Existentially closed C^* algebras, operator systems, and operator spaces

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East Coast Operator Algebras Symposium
Fields Institute, Toronto, ON
October 11, 2014

Goal of the talk: given one of the category of C^* algebras, operator systems, or operator spaces, define what the "algebraically closed" objects of that category are and examine what operator algebra-theoretic or operator space-theoretic properties these objects may or may not have.

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Defining existentially closed C* algebras

Definition

1. An *atomic formula* (in the language of C* algebras) is a formula of the form $\|P(\vec{x})\|$ for P some *polynomial with coefficients from $\mathbb{C}$.
2. A *quantifier-free formula* is a formula of the form $f(\varphi_1, \ldots, \varphi_n)$, where each φ_i is atomic and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.
3. If $\varphi(\vec{x}, \vec{y})$ is a quantifier-free formula and $\vec{a} \in A^{|\vec{y}|}$, we call $\varphi(\vec{x}, \vec{a})$ a *quantifier-free A-formula*.

Definition

A C* algebra A is *existentially closed* (e.c.) if, given any C* algebra $B \supseteq A$, any quantifier-free A-formula $\varphi(\vec{x})$, and any $k \geq 1$, we have

$$\inf\{\varphi(\vec{a}) \ : \ \vec{a} \in A_k\} = \inf\{\varphi(\vec{b}) \ : \ \vec{b} \in B_k\}.$$

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How many separable e.c. C^* algebras are there?

Lemma

Every separable C^* algebra is a subalgebra of a separable e.c. C^* algebra.

Corollary

There are uncountably many nonisomorphic separable e.c. C^* algebras.

Proof.

Otherwise, there would be a universal separable C^* algebra (namely the tensor product of the separable e.c. C^* algebras), contradicting a result of Junge and Pisier. □

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General properties of e.c. C* algebras

Lemma

Suppose that (P) is an $\forall\exists$-axiomatizable property of C* algebras such that every (separable) C* algebra can be emedded in a (separable) C* algebra with property (P). Then every (separable) e.c. C* algebra has property (P).

Corollary

A separable e.c. C* algebra is $\mathcal{O}_2$-stable, simple, and purely infinite.

Connection with nuclearity and exactness

- Not every separable e.c. C* algebra is exact, else every separable C* algebra would be exact.

Theorem (G.-Sinclair)

1. *e.c. + exact implies nuclear*
2. *$\mathcal{O}_2$ is the only possible separable e.c. nuclear C* algebra*
3. *$\mathcal{O}_2$ is e.c. if and only if the Kirchberg embedding problem (KEP) has a positive solution, that is, if and only if every separable C* algebra embeds into an ultrapower of $\mathcal{O}_2$.*

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An application-local criteria for KEP

Definition

For a C^* algebra A and an n-tuple $a \in A$, we define

$$\Delta^A_{\mathrm{nuc},n}(a) = \inf_{\phi,\psi} \|(\psi \circ \phi)(a) - a\|,$$

where $\phi : A \to M_k(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to A$ are u.c.p. maps. (So A is nuclear if and only if $\Delta^A_{\mathrm{nuc},n} \equiv 0$ for all n.)

A *condition* is a finite set of expressions of the form $\varphi(x) < r$, where $\varphi(x)$ is quantifier-free. A condition $p(x)$ has *good nuclear witnesses* if, for each $\epsilon > 0$, there is a C^* algebra A and a tuple $a \in A$ realizing $p(x)$ with $\Delta^A_{\mathrm{nuc},n}(a) < \epsilon$.

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KEP holds if and only if every satisfiable condition has good nuclear witnesses.

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Changing the language

- We are now going to change to the logic appropriate for dealing with operator systems.
- We won't get into the precise formulation here, but we will soon see an example of a quantifier-free formula in this new language, which should be enough to give you an idea of how the language should look. (There are some technical things that need to be added to the language in order to formulate Choi-Effros' abstract formulation of operator systems in our logic.)
- Of course, there is also an appropriate language for dealing with operator spaces.

Weakly injective operator systems

Definition

An operator system $X \subseteq \mathcal{B}(H)$ is *weakly injective* if there is a u.c.p. extension $\mathcal{B}(H) \to \overline{X}^{wk}$ of the identity map $X \to X$.

Theorem (G.-Sinclair)

If $X \subseteq \mathcal{B}(H)$ is an e.c. operator system, then X is weakly injective.

Definition

A C* algebra has the *weak expectation property* (WEP) if is weakly injective in its universal representation.

- Therefore, a C* algebra that is e.c. as an operator system has WEP.

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- Therefore, a C* algebra that is e.c. as an operator system has WEP.

Proof

- Consider the existential formulae

$$\phi_{n,k,\sigma}(a,b) := \inf_{x\in\mathcal{C}_n} \|[a_{ij} + \sigma_{ij}^1 b_1 + \cdots + \sigma_{ij}^k b_k] - x\|,$$

  where $[a_{ij}] \in M_n(\bullet)$ is a self-adjoint matrix and $\sigma = (\sigma^1, \ldots, \sigma^k) \in M_n(\mathbb{C})^k$ is self-adjoint.
- For $b = (b_1, \ldots, b_k) \in \mathcal{B}(H)^k$ self-adjoint, operator systems $X \subset Y \subset \mathcal{B}(H)$, and $b' \in Y^k$, note that the linear map

$$\eta : X + \mathbb{C}b_1 + \cdots + \mathbb{C}b_k \to Y, \quad \eta(x + \sum_l \lambda_l b_l) := x + \sum_l \lambda_l b_l'$$

  is u.c.p. if and only if, for every self-adjoint $a \in M_n(X)$ and every self-adjoint $\sigma \in M_n(\mathbb{C})^k$, we have $\phi_{n,k,\sigma}(a,b)^{\mathcal{B}(H)} = 0$ implies $\phi_{n,k,\sigma}(a,b')^Y = 0$.
- Call a formula $\phi_{n,k,\sigma}(a,y)$ *admissible* if $(\inf_y \phi_{n,k,\sigma}(a,y))^{\mathcal{B}(H)} = 0$.

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Proof (cont'd)

- Note that the set

$$W_{n,k,\sigma,a,\epsilon} := \{b \in X^k : \|b\| \leq 1,\ \phi_{n,k,\sigma}(a,b)^X < \epsilon\}$$

  is a bounded subset of X, so its weak closure is weakly compact.
- Since X is e.c., the family $(W_{n,k,\sigma,a,\epsilon})$, where we only consider admissible $\phi_{n,k,\sigma}(a,y)$, has the finite intersection property, whence the intersection of their weak closures is non-empty.
- This shows that for every $b \in \mathcal{B}(H)^k$, there is a u.c.p. map $\eta_b : X + \mathbb{C}b_1 + \cdots + \mathbb{C}b_k \to \overline{X}$ which extends the identity on X.
- Letting $\mathcal{F}$ be the net all of finite subsets of self-adjoint elements of $\mathcal{B}(H)$ directed by inclusion, we have that any weak cluster point η of $\{\eta_b : b \in \mathcal{F}\}$ is a u.c.p. map $\eta : \mathcal{B}(H) \to \overline{X}$ which extends the identity on X, whence X is weakly injective.

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PEC

- It turns out that WEP in turn implies some form of existential closedness.

Definition

- A quantifier-free formula $\varphi(x)$ is *positive* if, for any homomorphism $F : M \to N$ and any $a \in M$, we have $\varphi(F(a))^N \leq \varphi(a)^M$.
- M is said to be *positively existentially closed* (PEC) or *algebraically closed* if it is existentially closed with respect to positive formulae.

Observation

Formulae built from atomic formulae using increasing connectives are positive.

PEC vs. WEP

Observation

If M is a von Neumann algebra, then M has WEP if and only if M is PEC as an operator system.

Proof.

Suppose that M has WEP and $M \subseteq S \subseteq \mathcal{B}(H)$ is an operator system. Let $E : \mathcal{B}(H) \to M$ be a conditional expectation. If $\varphi(x, y)$ is a positive formula, $a \in M$ and $b \in S$, then $\varphi(a, E(b))^M \leq \varphi(a, b)^S$, whence $(\inf_y \varphi(a, y))^M = (\inf_y \varphi(a, y))^S$. □

Question

Does WEP=PEC as an operator system hold for any C* algebra?

WEP vs PEC (continued)

Lemma

If A has the WEP, then for every C^* algebra B containing A, every finite-dimensional subspace $E \subset B$, and every $n, \epsilon > 0$, there exists a map $\phi : E \to A$ with $\|\phi\|_n \leq 1$ and $\|\phi|_{E\cap A} - \mathrm{id}_{E\cap A}\| < \epsilon$.

Corollary

If A has the WEP, then A is PEC as an operator space.

Let's write $\mathrm{PEC}_{\mathrm{sp}}$ and $\mathrm{PEC}_{\mathrm{sys}}$ to denote being PEC as an operator space and as an operator system respectively.

Question 1

- We see that $\text{PEC}_{\text{sys}} \Rightarrow \text{WEP} \Rightarrow \text{PEC}_{\text{sp}}$.

Question 1

Do we have $\text{PEC}_{\text{sp}} \Leftrightarrow \text{PEC}_{\text{sys}}$?

- By Kirchberg, we know that CEP is equivalent to $C^*(\mathbb{F}_\infty)$ having WEP. In light of Question 1, it becomes interesting to check whether or not $C^*(\mathbb{F}_\infty)$ is PEC_{sys} or PEC_{sp}.
- We know $C^*(\mathbb{F}_\infty)$ is not PEC as a C^* algebra as we know PEC C^* algebras are $\mathcal{O}_2$-stable.

Approximate injectivity

Definition

A C^* algebra A is *approximately injective* if, for any finite-dimensional operator systems $E_1 \subseteq E_2$ and a completely positive map $\phi_1 : E_1 \to A$, there is a completely positive map $\phi_2 : E_2 \to A$ such that $\|\phi_2|E_1 - \phi_1\| < \epsilon$.

- Approximate injectivity implies $\mathrm{PEC}_{\mathrm{sys}}$.

Question 2

Are approximate injectivity and $\mathrm{PEC}_{\mathrm{sys}}$ equivalent?

- By work of Junge and Pisier, we cannot have positive answers to both Questions 1 and 2.

CP-stability

- The difference between WEP and approximate injectivity can be summarized as follows: if A has WEP, then given finite-dimensional operator systems $E_1 \subseteq E_2$ and a ucp map $\phi : E_1 \to A$, we can only find, for any n, an n-contractive approximate extension of ϕ to E_2 (rather than a ucp approximate extension).

Definition

An operator system X is said to be *CP-stable* if, for any finite-dimensional subspace $E_1 \subseteq X$ and $\delta > 0$, there is a finite-dimensional $E_1 \subseteq E_2 \subseteq X$ and $n, \epsilon > 0$ so that, for any unital map $\phi : E_2 \to A$, where A is a C* algebra, if $\|\phi\|_n < 1 + \epsilon$, then there is a ucp map $\psi : E_2 \to A$ such that $\|\phi|_{E_1} - \psi\| < \delta$.

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CP-stability (continued)

Proposition

If A has WEP, then A satisfies the conclusion of approximate injectivity for finite-dimensional pairs $E_1 \subseteq E_2$ that are contained in a CP-stable operator system.

Corollary

Not every existentially closed C^* algebra is CP-stable.

The local lifting property

Definition

A C* algebra *A* has the *local lifting property* (LLP) if, for any ucp map $\phi : A \to C/J$ and any finite-dimensional subspace $E \subseteq A$, there is a ucp map $\psi : E \to C$ with $\pi \circ \psi = \phi | E$, where $\pi : C \to C/J$ is the quotient map.

- Kirchberg proved that CEP is equivalent to the statement "LLP implies WEP."
- In light of Question 1, it becomes interesting to ask for the connection between LLP and existential closedness.

LLP and CP-stability

Proposition

If A is separable, then A is CP-stable if and only if A has the *local ultrapower lifting property*, meaning that for every unital C* algebra B and every ucp map $\phi : A \to B^\omega$, there is a ucp map $\phi' : A \to \ell^\infty(B)$ such that $\phi = \pi \circ \phi'$, where $\pi : \ell^\infty(B) \to B^\omega$ is the canonical quotient map.

Corollary

LLP implies CP-stable. Thus, not every e.c C* algebra has LLP.

Question 3

Can there exist an e.c. C* algebra with LLP?

LLP is an omitting types property

Very recently, Sinclair and I believe that we can show that LLP is an omitting types property in the language of operator systems. This leads to a notion of *good LLP witnesses* (analogous to the notion of good nuclear witnesses).

Theorem (G.-Sinclair)

If every satisfiable condition has good LLP witnesses, then there is an e.c. C^* algebra with LLP. This algebra is either nuclear (whence KEP holds) or is non-nuclear, providing the first example of a non-nuclear C^* algebra with both WEP and LLP.

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