CES production function and Goodwin model

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1 New Model

There are two modifications in this extension of basic Goodwin model:
(1)In the original Goodwin model, there is a constant capital-output ratio. This is relaxed in this extension by allowing firms maximize profit subject to a C.E.S production function.
(2)Consider the exogenous growth of the labor productivity, that is the efficiency of labor is also influenced by the size of the capital stock according to Kaldorian technical progress function.
The C.E.S (Constant elasticity of substitution) production function is a type of production function that displays constant elasticity of substitution. Given the from:

$$Y = A[\mu K^{-\eta} + (1-\mu)L_{ef}^{-\eta}]^{-\frac{1}{\eta}} \tag{1}$$

Where:
$A > 0$, Factor productivity
$0 < \mu < 1$, Share parameter
$\eta > 0$
$0 < s = \frac{1}{1+\eta} < 1$, Elasticity of substitution
$L_{ef} = Le^{\alpha t}K^{\gamma}$, the effective empoyed labour force
As for the Leontief prodction function,is a particular case of the above function:

$$\lim_{\eta \to \infty} A[\mu K^{-\eta} + (1-\mu)L_{ef}^{-\eta}]^{-\frac{1}{\eta}} = \min(AK, AL_{ef}) \tag{2}$$

As for the Cobb-Douglas:

$$\lim_{\eta \to 0} A[\mu K^{-\eta} + (1-\mu)L_{ef}^{-\eta}]^{-\frac{1}{\eta}} = AK^{\mu}L_{ef}^{1-\mu} \tag{3}$$

By the relationship that frims hire labour until the productivity of the marginal worker equals the worker's real wage, we get:

$$\frac{\partial Y}{\partial L} = \frac{\text{W}}{L} \tag{4}$$

$$\Rightarrow \frac{\partial Y}{\partial L_{ef}} = \frac{\text{W}}{L} e^{-\alpha t} K^{-\gamma} = \frac{\text{W}}{L_{ef}} \tag{5}$$

and since:

$$\frac{\partial Y}{\partial L_{ef}} = \frac{1-\mu}{A^{\eta}} (\frac{Y}{L_{ef}})^{\eta+1} \tag{6}$$

Thus the wage share of the national income is:

$$\begin{aligned} \omega &= \frac{\text{W}}{Y} \\ &= \frac{1-\mu}{A^{\eta}} (\frac{Y}{L_{ef}})^{\eta} \\ &= (1-\mu)[\mu K^{-\eta} + (1-\mu) L_{ef}^{-\eta}]^{-1} L_{ef}^{-\eta} \\ &= \frac{1-\mu}{\mu(\frac{K}{L_{ef}})^{-\eta} + (1-\mu)} \end{aligned} \tag{7}$$

Therefore we have the optimal factor demand ratio:

$$\frac{K}{L_{ef}} = (\frac{(1-\mu)(1-\omega)}{\mu\omega})^{-\frac{1}{\eta}} := g_1(\omega) \tag{8}$$

The capital-output ratio will be:

$$\begin{aligned} \nu &= \frac{K}{Y} \\ &= \frac{g_1(\omega) L_{ef}}{A[\mu K^{-\eta} + (1-\mu) L_{ef}^{-\eta}]^{-\frac{1}{\eta}}} \\ &= \frac{g_1(\omega)}{A(\frac{1-\mu}{\omega})^{-\frac{1}{\eta}}} \\ &= \frac{1}{A}(\frac{1-\omega}{\mu})^{-\frac{1}{\eta}} \end{aligned} \tag{9}$$

And

$$\frac{\dot{\nu}}{\nu} = \frac{\omega}{\eta(1-\omega)} \frac{\dot{\omega}}{\omega} \tag{10}$$

Labour's productivity will be given by

$$\begin{aligned} a &= \frac{Y}{L} \\ &= \frac{Y}{L_{ef}} e^{\alpha t} K^{\gamma} \\ &= A \frac{[\mu K^{-\eta} + (1-\mu) L_{ef}^{-\eta}]^{-\frac{1}{\eta}}}{L_{ef}} e^{\alpha t} K^{\gamma} \\ &= A (\frac{\omega}{1-\mu})^{\frac{1}{\eta}} e^{\alpha t} K^{\gamma} \end{aligned} \tag{11}$$

And

$$\frac{\dot{a}}{a} = \frac{1}{\eta} \frac{\dot{\omega}}{\omega} + \alpha + \gamma \frac{\dot{K}}{K} \tag{12}$$

After that, we are fine to build the differential equations. The Say's law is still hold in this extension, i.e. $\dot{K} = (1-\omega)Y - \delta K$. And we keep the assumption that using the Phillips curve to explain the behaviour of wages: $\dot{\mathrm{w}} = \Phi(\lambda)\mathrm{w}$.The labour force grows at the steady state: $N = N_0 e^{\beta t}$.
As

$$\omega = \frac{\mathrm{W}}{Y} = \frac{\mathrm{w}L}{aL} = \frac{\mathrm{w}}{a} \tag{13}$$

$$\lambda = \frac{L}{N} = \frac{Y}{aN} = \frac{K}{\nu a N} \tag{14}$$

We obtain:

$$\begin{aligned} \frac{\dot{\omega}}{\omega} &= \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{a}}{a} \\ &= \Phi(\lambda) - (\frac{1}{\eta} \frac{\dot{\omega}}{\omega} + \alpha + \gamma \frac{\dot{K}}{K}) \\ &= \Phi(\lambda) - [\frac{1}{\eta} \frac{\dot{\omega}}{\omega} + \alpha + \gamma (\frac{1-\omega}{\nu} - \delta)] \\ &= \Phi(\lambda) - [\frac{1}{\eta} \frac{\dot{\omega}}{\omega} + \alpha - \gamma\delta + (1-\omega)\gamma A (\frac{1-\omega}{\mu})^{\frac{1}{\eta}}] \end{aligned} \tag{15}$$

$$\Rightarrow \frac{\dot{\omega}}{\omega} = \frac{\eta}{1+\eta} [\Phi(\lambda) - \alpha + \gamma\delta - (1-\omega)\gamma A (\frac{1-\omega}{\mu})^{\frac{1}{\eta}}] \tag{16}$$

For the employment:

$$\begin{aligned}
\frac{\dot{\lambda}}{\lambda} &= \frac{\dot{K}}{K} - \frac{\dot{\nu}}{\nu} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} \\
&= \frac{(1-\omega)Y - \delta K}{K} - \frac{\omega}{\eta(1-\omega)}\frac{\dot{\omega}}{\omega} - (\frac{1}{\eta}\frac{\dot{\omega}}{\omega} + \alpha + \gamma\frac{\dot{K}}{K}) - \beta \\
&= (1-\gamma)(1-\omega)\frac{Y}{K} - \frac{1}{\eta(1-\omega)}\frac{\dot{\omega}}{\omega} - (\alpha + \beta + (1-\gamma)\delta) \\
&= (1-\gamma)(1-\omega)A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}} - \frac{1}{\eta(1-\omega)}\frac{\dot{\omega}}{\omega} - (\alpha + \beta + (1-\gamma)\delta) \\
&= [(1-\gamma)(1-\omega) + \frac{\gamma}{1+\eta}]A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}} - \frac{\Phi(\lambda) - \alpha + \gamma\delta}{(1-\omega)(1+\eta)} - \alpha - \beta - (1-\gamma)\delta
\end{aligned} \tag{17}$$

Putting all these together we have

$$\begin{cases} \frac{\dot{\omega}}{\omega} = \frac{\eta}{1+\eta}[\Phi(\lambda) - \alpha + \gamma\delta - (1-\omega)\gamma A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}}] \\ \frac{\dot{\lambda}}{\lambda} = [(1-\gamma)(1-\omega) + \frac{\gamma}{1+\eta}]A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}} - \frac{\Phi(\lambda)-\alpha+\gamma\delta}{(1-\omega)(1+\eta)} - \alpha - \beta - (1-\gamma)\delta \end{cases} \tag{18}$$

which are the new differential equations after the two modifications above.

2 Simulation

After we build the new 2-d Goodwin model, we want to simulate it to see whether it works or not. First, we let $\alpha = 0.025, \beta = 0.02, \delta = 0.01, A = 0.25, \mu = 0.5$

Then, specifically, we do not admit endogenous productivity growth, thus $\gamma = 0$. And We assume the Phillips curve to be a nonlinear function, i.e. $\Phi(\lambda) = -\phi_0 + \frac{\phi_1}{(1-\lambda)^2}$, where $\phi_0 = 0.040064, \phi_1 = 0.000064$. We chooes $\eta = 500$.Then we obtain the figure 1 and 2 which show a stable equilibrium case.

After that, we simulate the model by giving different value to γ. Figure 3 and 4 are obtained when $\gamma = 0.5$ while figure 5 and 6 are obtained when $\gamma = -0.5$. We can find that when $\gamma = 0.5$, there exits a limit cycle which implies that endogenous labor productivity could destabilize the original Goodwin model. And in the case when $\gamma = -0.5$, not surprisingly, the intial point tends to equilibrium point much faster.

In conclusion, the consideration of a more flexible production function is sufficient to stabilize Goodwin's model, while the endogenous labour productivity would have a destabilizing effect. The stabilizing effect of introducing some flexibility in production function is much stronger than the destabilizing effect of endogenous productivity growth. And only when the production function is extremely close to Leontief does the system generate perpetual oscillations.

3 Keen model

We now extend the model based on the previous modifications by introducing a banking sector to finance new investment. Defining D the amount of debt in real term, and the debt ratio in the economy $d = D/Y$, and r is a constant real interest rate. Thus we have the net profit Π given by the form $\Pi = Y - W - rD = Y(1-\omega-rd)$ and $\pi := \Pi/Y = 1-\omega-rd$.
Also we have the assumptions as following:

$$I = \kappa(\pi)Y \tag{19}$$

$$\dot{K} = I - \delta K = \kappa(\pi)Y - \delta K \tag{20}$$

$$\dot{D} = (\kappa(\pi) - \pi)Y \tag{21}$$

Combining them, we get the following three-dimensional system of differential equation:

$$\frac{\dot{\omega}}{\omega} = \frac{\eta}{1+\eta}(\Phi(\lambda) - \alpha - \gamma A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}}\kappa(\pi) + \gamma\delta) \tag{22}$$

$$\frac{\dot{\lambda}}{\lambda} = (1-\gamma-\frac{\gamma}{(1-\omega)(1+\eta)})(\frac{\kappa(\pi)}{\nu} - \delta) - \frac{1}{(1-\omega)(1+\eta)}(\Phi(\lambda)-\alpha) - \alpha - \beta \tag{23}$$

$$\frac{\dot{d}}{d} = \frac{\kappa(\pi)-\pi}{d} - A(\frac{1-\omega}{\mu})^{\frac{1}{\eta}}\kappa(\pi) + \delta + \frac{\omega}{\eta(1-\omega)} \tag{24}$$

We assume $\kappa(\pi)$ takes the form of sigmoid curve thus $\kappa(\pi) = \frac{1}{1+e^{-20(\pi-0.256)}}$ and we can see its shape from figure 7.
We then begin to simulate the new model by taking $\delta = 0.06, r = 0.03, \nu = 3$ and get the results shown in figure 8 and 9.

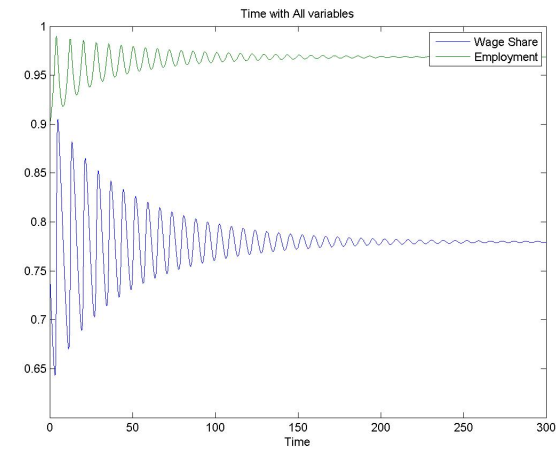

Figure 1: Time with Employment and Wage Share, $\gamma = 0$

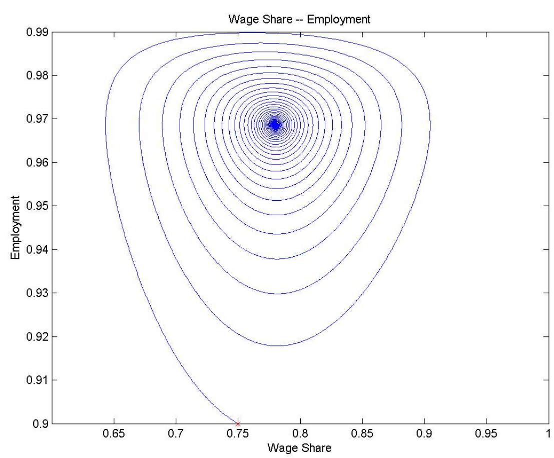

Figure 2: Employment – Wage Share, $\gamma = 0$

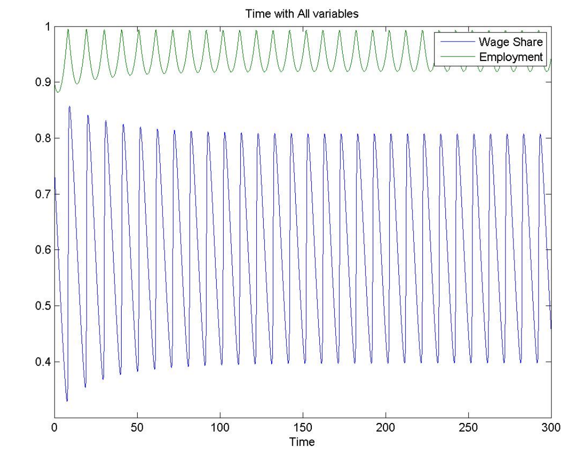

Figure 3: Time with Employment and Wage Share, $\gamma = 0.5$

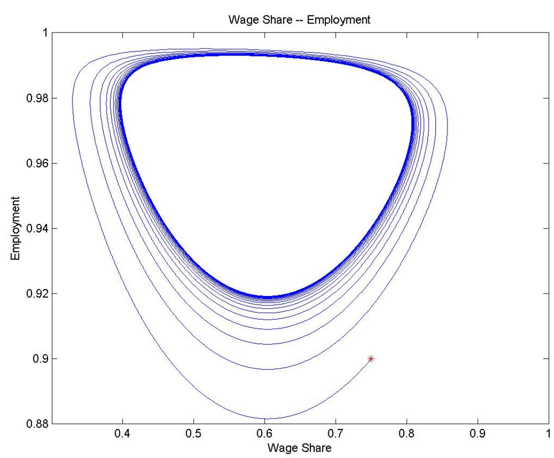

Figure 4: Employment – Wage Share, $\gamma = 0.5$

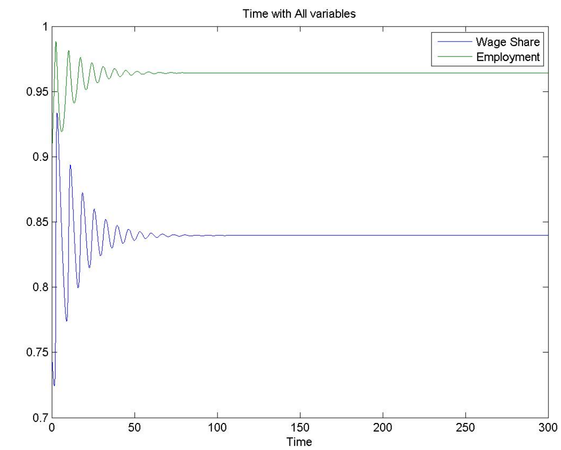

Figure 5: Time with Employment and Wage Share, $\gamma = -0.5$

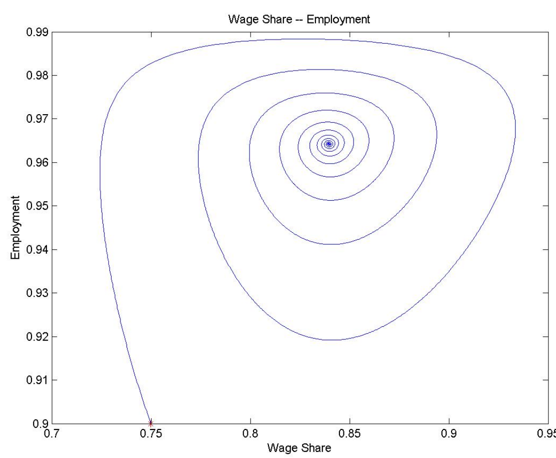

Figure 6: Employment – Wage Share, $\gamma = -0.5$

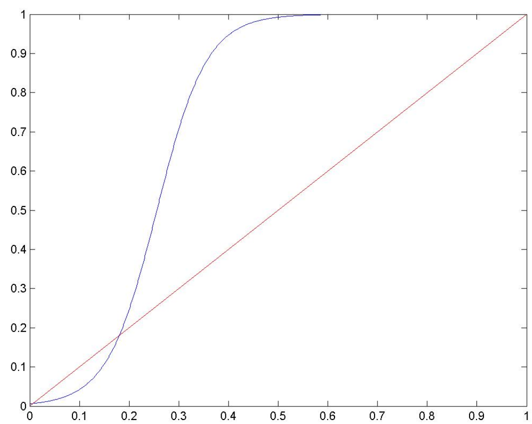

Figure 7: The investment function

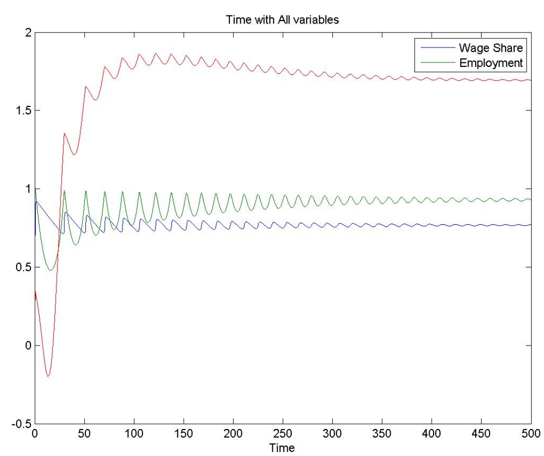

Figure 8: Time with Employment and Wage Share, $\omega = 0.7, \lambda = 0.9, d = 0.1$

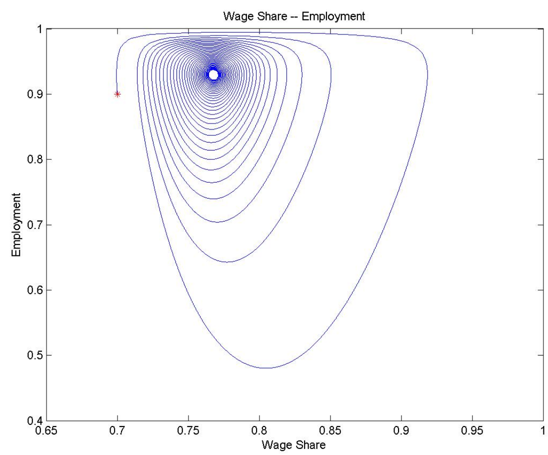

Figure 9: Employment – Wage Share, $\omega = 0.7, \lambda = 0.9, d = 0.1$