Fields Undergraduate Summer Research Program: Project on Logic and Operator Algebras

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The following is a report on a project concerning mathematical logic and operator algebras that took place as a part of the Fields Institute Undergraduate Summer Research Program in July and August 2013. A group of nine students participated in the project under the supervision of Dr. Bradd Hart and Dr. Ilijas Farah. The group worked on two seperate problems throughout the summer. The material covered in this report is introductory material as well as a description of the problems that were worked on. We would like to thank the Fields Institute and Mitacs and our supervisors.

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1 C*-algebras

Basics of C*-algebras

Definition: C*-algebra
A C-algebra A is an algebra over* $\mathbb{C}$ *with a norm* $a \to \|a\|$ *and an involution* $a \to a^*$ *such that A is complete with respect to the norm, and such that* $\|ab\| \leq \|a\|\|b\|$ *and* $\|a^*a\| = \|a\|^2$ *for every* $a, b \in A$.

A unital C*-algebra is a C*-algebra with a multiplicative identity.

Example
$M_n(\mathbb{C})$ with usual operations forms a unital C*-algebra.

Finite Dimensional Algebras

Finite dimensional algebras are in some sense a generalization of matrix algebras and will play a key role in what will follow.

Definition: Finite Dimensional C*-algebra
A C-algebra is finite dimensional if it is finite dimensional when considered as a complex vector space.*

Definition: Direct Sum
The direct sum of two C-algebras A, B is given by pairs (a,b) with pointwise operations and norm defined by* $\|(a,b)\| = max\{\|a\|,\|b\|\}$.

A key result about finite dimensional algebras is the following:

Every finite dimensional C*-algebra is isomorphic to a direct sum of full matrix algebras.

Example
$M_2(\mathbb{C}) \bigoplus M_3(\mathbb{C}) \bigoplus M_5(\mathbb{C})$

Homomorphisms

Definition: *-homomorphism
*A *-homomorphism* $\phi : A \to B$ *between two C*-algebras is a homormorphism that satisfies* $\phi(a^*) = \phi(a)^*$. *A *-homomorphism between two unital C*-algebras is called unital if it preserves the unit of the algebra.*

Definition: Conjugate *-homomorphisms
Two homomorphisms Φ, Ψ *from A to B are conjugate if* $\Phi = u\Psi u^*$ *for some unitary* $u \in B$. *(A unitary element of an algebra satisfies the property* $uu^* = u^*u = 1$*)*

We will now completely describe *-homomophisms between two matrix algebras and hence obtain the form of all *-homomorphisms between finite dimensional algebras. We start with the following lemma:

There is a unital *-homomorphism from $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$ if and only if n divides k. All unital *-homomorphisms from $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$ are conjugate.

Up to conjugacy, these homomorphisms are described by the following map $\Phi : A \to B$

$$\Phi(a) = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix}$$

All non-unital *-homomorphisms between matrix algebras are determined up to conjugacy by how many copies of a are put along the diagonal. Using this information we can construct all possible *-homomorphisms between finite dimensional algebras up to conjugacy. That is, any *-homomorphism between finite dimensional algebras is conjugate to a *-homomorphism which takes tuples of matrices and sends them to a tuple of matrices with copies of the original matrices along their diagonals. We will now present an example of a *-homomorphism between two finite dimensional algebras to make this idea more clear.

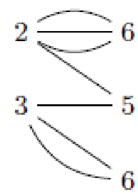
Example
A unital *-homomorphism from $M_2(\mathbb{C}) \bigoplus M_3(\mathbb{C})$ to $M_6(\mathbb{C}) \bigoplus M_5(\mathbb{C}) \bigoplus M_6(\mathbb{C})$ is described by the following map

$$\Phi(\text{(a,b)}) = \left(\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \right)$$

Bratteli Diagrams Bratteli Diagrams are a useful tool for pictorially describing *-homomorphisms between finite dimensional algebras. They will be used in the analysis of the work presented later in the paper. For a given *-homomorphism between finite dimensional algebras the diagram is constructed as follows:

- 1) The domain can be written as a direct sum of matrix algebras. The numbers corresponding to the dimensions of the matrix summands are placed on the left hand side of the diagram.
- 2) The range can be written as a direct sum of matrix algebras. The numbers corresponding to the dimensions of the matrix summands are placed on the right hand side of the diagram.
- 3) Lines are drawn from the numbers on the left to the numbers on the right. The number of lines from one number to another corresponds to the number of times a matrix in the domain is sent into a matrix in the range by putting copies of the original matrix along the diagonal.

The Bratteli Diagram that corresponds to the above example is the following:



Traces

A continuous linear functional $\phi : A \to \mathbb{C}$ is positive if $\phi(a) \geq 0$ for every positive $a \in A$. (An element of A is positive if it is of the form $a = b^*b$ for some $b \in A$)

Definition: Trace
A trace is a continuous linear positive functional $\tau : A \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(ab) = \tau(ba)$.

The usual trace of a matrix from linear algebra is a trace on a matrix algebra if it is normalized by dividing by the dimension of the matrix. This is called the normalized trace. It can be shown that any trace on a matrix algebra is actually the normalized trace and hence matrix algebras have unique trace.

Using the above fact we have the following lemma concerning traces on finite dimensional algebras:

Suppose A is a finite dimensional algebra of the form:

$$M_{n_1}(\mathbb{C}) \bigoplus M_{n_2}(\mathbb{C}) \bigoplus \cdots \bigoplus M_{n_k}(\mathbb{C})$$

Traces on A are all of the form:

$$\text{tr(a)} = \alpha_1\tau_{n_1} + \alpha_2\tau_{n_2} + \cdots + \alpha_k\tau_{n_k}$$

where τ_{n_i} is the unique trace on $M_{n_i}(\mathbb{C})$, $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$

Inductive Limits

Let $\{(A_i, \phi_{ij}) : i \leq j\}$ be an inductive system of C*-algebras.

Definiton: Inductive Limit
The algebraic inductive limit or direct limit of the above inductive system is denoted by $\lim_{i\to\infty}(A_i, \phi_{ij})$ or $\lim_{i\to\infty} A_i$, and is defined as $\bigcup_{i\in I} A_i \big/ \sim$, where $x_i \sim \phi_{ij}(x_i)$ for all $j \geq i$ for $x_i \in A_i$.

There is a canonical seminorm on algebraic direct limits of C*-algebras given as follows:
$\|a\| \equiv \lim_{j>i} \|\phi_{ij}(a)\| = \inf_{j>i} \|\phi_{ij}(a)\|$.

The completion of $\lim_{i\to\infty} A_i$ with elements of seminorm 0 divided out is a C*-algebra.

UHF Algebras

Definition: UHF Algebras
Uniformly hyperfinite (UHF) algebras are infinite tensor products of full matrix algebras. For seperable C-algebras this is equivalent to being a direct limit of full matrix algebras.*

For example, the *canonical anticommutation relations algebra* (**CAR algebra**), denoted $\mathbb{M}_{2^\infty}$, may be defined as the direct limit of the direct system illustrated below, where the unital embeddings are of the form $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$:

$$\mathbb{M}_2(\mathbb{C}) \hookrightarrow \mathbb{M}_4(\mathbb{C}) \hookrightarrow \mathbb{M}_8(\mathbb{C}) \hookrightarrow \ldots$$

2^∞ is the **generalized integer** associated to the CAR algebra.

Similarly, 3^∞ is the generalized integer (or *supernatural number*) associated to the UHF algebra $\mathbb{M}_{3^\infty}$.

Observe that $\mathbb{M}_{2^\infty} \ncong \mathbb{M}_{3^\infty}$.

Two separable UHF algebras are isomorphic iff they have the same generalized integer.

Consequently, there are uncountable infinite isomorphism classes of UHF algebras.

AF Algebras

Definition: AF Algebra
A C-algebra is* approximately finite *(AF) iff it is an inductive limit of finite-dimensional C*-algebras.*

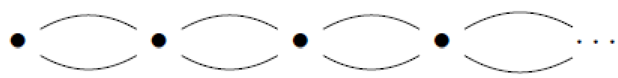
For example, UHF algebras are AF algebras.

Bratteli diagrams of finite-dimensional C*-algebras naturally generalize to *unital* AF algebras. Simply concatenate the Bratteli digrams corresponding to the unital embeddings $A_n \hookrightarrow A_{n+1}$ for a direct limit of the form $\lim_{n\to\infty} A_n$.

Again consider the unital embedding $\mathbb{M}_2 \hookrightarrow \mathbb{M}_4$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

The **multiplicity** of this embedding is 2, and thus corresponds to the following Bratteli diagram:

So the Bratteli diagram for the CAR algebra, given by the unital inductive system of the form $\mathbb{M}_2 \hookrightarrow \mathbb{M}_4 \hookrightarrow \mathbb{M}_8 \hookrightarrow \ldots$, is of the following form:



2 Project 1

2.1 K-Theory

In this section, we only consider unital AF C*-algebras.

Murray-von Neumann equivalence of projections.
We define projections and their equivalence relation.
Definition. $p \in A$ is a *projection* if $p^2 = p^* = p$.
Definition. Two projections $p, q \in A$ are *Murray-von Neumann equivalent* if there exists $v \in A$ such that $p = v^*v, q = vv^*$.
Example. Consider projections in $M_2(\mathbb{C})$ such as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \ldots$$

One can observe that any projection is equivalent to exactly one of the following:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In general, two projections are Murray-von Neumann equivalent if and only if their ranges have the same dimension, which is their rank in matrix algebra case.

K_0 groups of unital C*-algebras.
Denote $\lim_{n\to\infty} M_n(A)$ by $M_\infty(A)$, where the connecting maps are given naturally:

$$a \in M_n(A) \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(A).$$

Let $\mathcal{P}(M_\infty(A))$ be the set of all projections in $M_\infty(A)$, and define $V(A) = \mathcal{P}(M_\infty(A))/\sim$.

Next, we define addition on $V(A)$. For $[p], [q] \in V(A)$, define

$$[p] \oplus [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

Since

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}^* \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}^* = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix},$$

$V(A)$ is an abelian semigroup.

Definition. $K_0(A)$ is the Grothendieck group of $(V(A), \oplus)$.
Grothendieck group construction allows extending the addition on $V(A)$ to an abelian group operation. Indeed, roughly, $K_0(A) = V(A) - V(A)$.
In fact, $K_0(A)$ is an ordered group $(K_0(A), K_0(A)^+, [1_A])$ with $K_0(A)^+$ being the image of $V(A)$ and $[1_A]$ the order unit.

Example. Here are some K_0 groups in simple cases, given explicitly.

(1)$K_0(\mathbb{C}) \cong (\mathbb{Z}, \mathbb{Z}_{\geq 0}, 1)$ and $K_0(M_n(\mathbb{C})) \cong (\mathbb{Z}, \mathbb{Z}_{\geq 0}, n)$. In $K_0(\mathbb{C})$, a projection with rank n corresponds to the nonnegative integer n. Hence $V(\mathbb{C}) \cong \mathbb{Z}_{\geq 0}$, and by the description above, $K_0(\mathbb{C}) \cong \mathbb{Z}_{\geq 0} - \mathbb{Z}_{\geq 0} = \mathbb{Z}$. Since the multiplicative unit I_n in $\boldsymbol{(}M_n(\mathbb{C})$ has rank n, the order unit of $K_0(M_n(\mathbb{C}))$ is n.

(2)K_0 can be viewed as a functor between C*-algebras and abelian groups. One important property is that $K_0(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} K_0(A_n)$, which is useful when computing K_0 groups of UHF algebras.

(3)$K_0(M_{2^\infty}) \cong \{\frac{m}{2^k} \mid m \in \mathbb{Z}, k \in \mathbb{N}\}$ and $K_0(M_{3^\infty}) \cong \{\frac{m}{3^k} \mid m \in \mathbb{Z}, k \in \mathbb{N}\}$. In general, for a UHF algebra $\mathcal{A}$ with generalized integer k,
$K_0(\mathcal{A}) \cong \{\frac{m}{l} \mid m \in \mathbb{Z}, l \mid k\}$.

The following essential theorem demonstrates that K_0 can be a powerful invariant in specific cases.

Theorem (Elliott, 1976) *Two separable unital AF algebras are isomorphic iff their ordered K_0 groups are isomorphic.*

The proof to Elliott's theorem uses the so-called intertwining argument, which appears frequently throughout mathematics and logic.

2.2 Continuous Model Theory

The first notion to define is that of a metric structure:

Given a metric space M, we have the following types of functions:

1) A **predicate** $P : M^n \to \mathbb{R}$ is a uniformly continous function, where the natural number n is called the arity of P.
2) A **function** $f : M^k \to M$ is a uniformly continuous function, where the natural number k is called the arity of f. A constant is a funtion with arity 0.

Definition: Signature

A signature L is a triple $(P_i, f_j, c_k), i \in Ij \in J, k \in K$ *where I, J and K are index sets and the* $P_i's$ *are predicates, the* $f_j's$ *functions and the* $c_k's$ *are constants. The predicates, formulas and constants are to be thought of formally as abstract symbols. These symbols are waiting to be interpreted (i.e. given a domain and range). A signature is also assigned a countably infinite set of variables.*

Definition: Metric Structure
A metric structure M is a triple(X_s*, L, I) where each* X_s *is a complete metric space and s in some index set S, L is a signature and I is an interpretation function. Each* X_s *is commonly referred to as a sort. An interpretation function takes a symbol from the signature L and sends it to a uniformly continuous function whose domain is some product of sorts specified in the metric structure.*

That is, $f \to f^M, P \to P^M, c \to c^M$. For example $I(p) = P^M : X_1 \dots X_n \to \mathbb{R}$. This is a uniformly continous function. Metric structures assign meaning (or interpret) the symbols within the signature in this way.

A C*-algebra is a metric structure for which the sorts are just balls. Note that in the signature L, for a C*-algebra A, there is a symbol for the norm, $\| - \| : A \to \mathbb{R}_{\geq 0}$.

Terms, Formulas and Sentences

Given a signature L, one can now define what terms, formulas and sentences are. For our purposes, however, these notions will only be defined in the context of C*-algebras.

Suppose that we have a C* algebra A = (X_s, L, I), which is nothing but a metric structure. We now use L to generate some more purely syntactical objects:

Definition: Term
*A term in the signature is just a *-polynomial in the variables* $x_1, \dots, x_n, n \in N$.

Defintion: Formula
Formulas are defined inductively as follows:
1) Atomic formulas are of the form $\|t\|$ *where t is a term.*
2) If $\Phi_1, \dots \Phi_n$ *are formulas and* $f : [0, \infty)^n \to [0, \infty)$ *is continous then* $f(\Phi_1, \dots \Phi_n)$ *is a formula.*
3) If Φ *is a formula and x is a variable then* $\inf_x \Phi$ *and* $\sup_x \Phi$ *are formulas.*

Example
$\Phi(x) = \|x - x^*\| + \|x - x^2\|$

All formulas get interpreted (in a metric structure) as uniformly continous functions. Given a C*-algebra A, the above example is interpreted in A as the uniformly continous funcion $\Phi^A(x) = \|x - x^*\| + \|x - x^2\|$. Formulas, under interpretation, are evaluated to real numbers in this way.

A variable inside a function is said to be free if it is not quantified by a sup/inf.

Definition: Sentence
A sentence is a formula with no free variables.
$\sigma(x) = \inf_x \|x - x^*\| + \|x - x^2\|$*is an example of a sentence because the only variable is quantified by the infimum.*

A sentence σ evaluates to a number which is denoted by σ^A.
Note that the zero set of Φ^A above is exactly the set of projections in the C*-algebra. Since 0 is a projection for any algebra A, $\sigma^A = 0$.

Definition: Theory
The theory of a C-algebra, denoted Th(A), is the set of all sentences which evaluate to zero. That is,* $Th(A) = \{\sigma : \sigma^A = 0\}$.

Two C*-algebras A and B are said to be **elementarily equivalent** (denoted $A \equiv B$) if Th(A) = Th(B).

Ehrenfeucht-Fraïssé (EF) Games

Given two C*-algebras, how do we determine if they are elementarily equivalent? EF games can provide an answer to this question. An EF game is a game between two players. Given two algebras A and B the game works in the following way:

- The following are given at the beginning of the game: length of game $n \in \mathbb{N}$, k formulas of the form $\|p_1(x_1, \dots, x_n)\|, \dots, \|p_k(x_1, \dots, x_n)\|$ where each p_i is a *-polynomial, and $\varepsilon > 0$.
- Player 1 picks an element from either A or B.
- Player 2 picks an element from the other algebra.
- After n rounds there are 2 sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$.
- Player 2 wins if $|\|p_i^A(a_1, \dots, a_n)\| - \|p_i^B(b_1, \dots, b_n)\|| \leq \varepsilon$ for all i.

The following theorem was of great importance to the first project:

Theorem
$A \equiv B \Leftrightarrow$ Player 2 has a winning strategy for all EF games.

2.3 The Project

We are now ready to discuss the first of two projects attempted this summer.

The Crazy Conjecture

Project one was an attempt to prove the following conjecture:
Conjecture. If A and B are separable unital AF-algebras with $K_0(A) \equiv K_0(B)$ then $A \equiv B$.
The conjecture still remains open.

Many of examples of equivalent ordered abelian groups (e.g.$\mathbb{Q} \equiv \mathbb{Q}(\sqrt{2})$) are known. A concrete example of nonisomorphic equivalent AF-algebras is not known (the existence of such a pair is known).

$K_0(A) \equiv K_0(B)$ implies that A and B are "locally" isomorphic.
For example if D is a finite-dimensional subalgebra of A then B contains an isomorphic copy of D.
If $D_1 \subset D_2, D_1 \subset D_3$ are finite dimensional subalgebras of A then B has isomorphic copies of D_1, D_2, D_3 such that $D_1' \subset D_2', D_1' \subset D_3'$ and types of corresponding embeddings are the same.
The same is true for any tree of finite-dimensional subalgebras.

Profiles and EF-games

Let's fix the setup for EF-game: polynomials $\varphi_1, ..., \varphi_n$ and ε.

During the project the following theorem was proved:
Theorem. There exists a combinatorial invariant of C^*-algebras (called "profile") such that:
If A and B are AF algebras with the same profiles then the $\varphi_1, ..., \varphi_n, \varepsilon$ EF-game can be won.

There are finitely many different possible profiles.

Using this technique we showed that for one-sided EF-games (i.e. the games in which player one always plays in the same algebra) there is a strategy for the second player.

3 Project 2

3.1 Fraïssé limits

In this section, we develop the theory of Fraisse class and limit in the discrete case, with a mind towards the goal of introducing the rudiments of the theory in the continuous logic case and discussing the connections to our work with particular classes of C^*-algebras.
The Fraisse construction, originally introduced in the discrete logic setting in the 1950s, is summarily a construction technique in model theory which takes a countable collection K of finitistic structures of a language, and melds them together in a particular way to form a unique limit object Of countable cardinality which satisfies the additional property of ultrahomogeneity.
Throughout, we limit ourselves to the case of models of a countable language L. Definitions follow.

Definition. A category K of finitely-generated models M of signature L is called a Fraisse class if it satisfies to following properties:

- Hereditary Property (HP) K is closed under the process of taking substructures.
- Joint Embedding Property (JEP) For any two objects $A, B \in \mathrm{Ob}(K)$, there is an additional object C and two morphism f, g with sources A and B respectively and target C. A and B are jointly embedded into C.
- Amalgation Property (AP) For objects A, B, C of K, and for morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(A, C)$, there exists an object D in K and further morphisms $f' \in \mathrm{Hom}(B, D)$, $g' \in \mathrm{Hom}(C, D)$ such that the associated paths commute: $f' \circ f = g' \circ g$.

We will throughout restrict ourselves to the case where the morphisms of the category are required to be embeddings of the structures of the category. Under these circumstances, AP simply states that embeddings of a base object A into two others B, C can be made to meet back by means of embeddings into a further structure D. As we shall see, this property is fundamental to the construction of our limit object, the Fraisse limit.

Example. The category G of finite graphs with morphisms all graph embeddings between them is a Fraisse class. HP and JEP are trivially verified, while to see AP, simply observe that images of two embeddings f, g can be brought together in the upper object D, and the elements of B, C not in the images of f, g can be sent to new elements of D. The result is a finite graph.

We now define an easy notion of mostly abbreviational value.
Definition. By the age of a model M of signature L, we mean the collection of finitely-generated substructures of M.
Thus, the age of some structure can be regarded as the class of finitistic objects it contains. We will be interested in the relation of a Fraisse class to the age of its Fraisse limit.

We finally define the ultrahomogeneity property we demand of our limit object.
Definition. A structure M is said to be ultrahomogeous (UH) if every isomorphism of finitely-generated substructures of M extends to an automorphism of M.

We now relate the previous notions with a classical result of model theory.
Theorem (Fraisse, 1954). For any countable Fraisse class K over a countable language L, there is a unique (to isomorphism) structure F of signature L such that

- F is at most countable.
- The age of F is K (modulo isomorphisms of structures of K).
- F is ultrahomogeneous.

A sufficient amount of staring at the theorem will convince the reader that the Fraisse limit is in fact a special object. It collects the entirety of K into one, in such a way that yields a tremendous amount of symmetry in the limit object F (UH), such that only K is contained in F, and in the unique way of the countable cardinality.

Example. For the Fraisse class G of finite graphs described earlier, the Fraisse limit structure is the generic Rado graph R, well known in graph theory. This countable has (for instance), the remarkable property that it is isomorphic to any countable extension of itself.
While we will not attempt to prove the theorem, we briefly indicate the method of construction of F. The key observation is that there are only countable many morphisms f between any two objects of the category. Thus we can order them. We will plan to form a chain of L-structures $\{B_n\}_{n\in\omega}$, such that for every morphism $f : A \to A'$, if A can be embedded at some step of the chain B_j, then we will choose a higher up B_k such that such that A' will embed into B_k in a way that extends the embedding of A into B_j. Not worrying to much about details, by the countability of our set of morphisms, we can ensure that every embedding f is put in the chain in this way at some step B_n. We furthermore, observe that the existence of the structures B_n in our class K is a consequence of the amalgamation property, which is thereby fundamental.
From here, it's clear that we can take F to be the direct limit $\bigcup_{n\in\omega} B_n$. Fraisse goes on to prove that this object satisfies our demands.
We now transfer the theory of Fraisse limits of Fraisse classes to the continuous model theory settings. The development is mired in details. We do not attempt to describe it in detail, rather we give the basic modification and notions necessary, and refer readers looking for depth to ben Yaacov.

Definition. A category K of finitely-generated continuous structures is said to be an incomplete Fraisse class if it satisifies HP and JEP, and the following:

- Near Amalgamation Property (NAP) For every $\epsilon > 0$, and for A, B, C objects and f, g morphisms in the category as before, we can be find object D and morphisms f', g' with source and targets as before, such that the composite maps $f' \circ f$ and $g' \circ g$ meet in D up to ϵ. In other words, for all $x \in A$, $d(f'f(x), g'g(x)) < \epsilon$.

- Polish Property (PP) The metric of each object in the category is separable.

- Cauchy Continuous Property (CCP) Every symbol of the signature is Cauchy continuous on the metric space of each model (it carries Cauchy sequences to Cauchy sequences).

Definition. A continuous structure M is said to be approximately ultrahomogoneous (AUH) if for every $\epsilon > 0$ and every isomorphism ϕ_0 of finitely-generated substructures of M, there is an automorphism ϕ of M which is within ϵ of ϕ_0. In other words, for all $x \in \text{dom}(\phi_0)$, we have $d(\phi_0(x), \phi(x)) < \epsilon$.

We finish with the theorem of ben Yaacov, leaving the many details obscured. **Theorem. (ben Yaacov)** An countable incomplete Fraisse class K admits a unique up to isomorphism separable continuous structure F, called the Fraisse limit of K, such that the age of F is K, and such that F is approximately ultrahomogeneous.

Our desire was, as we shall see, to show that certain categories of dimension drop algebras form incomplete Fraisse classes (by showing that they would amalgamate), and thereby to show that the Jiang-Su algebra, for instance, can be expressed as a Fraisse limit.

3.2 Dimension-drop algebras

The algebra of continuous functions $C([0,1], \mathbb{C})$ is frequently studied in mathematics. This algebra is generalized in the same sense as the square matrices generalize the complex numbers when we consider $C([0,1], M_n(\mathbb{C}))$. Recall that we may define addition, multiplication, and the *-involution pointwise on $C([0,1], M_n(\mathbb{C}))$. Furthermore, the C*-identity holds, as for all $f \in C([0,1], M_n(\mathbb{C}))$, we have

$$\|f^* f\| = \sup_{x \in [0,1]} |f^*(x) f(x)| = \sup_{x \in [0,1]} |f(x)|^2 = \|f\|^2$$

So these algebras are indeed C*-algebras.

One class of algebras of this form that we will draw particular attention to is the class of *dimension-drop algebras*. Recall that for any given $p, q \in \mathbb{N}$, we have $M_p(\mathbb{C}) \otimes M_q(\mathbb{C}) \cong M_{pq}(\mathbb{C})$.

Definition 1 (Dimension-drop algebra)**.** *$\mathcal{A} \subset C([0,1], M_{pq}(\mathbb{C}))$ is a dimension-drop algebra if every $f \in \mathcal{A}$ is such that $f(0) \in M_p(\mathbb{C}) \otimes 1_q$ and $f(1) \in 1_p \otimes M_q(\mathbb{C})$. We then denote $\mathcal{A} = \mathcal{Z}_{pq}$.*

Notice that for all $f, g \in \mathcal{Z}_{pq}$,

1. $(f+g)(0) = f(0) + g(0) \in M_p \otimes 1_q$ and
   $(f+g)(1) = f(1) + g(1) \in 1_p \otimes M_q$
2. $(fg)(0) = f(0)g(0) \in M_p \otimes 1_q$ and $(fg)(1) = f(1)g(1) \in 1_p \otimes M_q$
3. $f^*(0) = f(0)^* \in M_p \otimes 1_q$ and $f^*(1) = f(1)^* \in 1_q \otimes M_q$

So $\mathcal{Z}_{pq}$ is a C*-subalgebra of $C([0,1], M_{pq}(\mathbb{C}))$.

Definition 2 (Prime dimension-drop algebra)**.** *If we have $gcd(p,q) = 1$, then we call the resulting dimension-drop algebra $\mathcal{Z}_{pq}$* prime*.*

Jiang and Su, in their paper [1], define morphisms (which we will henceforth refer to as Jiang-Su morphisms) as follows. For a given prime dimension-drop algebra $\mathcal{Z}_{pq}$, choose integers k_p, k_q so that

$$k_p p,\ k_q q > 2pq \text{ and } (k_p p, k_q q) = 1$$

Let $p_1 = k_p p$ and $q_1 = k_q q$, and write $k = k_p k_q$.
We then embed $\mathcal{Z}_{pq}$ unitally into $\mathcal{Z}_{p_1 q_1}$ by the map $f \mapsto \text{diag}_k(f, ..., f)$. It is shown in the paper of Jiang and Su that the direct limit of any system of prime dimension drop algebras with connecting Jiang-Su morphisms forms a simple unital C*-algebra with trivial projections and an unique trace, usually denoted $\mathcal{Z}$. In particular, the trace is the one induced by the Lebesgue measure by the Riesz-Markov representation theorem, i.e.

$$\tau(f) = \int_0^1 f(x) d\mu(x)$$

where μ denotes Lebesgue measure.
One goal of our project was to show that the Jiang-Su algebra $\mathcal{Z}$ is a Fraïssé limit of a particular class of dimension-drop algebras and connecting morphisms.
We spent some time attempting to demonstrate the NAP when the category is taken to be the class of prime-dimension drop algebras where the connecting morphisms are exactly of Jiang-Su type.

Suppose we have Jiang-Su embeddings $\phi_i : \mathcal{Z}_{p,q} \hookrightarrow \mathcal{Z}_{p_i,q_i}$, $i \in \{1, 2\}$. Up to unitary equivalence, morphisms of Jiang-Su type are specified by a choice of integers. Choose $k_p^i, k_q^i \in \mathbb{Z}$ (corresponding to the morphism ϕ_i) such that

$$k_p^i p,\ k_q^i q > 2pq$$

Let $k^i = k_p^i k_q^i$. The images of the ϕ_i contain some number of copies of compositions of a given input $a \in \mathcal{Z}_{pq}$ with functions $\xi_j,\ 1 \leq j \leq k^i$, where $\xi_j(x) \in \{f_1(x), f_2(x), f_3(x)\} = \{\frac{x}{2}, \frac{1}{2}, \frac{1+x}{2}\}$ depends on functions of k_p^i, k_q^i. We can write $\phi_i : a \mapsto \text{diag}(a \circ \xi_1, ..., a \circ \xi_{k^i})$.

Under ϕ_1, let's say we have α_1 copies of $a \circ f_1$, α_2 copies of $a \circ f_2$, and α_3 copies of $a \circ f_3$. These α's are defined via integers that Jiang and Su call r_0, r_1, and depend only on the choice of k_p^i, k_q^i.
On the other hand, under ϕ_2 we have β_j copies of $a \circ f_j$, for $j \in \{1, 2, 3\}$.

We can amalgamate if we can find morphisms ψ_i of Jiang-Su type such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$. These morphisms make some number of copies of compositions of a given element of $\mathcal{Z}_{p_i,q_i}$ with some f_i, as it was with the ϕ's. Let γ_j correspond to ψ_1 and δ_j correspond to ψ_2.

So for $i, j \in \{1, 2, 3\}$, $\alpha_i \gamma_j$ is the number of copies of $a \circ f_i \circ f_j$ given by the composition $\psi_1 \circ \phi_1$, while $\beta_i \delta_j$ is the number of such copies given by $\psi_2 \circ \phi_2$. So amalgamation is equivalent to there existing δ's and γ's such that $\alpha_i \gamma_j = \beta_i \delta_j$ for any α's and β's constructed as above.

This leaves plenty of future work to be done. In particular, we would like to resolve the present situation with amalgamating prime dimension-drop algebras with Jiang-Su connecting morphisms. Assuming such a resolution, we would then like to consider whether all unital morphisms of prime dimension-drop algebras can be approximately unitarily represented by morphisms of Jiang-Su type. Additionally, we thought about the case when the category was chosen to be precisely those prime dimension-drop algebras and connecting morphisms where the trace induced by the Lebesgue measure is preserved, but did not have time to work much on this particular branch of the project.

References

[1] On a Simple Unital Projectionless C*-Algebra Hongbing Su, Xinhui Jiang

From: American Journal of Mathematics Volume 121, Number 2, April 1999 pp. 359-413 — 10.1353/ajm.1999.0012