
GMC
C STARR

Elliott's program and descriptive set theory I

Ilijas Farah

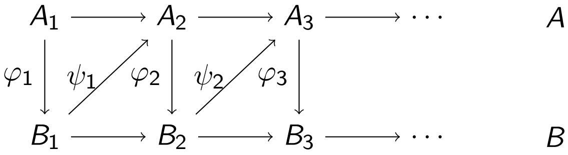
LC 2012, Manchester, July 12

a, a, a, a, the, the, the, the.

I shall need this exercise later, someone please solve it

Exercise

If $A = \varinjlim_n A_n$, $B = \varinjlim_n B_n$ *and there are morphisms* φ_j, ψ_j *such that the following diagram commutes*



then $A \cong B$.

The plan

1. Today:
    1.1 Basic properties of C*-algebras.
    1.2 Classification: UHF and AF algebras.
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    1.2 Classification: UHF and AF algebras.
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2. Friday: Applying logic to 1.2–1.3.
3. Saturday: Convincing you that 1.2–1.3 is logic.

Prologue

A topological space X is *Polish* if it is separable and completely metrizable.
A subset of X is *analytic* if it is a continuous image of a Borel set.
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In almost all cases, the space of invariants has a Polish topology and the computation of invariants is given by a Borel-measurable function.

Hilbert space, inner product

$$v = (v_0, v_1, \dots, v_n, \dots) \in \mathbb{C}^{\mathbb{N}}$$

$$(v|u) = \sum_n v_n \bar{u}_n \qquad \text{inner product}$$

$$\|v\| = \sqrt{(v|v)} \qquad \text{norm}$$

$$\ell_2 = \{v \mid \|v\| < \infty\}$$

$$L_2(\mu) = \{f : [0,1] \to \mathbb{C} \mid \int |f|^2 \, d\mu < \infty\}$$

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Fact

Two complex Hilbert spaces are isomorphic iff their dimensions are equal. In particular, $\ell_2 \cong L_2(\mu)$.

C*-algebras

H: a complex Hilbert space, ℓ_2
If $a \colon H \to H$ is linear let

$$\|a\| = \sup_{\|\xi\|=1} \|a\xi\|.$$

Define a^* implicitly via

$$(a\eta|\xi) = (\eta|a^*\xi)$$

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Example

If $\dim(H) = n$ then $\mathcal{B}(H)$ is $M_n(\mathbb{C})$: $n \times n$ complex matrices.

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1. $(a^*)^* = a$
2. $(ab)^* = b^* a^*$
3. $\|a\| = \|a^*\|$
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Theorem (Gelfand–Naimark–Segal, 1942)

Every abstract C-algebra is isomorphic to a concrete C*-algebra.*

Examples of C*-algebras

1. $\mathcal{B}(H)$, $M_n(\mathbb{C})$.
2. If X is a compact Hausdorff space,

$$C(X) = \{f : X \to \mathbb{C} | f \text{ is continuous}\}.$$

Gelfand–Naimark duality

A C*-algebra is *unital* if it has a multiplicative unit.

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Every unital abelian C-algebra is of the form* $C(X)$ *for some compact Hausdorff space.*

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Theorem

Categories of unital abelian C-algebras with *-homomorphisms and compact Hausdorff spaces with continuous maps are equivalent.*

$$\mathrm{X} \xrightarrow{\quad \varphi \quad} \mathrm{Y}$$

$$\mathrm{C(X)} \xleftarrow{\; g \mapsto g \circ \varphi \;} \mathrm{C(Y)}$$

Automatic continuity

Proposition

*Every *-homomorphism $\Phi\colon A \to B$ between C*-algebras satisfies*

$$\|\Phi(a)\| \leq \|a\|$$

(C-algebraists call such map a contraction).*

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Then use the C*-equality, $\|a\|^2 = \|a^*a\|$. □

Direct (or inductive) limits

Assume A_λ, for $\lambda \in \Lambda$, is a directed system of C*-algebras with

$$\Phi_{\kappa\lambda} \colon A_\kappa \to A_\lambda$$

for $\kappa < \lambda$ in Λ a commuting system of *-homomorphisms. Since all $\Phi_{\kappa\lambda}$ are contractions, on the 'algebraic' direct limit we have a uniquely defined C*-norm.
Let $\lim_\lambda A_\lambda$ be the closure of the 'algebraic' direct limit.

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$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$$

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$$M_{2^\infty}(\mathbb{C}) = \varinjlim M_{2^n}(\mathbb{C}).$$

The structure of full matrix algebras

Lemma

*There is a unital *-homomorphism of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ if and only if m divides n. Moreover, such *-homomorphism is unique up to a unitary conjugacy.*

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Partial proof.

If $m|n$ then $a \mapsto \mathrm{diag}(a, a, \ldots, a)$ (a repeated m/n times). □

Uniformly HyperFinite (UHF) algebras are direct limits of full matrix algebras

If $A = \lim_j M_{n(j)}(\mathbb{C})$ is unital, let the *generalized integer* associated to A be the formal infinite product

$$GI(A) = \prod_{p \text{ prime}} p^{k(p)}$$

where $k(p)$ is the supremum of all k such that p^k divides $n(l)$ for some l.

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It seems that

$$GI(M_{2^\infty}) = 2^{\aleph_0},$$

but one needs to check that $GI(A)$ is well-defined.

Stability

We need to show that $A = \lim_j M_{n(j)}(\mathbb{C}) = \lim_j M_{k(j)}(\mathbb{C})$ implies

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$$C \subseteq_\varepsilon D$$

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Lemma

$(\forall k \in \mathbb{N})(\exists \varepsilon > 0)$ *such that for all A and subalgebras C and D of A such that $C \subseteq_\varepsilon D$ and $C \cong M_k(\mathbb{C})$, then there exists an inner automorphism Φ of A such that $\Phi[C] \subseteq D$.*

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Corollary

If A is separable then LM implies UHF.

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Theorem (Glimm, 1960)

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Then we have φ_j and ψ_j for all $j \in \mathbb{N}$ so that

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commutes, and by the exercise we have $A \cong B$. $\square$

AF algebras

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A C*-algebra is *AF (Approximately Finite)* if it is a direct limit of finite-dimensional C*-algebras.

Lemma

Every finite-dimensional C-algebra is a direct sum of full matrix algebras.*

Bratteli diagrams

Consider unital *-homomorphism

$$\Phi\colon M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_6(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_6(\mathbb{C})$$

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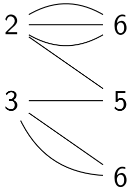
The *Bratteli diagram* describing this map is the following:



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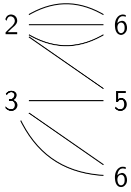
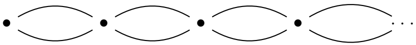


Bratteli diagram determines *-homomorphism Φ uniquely, up to the unitary conjugacy.

Examples of Bratteli diagrams that describe AF algebras



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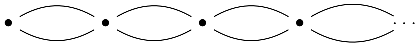

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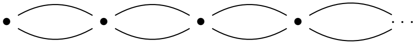

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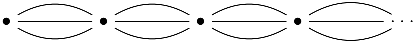

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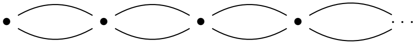

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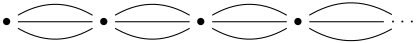

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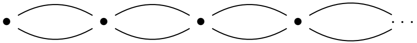

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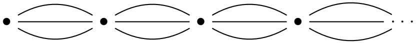

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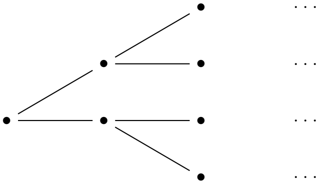

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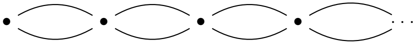

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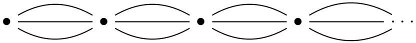

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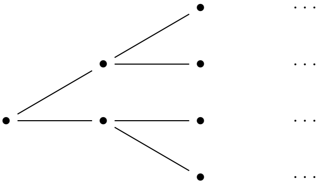

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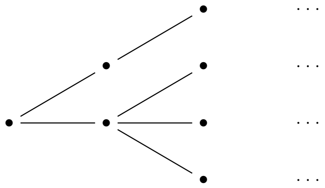
More examples of Bratteli diagrams



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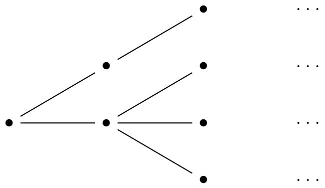

$C(\omega + 1)$

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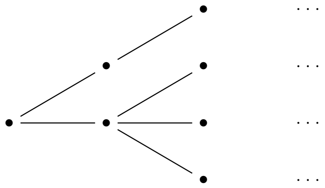

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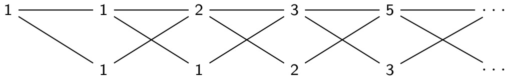

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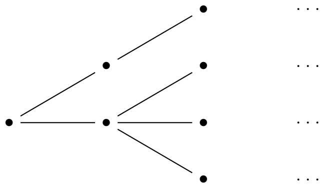

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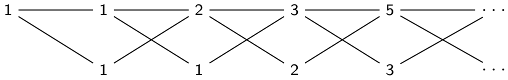

Fibonacci algebra: a simple unital AF algebra with a unique trace that is not UHF.

Classification of AF algebras: Stabilization

If $n \in \mathbb{N}$ and A is a C*-algebra, then so is $M_n(A)$: $n \times n$ matrices of elements of A with respect to the matrix operations and the operator norm.

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Consider the direct limit of $M_n(A)$, for $n \in \mathbb{N}$, with (non-unital) *-homomorphism

$$\Phi_n \colon M_n(A) \to M_{n+1}(A)$$

defined via

$$\Phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $M_\infty(A) = \lim_n M_n(A)$ is the *stabilization of A*.

Classification of AF algebras: Murray-von Neumann equivalence

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Let $V(A)$ be the set of projections on $M_\infty(A)$ modulo $\sim$, equipped with the operation $\oplus$ defined by

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K-theoretic classification

Example

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Categories of AF algebras and their K_0 groups are equivalent.

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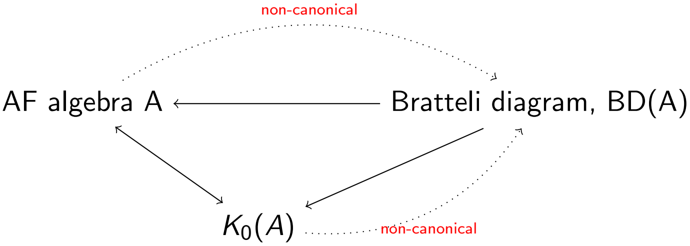

Elliott's program

Conjecture (Elliott, 1990's)

All nuclear,[1] *separable, simple, unital, infinite-dimensional C*-algebras are classified by the K-theoretic invariant,*

$$\mathrm{Ell}(A): \qquad ((K_0(A), K_0(A)^+, 1), K_1(A)).$$

[1] I shall define nuclear C*-algebras on Saturday. All algebras mentioned today (except $\mathcal{B}(H)$) are nuclear.

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This conjecture has lead to some spectacular developments.

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Examples

If $\mathbb{K}$ is a class of compact Hausdorff spaces, then $A\mathbb{K}$ *algebras* have building blocks of the form

$$C(X, M_m(\mathbb{C})) = \{f : X \to M_m(\mathbb{C}) | X \in \mathbb{K} \text{ and } f \text{ is continuous}\}$$

With $\mathbb{K} = \{[0,1]\}$ we have AI algebras, if $\mathbb{K} = \{\{z \in \mathbb{C} : |z| = 1\}\}$ we have A$\mathbb{T}$ algebras, if $\mathbb{K}$ is the class of all compact metric spaces then we have AH algebras.

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It actually suffices to have algebras of *slow dimension growth*.

Nuclear, separable, simple, unital counterexamples

Jiang–Su, 1999

There exists an infinite-dimensional C*-algebra $\mathcal{Z}$ such that $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$ and $\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(A)$ for all A.

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The algebra A constructed by Toms cannot be distinguished from $A \otimes \mathcal{Z}$ by any 'reasonable' invariant.

Elliott's conjecture recast

Conjecture (Toms–Winter, 2009)

All nuclear, separable, simple, unital algebras A *such that* $A \otimes \mathcal{Z} \cong A$ *are classified by Elliott's invariant.*

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Next time:

1. What is the descriptive complexity of the isomorphism relation of C*-algebras?