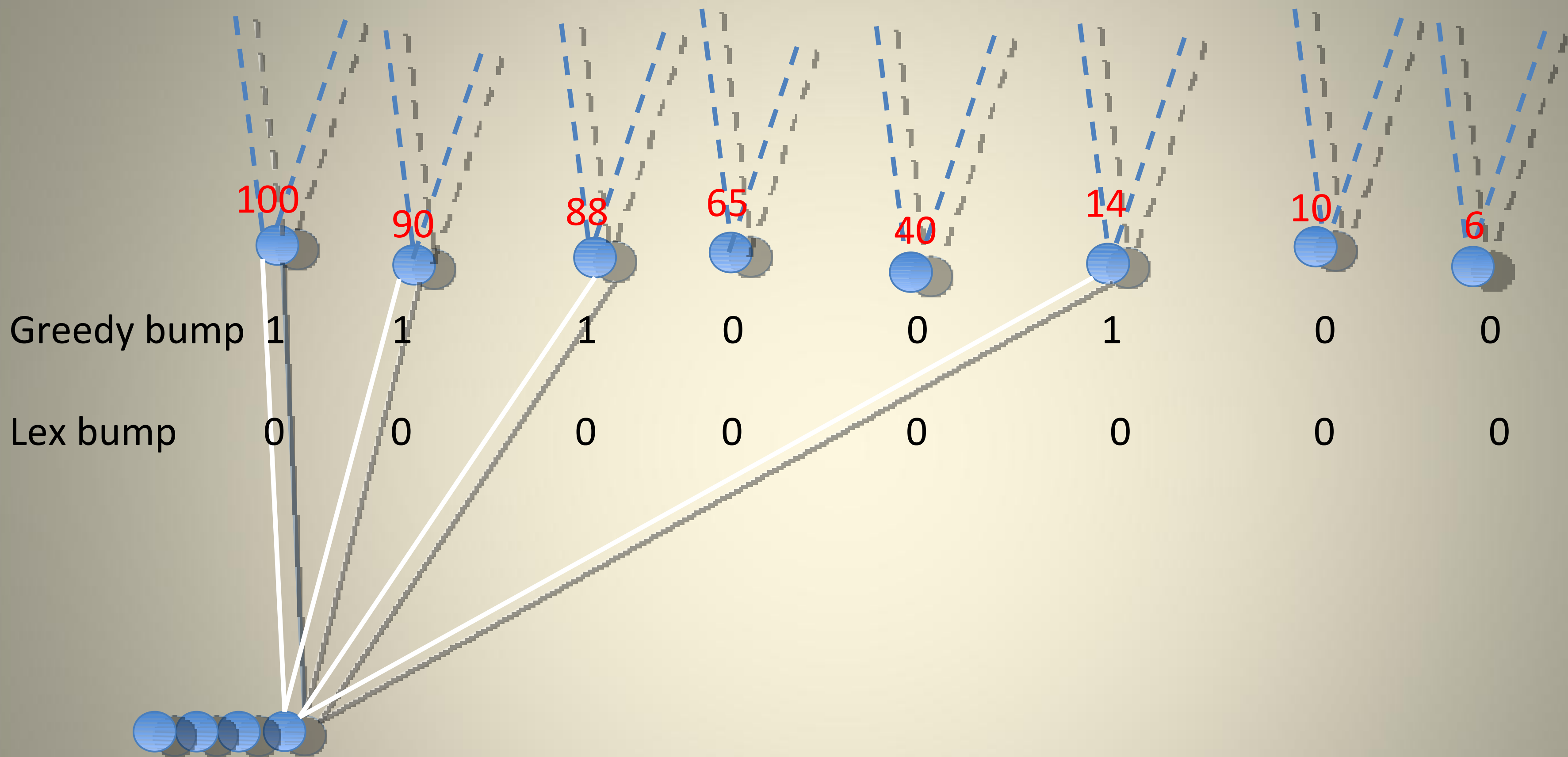

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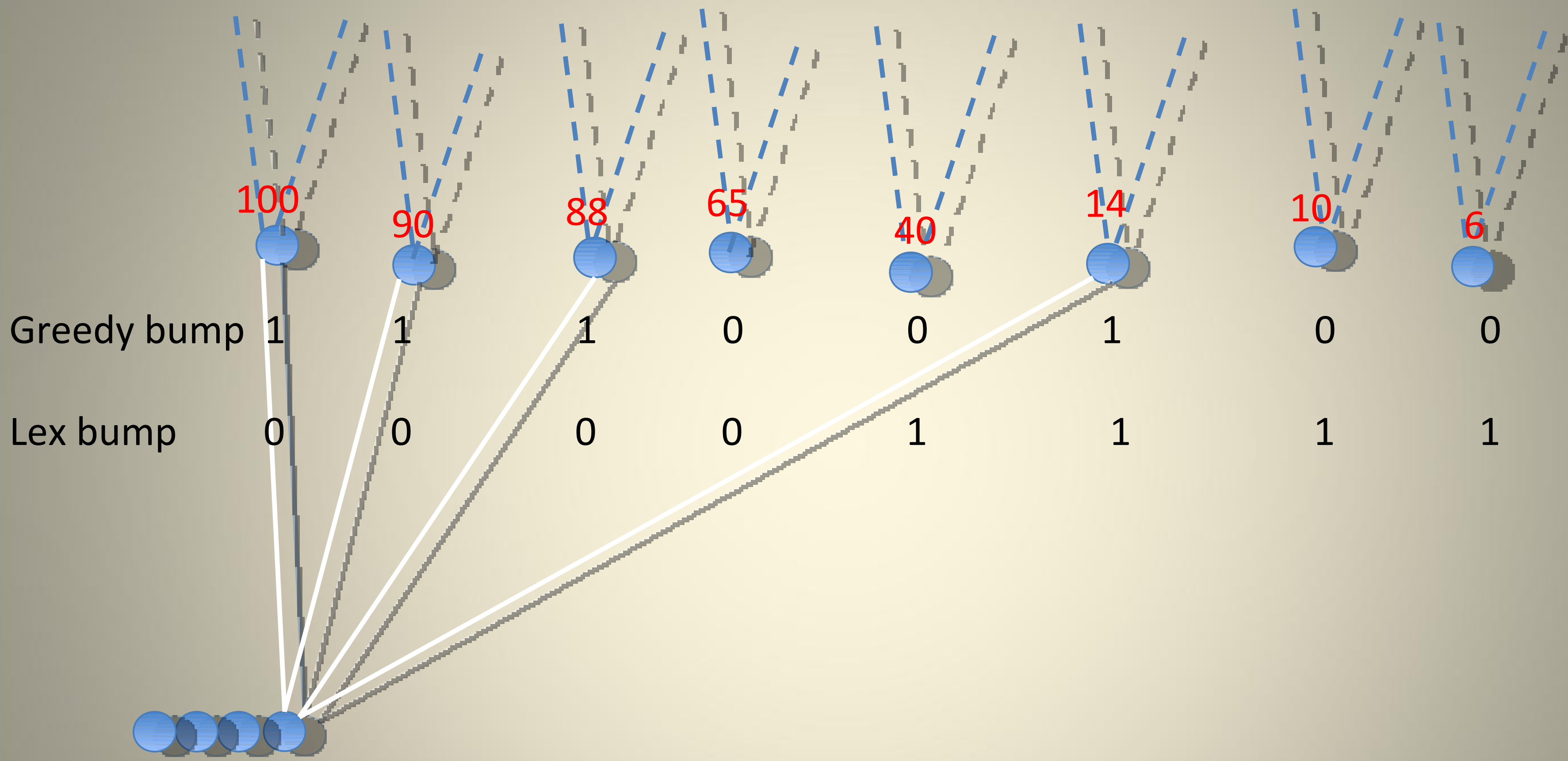

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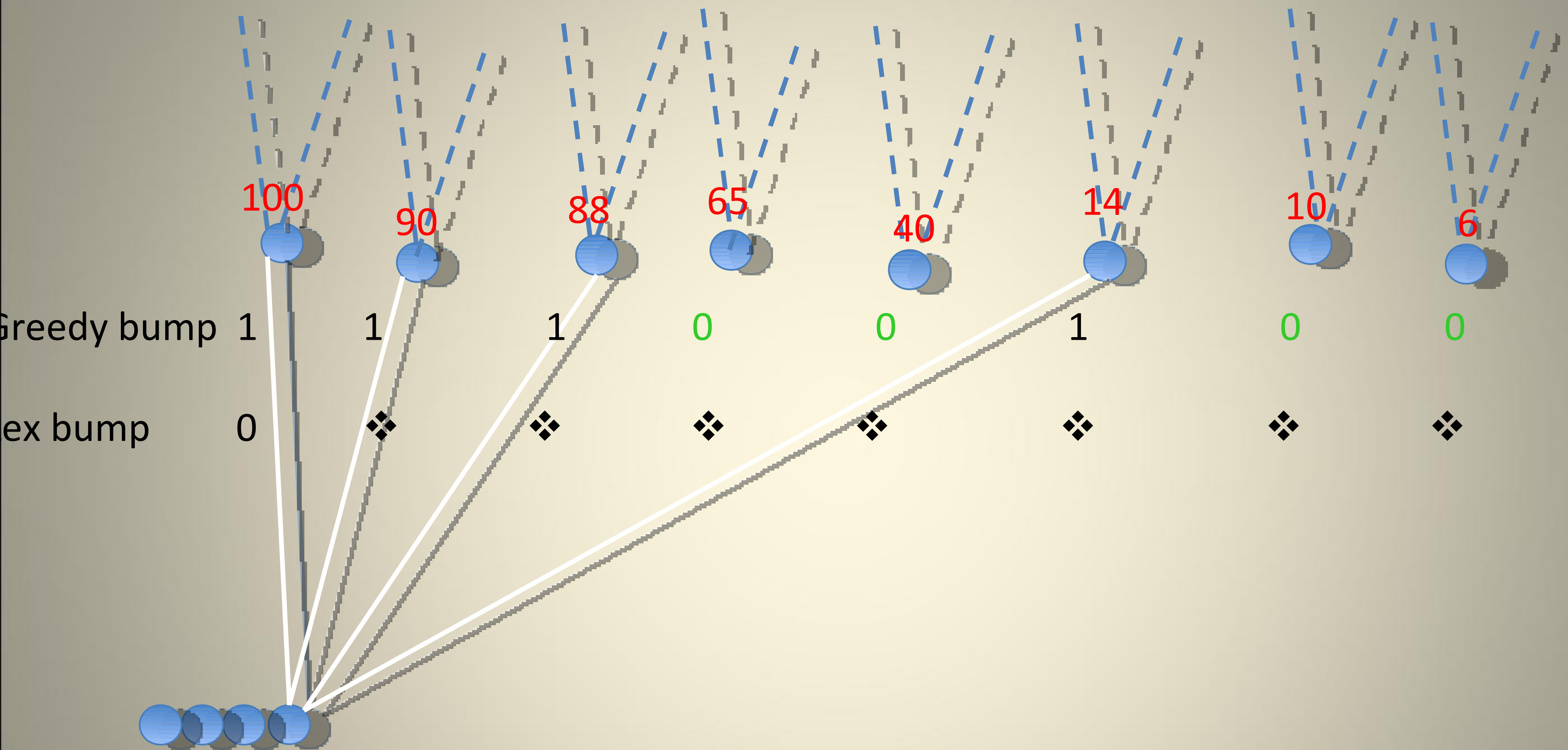

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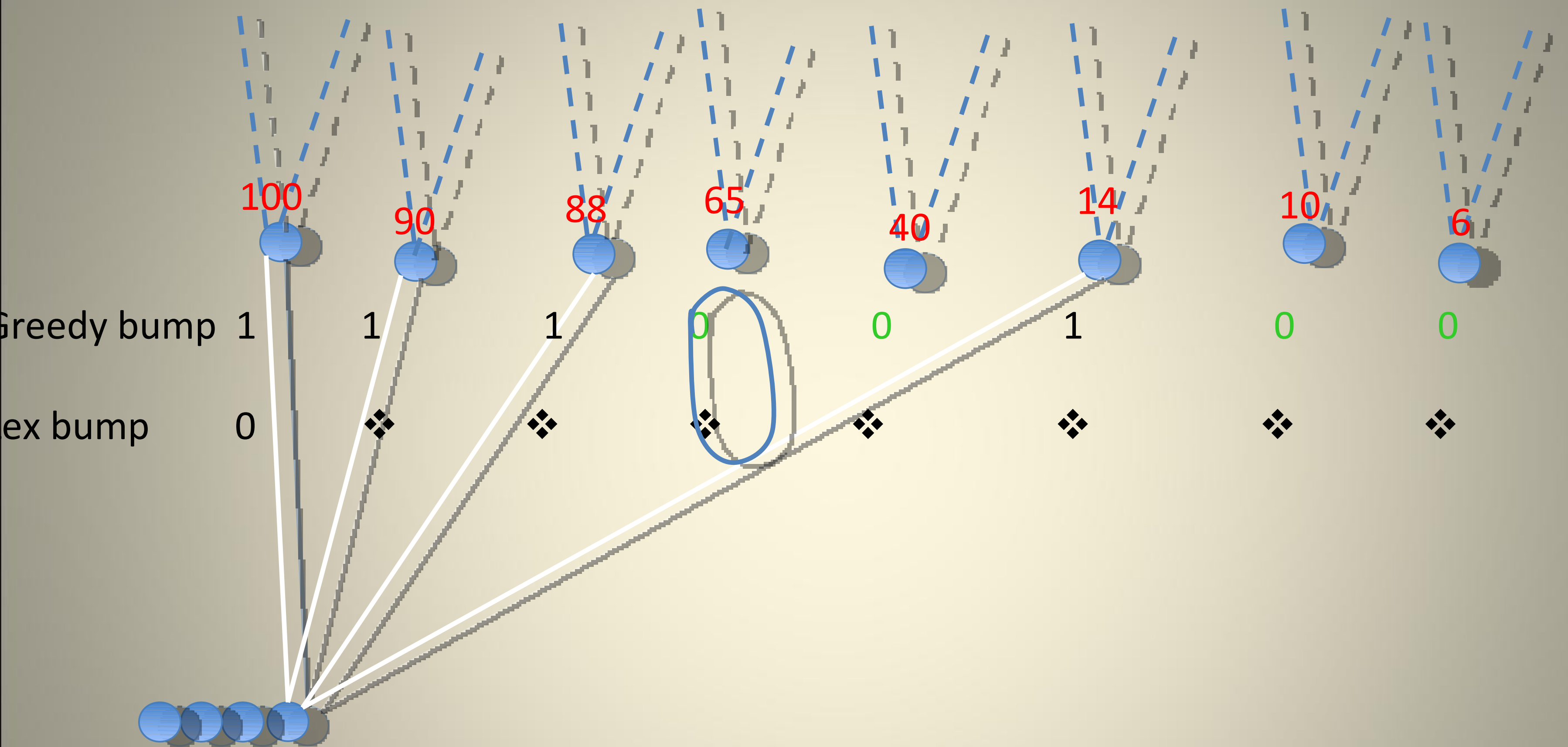
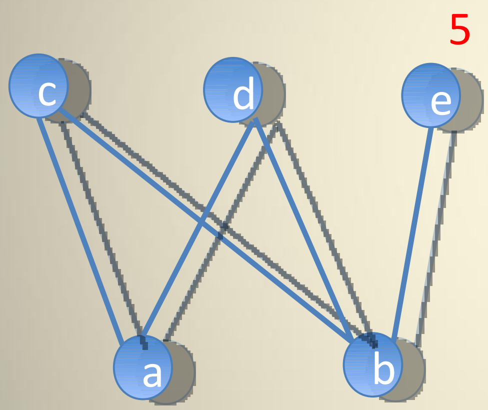


LexYanking Lemma ➔ the Greedlex Algorithm will always yield min-bump l.e.

Proof of LexYanking Lemma

b x x x a x x x   has k bumps, lex(a) ≥ lex(b), a is minimal

⇒ a x’x’x’ b x’x’x’   has ≤ k bumps



If lex(a)≥lex(b) and b has a private neighbour…

Proof of LexYanking Lemma

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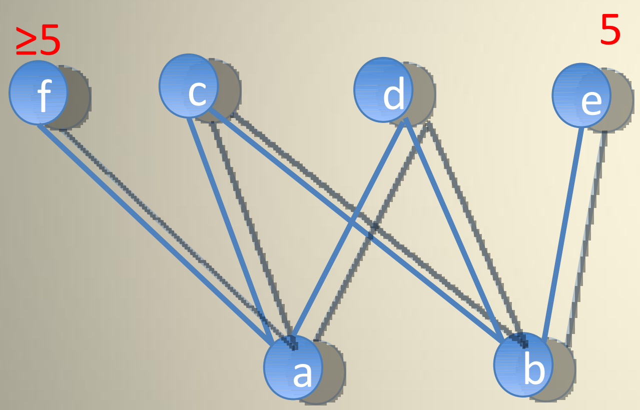


If lex(a)≥lex(b) and b has a private cover (not covering a)…

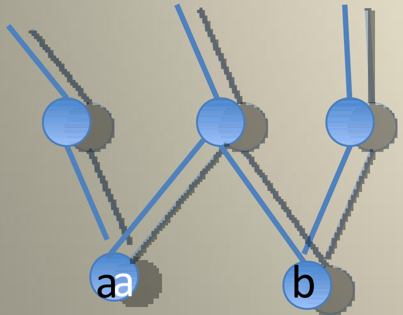
Then a has a private cover with lex# at least as large.

Proof of LexYanking Lemma

By induction on $n=|V(P)|$. Base cases $n=0,1$ are trivial.

Let P be a poset on $n>1$ elements, and suppose LexYanking Lemma holds for all smaller posets. (Then also Greedlex works on smaller posets.)

b xxx a xxxx   a l.e. with k bumps, lex(a)≥lex(b), a and b min



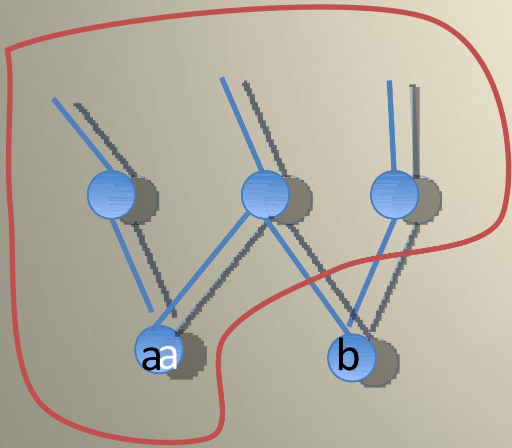
Proof of LexYanking Lemma

b xxx a xxxx   a l.e. with k bumps, lex(a)≥lex(b), a and b min

The poset \ {b} is smaller, so by Ind. Hyp., LexYanking holds, and Greedlex produces a min-bump suffix to follow b

b yyy a yyyy

All these elements have lex# > lex(a)



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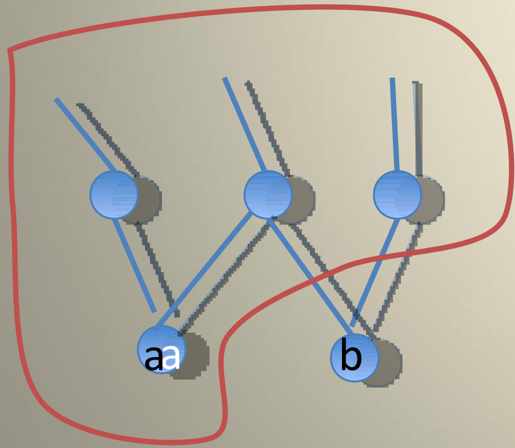

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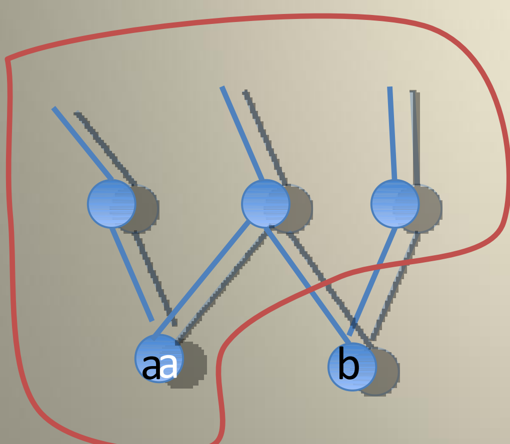

All these elements have lex# > lex(a) ≥ lex(b)
Hence all are incomparable with b
They are also incomparable with a

Swap: a yyy b yyyy

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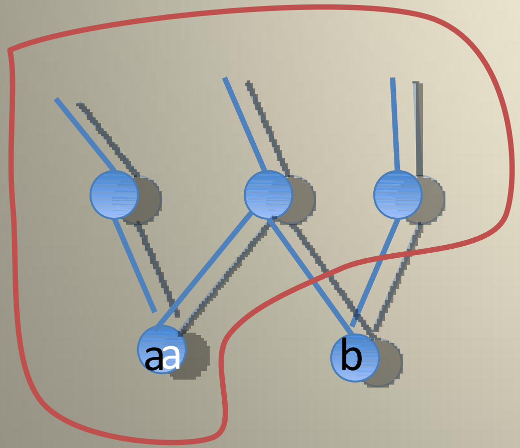

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Swap: a yyy b yyyy
May have introduced a bump

Proof of LexYanking Lemma

a yyy b yyyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Proof of LexYanking Lemma

a yyy b y1yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y1 is a private cover of b (with respect to a).

Proof of LexYanking Lemma

a yyy b y1yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y1 is a private cover of b (with respect to a).

Then a has some private cover c (w.r.t. b), with lex(c)≥ lex(y_1).

a yyy b y1y…c..y

Proof of LexYanking Lemma

a yyy b y1yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y1 is a private neighbour of b (with respect to a).

Then a has some private neighbour c (w.r.t. b), with lex(c)≥ lex(y_1).

a yyy b y1y…c..y

Then c can be yanked forward in the suffix, by the Ind. Hyp., without increasing bumps

a yyy b c z…y1..z

Proof of LexYanking Lemma

a yyy b y1yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y1 is a private neighbour of b (with respect to a).

Then a has some private neighbour c (w.r.t. b), with lex(c)≥ lex(y_1).

a yyy b y1y…c..y

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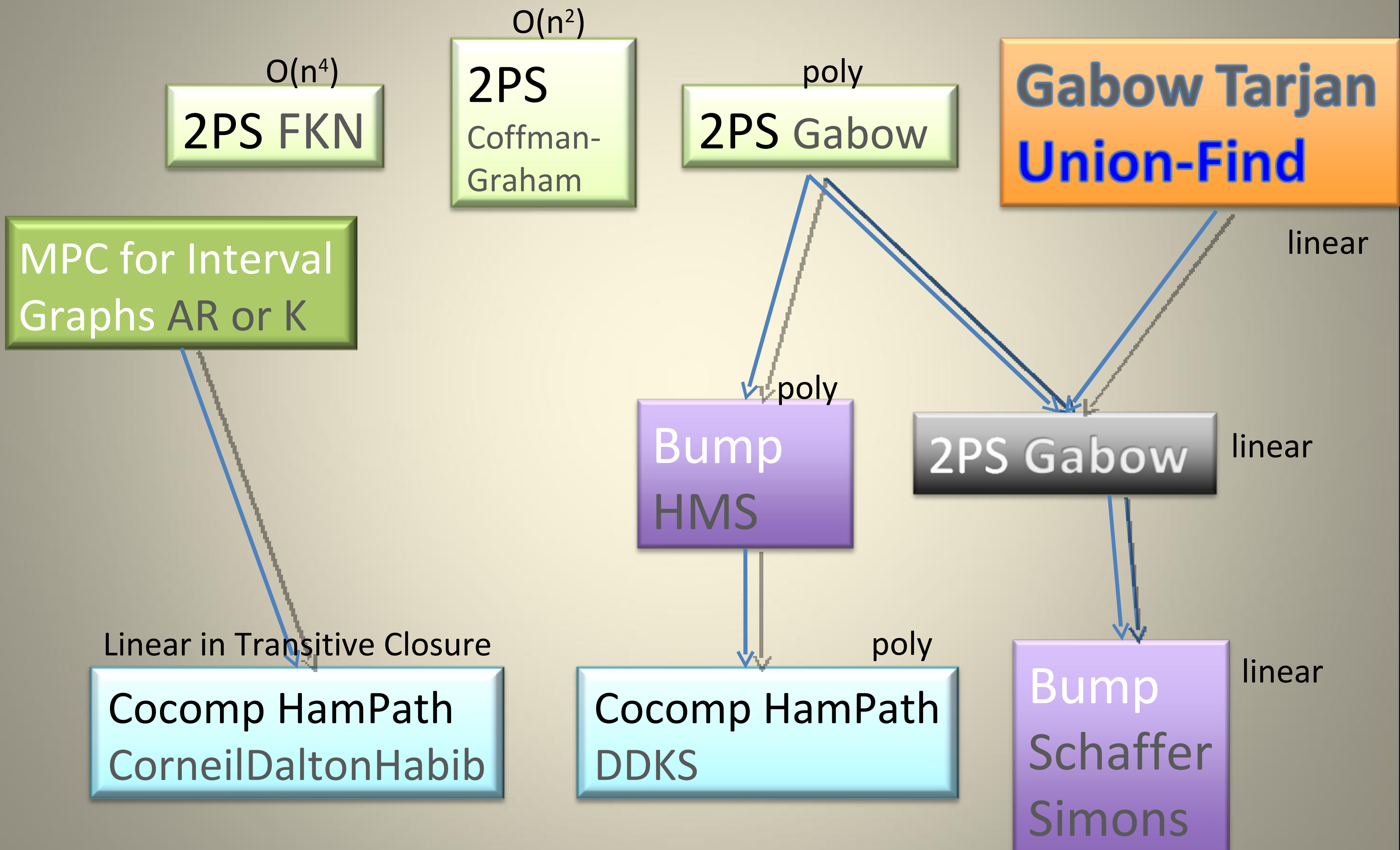
Then c can be yanked forward in the suffix, by the Ind. Hyp., without increasing bumps and destroying the bump after b.
[if c is not a min, take c's descendent].

Where does this work fit in?



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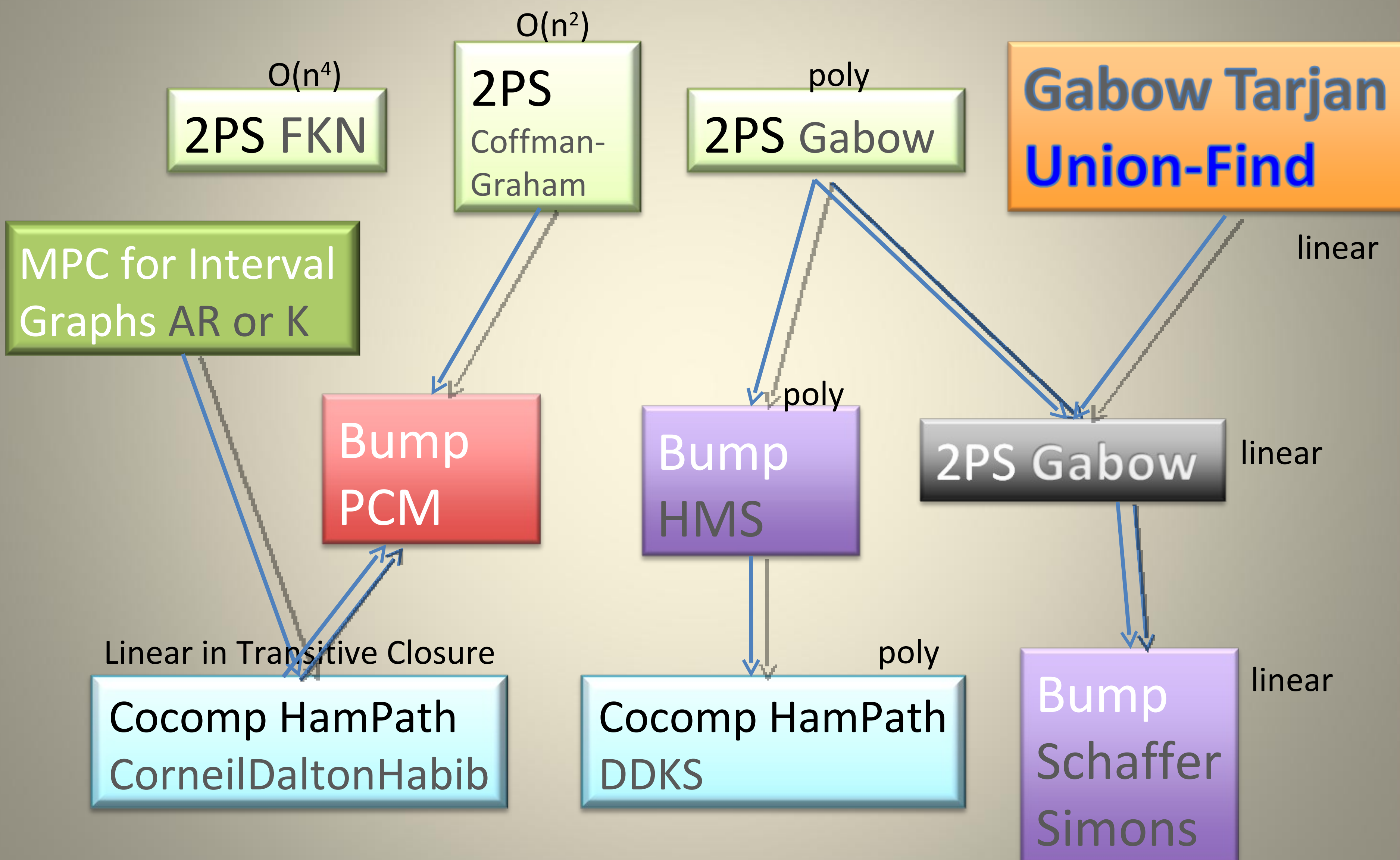

Further Work

Completed:

- Solve 2-Proc Sched using Greedlex
- Greedlex can work on either transitive closure or transitive reduction
- Greedlex can generate all min-bump linear extensions (all MinPath Covers in Cocomp graphs)

Open:

- Terminal elements in the poset.... (see Garth Isaak's work on Path Partitions)
- What about representations that are in between transitive closure and reduction?
- What about AT-free graphs?
  - Contains the cocomp graphs

Thank You!

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Coauthors:

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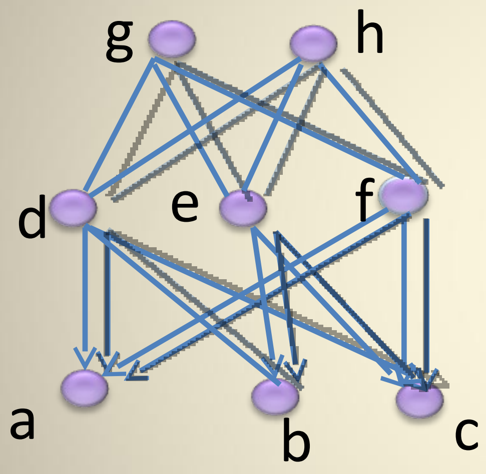
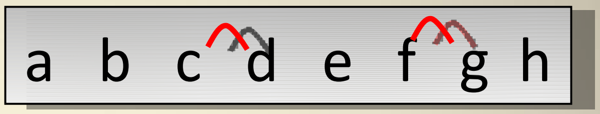
Lalla Mouatadid

University of Toronto

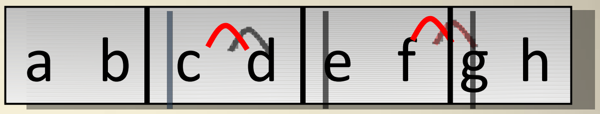
2-Processor Schedules

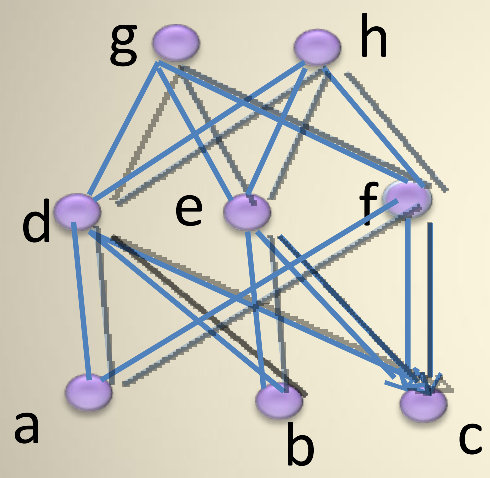




Want to schedule these unit-length jobs on two identical processor so that no job is executed before all of its lower covers have completed execution.

2-Processor Schedules





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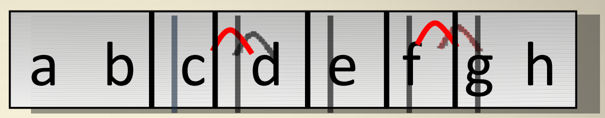

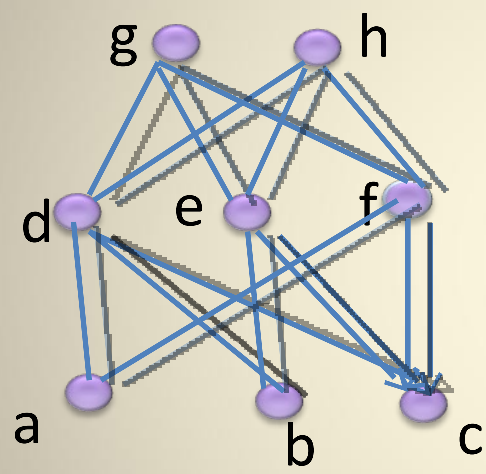

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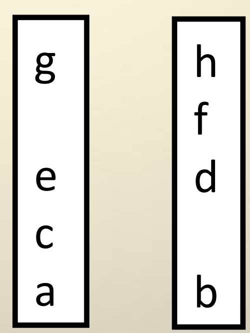
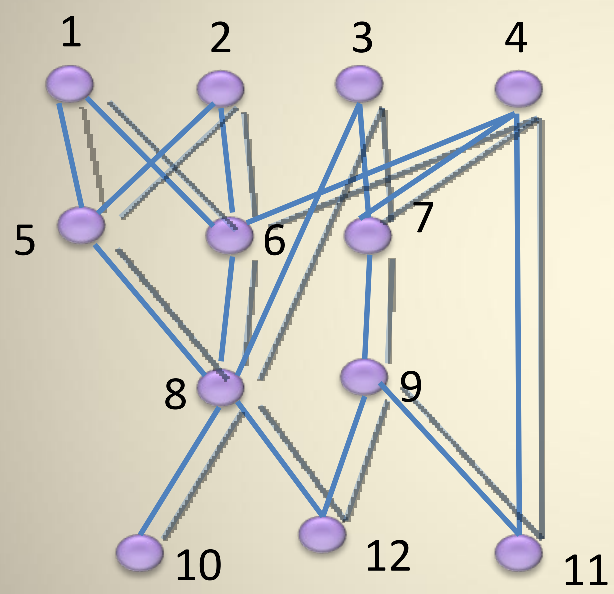


Coffman-Graham Lexicographic Labelling



- Give t minima ~~arbitrary~~ lex#'s 1…t arbitrarily
- Assign lex#s t+1…n so that lex(u)<lex(v) whenever {lex(u'): u'covers u} $<_{lexico}$ {lex(v'): v' covers v}, breaking ties arbitrarily

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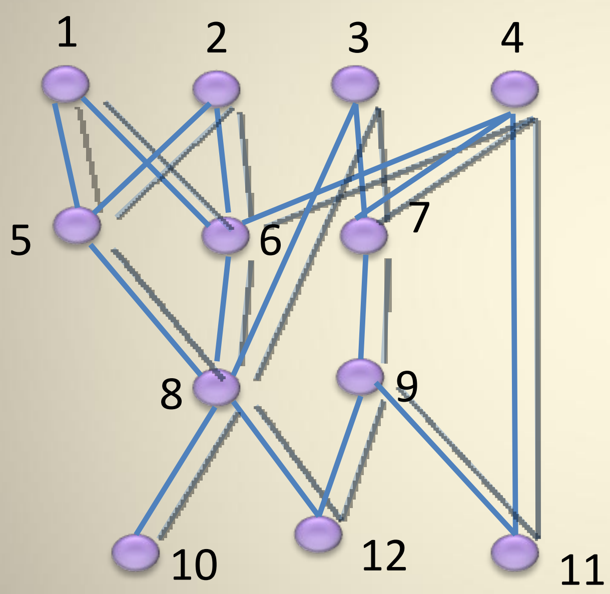

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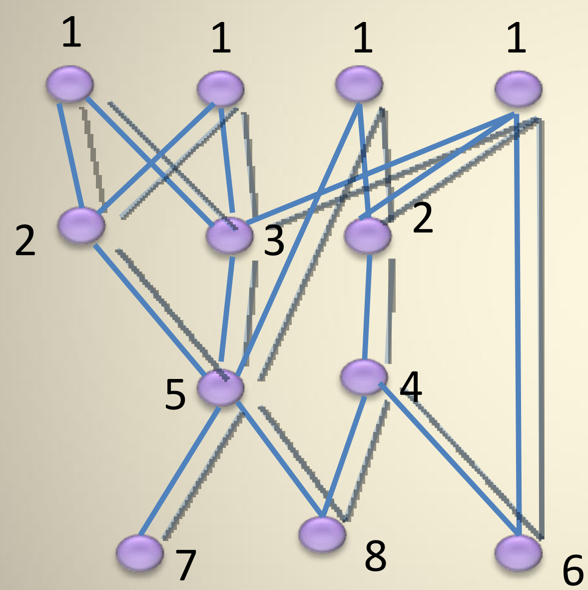
(Sethi, 1986) O(n+m) algorithm for C-G lex labelling

Lexicographic Labelling and 2PS



- Coffman and Graham '72 used it for 2-proc scheduling $O(n^2)$
- Sethi '76 also used it for a 2PS; lex labelling takes $O(n + m)$ though the remainder of the 2PS alg takes $O(n\ \alpha(n) + m))$