The Gromov-Hausdorff Propinquity

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Noncommutative Metric Geometry

Theorem (Gel'fand-Naimark duality)

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Motivation

Noncommutative metric geometry aims at providing a foundation for constructions of approximations in quantum physics based upon quantum spaces, and provides a new approach to developing a geometry for quantum spaces from the metric geometry of their state spaces. *The key tools are metrics on classes of quantum metric spaces.*

Structure of this Presentation

1. *Quantum Compact Metric Spaces*
   - The Monge Kantorovich distance
   - Compact Quantum Metric Spaces

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   - The quantum Gromov-Hausdorff distance
   - The dual propinquity
   - The Quantum Propinquity
3. *Locally Compact Quantum Metric Spaces*
   - Topographies
   - Convergence for locally compact quantum metric spaces

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What should a noncommutative analogue of a Lipschitz algebra be?

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What should a noncommutative analogue of a Lipschitz algebra be? For a locally compact metric space, Gel'fand duality suggests that a noncommutative Lipschitz algebra be based on a C*-algebra. What extra structure does the metric provide?

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We begin with the classical picture as a guide.

Lipschitz Seminorms

A natural dual object to a metric is the Lipschitz seminorm:

Definition

Let (X, m) be a metric space. For any function $f : X \to \mathbb{R}$, define:

$$\mathsf{L}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y \right\}.$$

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Questions

1. Can we recover the metric from its Lipschitz seminorm?
2. What makes a Lipschitz seminorm special among all seminorms?

A distance on the state space

The self-adjoint part of a C*-algebra $\mathfrak{A}$ is denoted by $\mathfrak{sa}\left(\mathfrak{A}\right)$ while its state space is denoted by $\mathscr{S}\left(\mathfrak{A}\right)$ and the smallest unital C*-algebra containing $\mathfrak{A}$ is denoted by $\mathfrak{u}\mathfrak{A}$.

Definition

A *Lipschitz pair* $(\mathfrak{A}, \mathsf{L})$ is a C*-algebra $\mathfrak{A}$ and a densely defined seminorm L on $\mathfrak{sa}\left(\mathfrak{u}\mathfrak{A}\right)$ such that $\{a \in \mathfrak{sa}\left(\mathfrak{u}\mathfrak{A}\right) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$.

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Definition (Kantorovich (1940), Kantorovich-Rubinstein (1958), Wasserstein (1969), Dobrushin (1970), Connes (1989), Rieffel (1998))

The *Monge-Kantorovich metric* mk_{L} on $\mathscr{S}(\mathfrak{A})$ associated with a Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ is defined for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}.$$

The classical Monge-Kantorovich metric

Theorem

Let (X, m) be a compact metric space and identify X with the pure state space of $C(X)$ (i.e. the Gel'fand spectrum of $C(X)$). Let L be the Lipschitz seminorm for m. Then:

$$\forall x, y \in X \quad \mathsf{m}(x, y) = \mathsf{mk}_{\mathsf{L}}(x, y).$$

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The Monge-Kantorovich metric is well-behaved when working over *compact* metric spaces:

Theorem (Wasserstein, Dobrushin (1970))

Let (X, m) be a compact metric space. The Monge-Kantorovich metric mk_{L} associated with m is a metric which metrizes the weak topology on the state space $\mathscr{S}(C(X))$ of $C(X)$.*

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Compact Quantum Metric Spaces

Based on this observation, Rieffel introduced:

Definition (Rieffel, 1998)

A *compact quantum metric space* $(\mathfrak{A}, \mathsf{L})$ consists of an order-unit space $\mathfrak{A}$ and a seminorm L densely defined on $\mathfrak{A}$, satisfying:

$$\{a \in \mathfrak{A} : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}},$$

and such that the distance:

$$\mathsf{mk}_{\mathsf{L}} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}) \mapsto \sup\{|\varphi(a) - \psi(a)| : a \in \mathfrak{A}, \mathsf{L}(a) \leqslant 1\}$$

metrizes the weak* topology on the state space $\mathscr{S}(\mathfrak{A})$. The seminorm L is then called a *Lip-norm*.

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We shall call a quantum compact metric space a unital Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ such that $(\mathfrak{sa}(\mathfrak{A}), \mathsf{L})$ is a compact quantum metric space.

Characterization of Compact Quantum Metric Spaces

The key observation of Rieffel is that one may characterize compact quantum metric spaces in C*-algebraic terms:

Theorem (Rieffel, 1998)

A unital Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ with $\mathfrak{A}$ unital is a compact quantum metric space if and only if:

1. $r = \operatorname{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}\right) < \infty$,
2. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant r\}$ *is precompact in norm.*

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Proof.

Use Kadison functional representation and Arzéla-Ascoli theorems. □

Examples: Ergodic Actions of Compact Groups with continuous Lengths

For any C*-algebra $\mathfrak{A}$, let $\mathfrak{sa}(\mathfrak{A})$ be its self-adjoint part and $\|\cdot\|_{\mathfrak{A}}$ be its norm.

Theorem (Rieffel, 1998)

Let α be a strongly continuous action of a compact group G on a unital C-algebra $\mathfrak{A}$ and ℓ be a continuous length function on G. Let $e \in G$ be the unit of G. For all $a \in \mathfrak{A}$, define:*

$$\mathsf{L}(a) = \sup\left\{ \frac{\|\alpha_g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{e\} \right\}.$$

If $\{a \in \mathfrak{A} : \forall g \in G \quad \alpha_g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$, then $(\mathfrak{sa}(\mathfrak{A}), \mathsf{L})$ is a compact quantum metric space.

This result uses the fact that spectral subspaces for such actions are finite dimensional (Hoegh-Krohn, Landstad, Stormer, 1981).

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Convergence of Compact Metric Spaces

Definition

Let (X, m_X) and (Y, m_Y) be two compact metric spaces. A distance m on $X \coprod Y$ is *admissible* for $(\mathsf{m}_X, \mathsf{m}_Y)$ when the canonical injections $(X, \mathsf{m}_X) \hookrightarrow (X \coprod Y, \mathsf{m})$ and $(Y, \mathsf{m}_Y) \hookrightarrow (X \coprod Y, \mathsf{m})$ are isometries.

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Notation

The Hausdorff distance on the compact subsets of a metric space (X, m) is denoted by $\mathsf{Haus}_{\mathsf{m}}$.

Definition (Gromov, 1981)

The *Gromov-Hausdorff distance* between two compact metric spaces (X, m_X) and (Y, m_Y) is the infimum of the set:

$$\left\{ \mathsf{Haus}_{\mathsf{m}}(X, Y) : \mathsf{m} \text{ is admissible for } (\mathsf{m}_X, \mathsf{m}_Y) \right\}.$$

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Theorem (McShane, 1934)

Let (Z, m) be a metric space and $X \subseteq Z$. If $f : X \to \mathbb{R}$ has Lipschitz constant l, then there exists $g : Z \to \mathbb{R}$ with Lipschitz constant l and whose restriction to X is f.

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$$\mathsf{L}_X(f) = \inf\{\mathsf{L}_Z(g) : g \in C(Z \to \mathbb{R}), g \circ \iota = f\}$$

for all $f \in C(X \to \mathbb{R})$.

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for all $f \in C(X \to \mathbb{R})$. This result requires that we work with *$\mathbb{R}$-valued* Lipschitz functions.

The quantum Gromov-Hausdorff distance

Definition (Rieffel, 2000)

Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two compact quantum metric spaces. A Lip-norm L on $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is *admissible* for $(\mathsf{L}_1, \mathsf{L}_1)$ when, for all $\{j,k\} = \{1,2\}$ and $a_j \in \mathfrak{sa}\left(\mathfrak{A}_j\right)$:

$$\mathsf{L}_j(a) = \inf\{\mathsf{L}(a_1, a_2) : a_k \in \mathfrak{sa}\left(\mathfrak{A}_k\right)\}.$$

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Proposition (Rieffel, 1999)

If L is an admissible Lip-norm for $(\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$ then the canonical injections $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}) \hookrightarrow (\mathscr{S}(\mathfrak{A} \oplus \mathfrak{B}), \mathsf{mk}_{\mathsf{L}})$ is an isometry (and similarly with $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$).

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Definition (Rieffel, 2000)

The *quantum Gromov-Hausdorff distance* $\mathsf{dist}_q((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ between two compact quantum metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is the infimum of the set:

$$\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}}}(\mathscr{S}(\mathfrak{A}), \mathscr{S}(\mathfrak{B})) : \mathsf{L} \text{ is admissible for } (\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

Basic Properties of dist_q

Theorem (Rieffel, 2000)

For any three quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *and* $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$, *we have:*

1. $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right) + \mathrm{diam}\left(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}\right) \geqslant \mathsf{dist}_q((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \mathsf{dist}_q((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})) \geqslant 0$,

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5. $\mathsf{dist}_q((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ *iff there exists a order-unit-space isomorphism from* $\mathfrak{sa}(\mathfrak{A})$ *to* $\mathfrak{sa}(\mathfrak{B})$ *whose dual map is an isometry from* $\left(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}\right)$ *to* $\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right)$.

The Distance Zero Problem

How to get *-isomorphism as necessary for distance zero?

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   - Li's *nuclear distance* based on Lip-balls,
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     1. *FL's quantum propinquity* based on Lip-balls.
     2. *FL's dual propinquity* based on state space.

Thus, our new approach focuses on keeping the noncommutative Monge-Kantorovich metric and shift the focus to the relationship between Lip-norms and multiplicative structure.

The Leibniz inequality

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Definition

A seminorm L on a C*-algebra $\mathfrak{A}$ has the *Leibniz property* when:

$$\forall a, b \in \mathfrak{A} \quad \mathsf{L}(ab) \leqslant \|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}.$$

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In most cases, the Lip-norms of quantum locally compact metric space comes from derivations, spectral triples or similar constructions which gives the Leibniz property. This is *a natural connection between metric and multiplicative structures of quantum locally compact metric space*.

The role of the Leibniz inequality

- The Leibniz inequality plays a central role in Rieffel's recent work on convergence of vector bundles. It appears that one should work within the framework of C*-metric spaces, where Lip-norms are defined on C*-algebras and satisfy a strong form of the Leibniz property (cf Rieffel's work on convergence of matrix algebras to spheres, for instance).

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- *Yet,* the quotient of a Leibniz seminorm is not Leibniz in general. This means that if one asks for admissible Lip-norms to be Leibniz in the definition of dist_q, one only gets a *pseudo-semi-metric* (Rieffel's proximity).

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Hard Problem

How does one define a non-trivial *metric* on *-isomorphic, quantum isometric classes of C*-metric spaces?

GPS

Leibniz quantum compact metric spaces

We first choose a category of quantum compact metric spaces.

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For a, b elements of a C*-algebra $\mathfrak{A}$, let $a \circ b = \frac{ab+ba}{2}$ be the Jordan product of a, b and $\{a, b\} = \frac{ab-ba}{2i}$ be the Lie product of a, b.

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Definition (Latrémolière, 2013)

A quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ is a *Leibniz quantum compact metric space* when, for all $a, b \in \mathfrak{sa}(\mathfrak{A})$ we have:

$$\mathsf{L}(a \circ b) \leqslant \|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}$$

and

$$\mathsf{L}(\{a, b\}) \leqslant \|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}},$$

while L is *lower semi-continuous.*

Bridges and Tunnels

We propose the following notion of a pair of isometric embeddings of Leibniz quantum compact metric spaces:

Definition (Latrémolière, 2013)

Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two Leibniz quantum compact metric spaces. A *tunnel* $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ is a Leibniz quantum compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ together with two surjective *-morphisms π_1 and π_2 such that:

$$\mathsf{L}_j(a) = \inf\left\{\mathsf{L}_{\mathfrak{D}}(d) \middle| \pi_j(d) = a\right\}$$

for all $j \in \{1, 2\}$ and $a \in \mathfrak{sa}\left(\mathfrak{A}_j\right)$.

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We do not require the tunnel to be of the form $(\mathfrak{A} \oplus \mathfrak{B}, \mathsf{L}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ with $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ canonical surjections.

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We can add various conditions on the Leibniz quantum compact metric space of a tunnel: strong Leibniz Lip-norm, compact C*-metric space, etc...

Bimodules and Bridges

A particular, common type of tunnels is given by the following construction for two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$:

1. Let Ω be a $\mathfrak{A}$-$\mathfrak{B}$-bimodule, with a norm $\|\cdot\|_\Omega$ such that:

$$\|a\omega b\|_\Omega \leqslant \|a\|_{\mathfrak{A}}\|\omega\|_\Omega\|b\|_{\mathfrak{B}}$$

for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and $\omega \in \Omega$.

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   for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and $\omega \in \Omega$.
2. Choose $\omega_0 \in \Omega$ and $\gamma > 0$ such that, if we set:

   $$\mathsf{bn}_{\omega_0,\gamma}(a,b) = \|a\omega_0 - \omega_0 b\|_{\Omega}$$

   and then:

   $$\mathsf{L}(a,b) = \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\gamma}\mathsf{bn}_{\omega_0,\gamma}(a,b)\right\}$$

   for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, then $(\mathfrak{A} \oplus \mathfrak{B}, \mathsf{L}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a tunnel (where $\pi_{\mathfrak{A}}$,$\pi_{\mathfrak{B}}$ are canonical surjections).

Bridges

The bimodule approach to the construction of Lip-norm is particularly interesting when the bimodules are C*-algebras. We thus propose:

Definition (Latrémolière, 2013)

Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two Leibniz quantum compact metric spaces. A *bridge* $(\mathfrak{D}, \omega, \rho_1, \rho_2)$ is a unital C*-algebra $\mathfrak{D}$ and two unital *-monomorphisms $\rho_j : \mathfrak{A}_j \hookrightarrow \mathfrak{D}$ $(j = 1, 2)$ and $\omega \in \mathfrak{D}$ such that there exists $\varphi \in \mathscr{S}(\mathfrak{D})$ with $\varphi((1-\omega)^*(1-\omega)) = 0$ and $\varphi((1-\omega)(1-\omega)^*) = 0$.

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To every bridge, we can associate a tunnel. The question is to choose the constant γ such that:

$$\mathsf{L} : (a, b) \in \mathfrak{sa}\left(A \oplus \mathfrak{B}\right) \mapsto \max\left\{\mathsf{L}_1(a), \mathsf{L}_2(b), \frac{1}{\gamma}\|a\omega - \omega b\|_\Omega\right\}$$

is admissible (difficulties arise: *Rieffel, 0910.1968*)

Defining a Distance from Tunnels: reach

How do we define a distance from tunnels?

Defining a Distance from Tunnels: reach

How do we define a distance from tunnels? We associate numerical quantities to a tunnel. The first is:

Definition (Latrémolière, 2013)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two Leibniz quantum compact metric spaces and $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. The *reach $\rho(\tau)$ of τ* is:

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\pi_{\mathfrak{A}}^{*}\left(\mathscr{S}(\mathfrak{A})\right), \pi_{\mathfrak{B}}^{*}\left(\mathscr{S}(\mathfrak{B})\right)\right),$$

where $\mathsf{Haus}_{\mathrm{m}}$ is the Hausdorff distance on compact subsets of a metric space (E, m).

Defining a Distance from Tunnels: depth

We must also account for the greater level of generality from Rieffel's admissibility.

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We must also account for the greater level of generality from Rieffel's admissibility. The key is the quantity:

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$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \overline{\mathrm{co}}\left(\pi_{\mathfrak{A}}^{*}\left(\mathscr{S}(\mathfrak{A})\right) \cup \pi_{\mathfrak{B}}^{*}\left(\mathscr{S}(\mathfrak{B})\right)\right)\right),$$

where $\overline{\mathrm{co}}(A)$ is the weak* closure of the convex hull of any subset A of $\mathscr{S}(\mathfrak{D})$.

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We must also account for the greater level of generality from Rieffel's admissibility. The key is the quantity:

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where $\overline{\mathrm{co}}(A)$ is the weak* closure of the convex hull of any subset A of $\mathscr{S}(\mathfrak{D})$.

This quantity will prove useful in dealing with the triangle inequality property of our new metric. No other approach has ever involved our more general tunnels and only look at $\mathfrak{A} \oplus \mathfrak{B}$, for which the depth is always 0.

Putting it together

Originally, we define the *length* of a tunnel by:

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The *length* of a tunnel τ is the maximum of its reach and its depth.

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A better, equivalent, synthetic quantity, however, is:

Definition (Latrémolière, 2014)

Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a tunnel between two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. The *extent* $\chi(\tau)$ *of* τ is:

$$\max\Big\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}\left(\mathscr{S}(\mathfrak{A})\right),\right) \\ \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}\left(\mathscr{S}(\mathfrak{B})\right)\right)\Big\}.$$

The Dual Propinquity

We can define a new distance between Leibniz quantum compact metric spaces:

Definition (Latrémolière, 2013, 2014)

The *dual propinquity* $\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ between two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is:

$$\inf \left\{ \chi(\tau) \middle| \tau \text{ is a tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \right\}.$$

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We originally defined the dual propinquity in terms of lengths of tunnels, though this requires more care; the resulting metrics are equivalent.

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We may restrict our attention to some specific classes of tunnels, and define specialized versions of the dual propinquity, e.g. to compact C*-metric spaces.

Triangle Inequality

Theorem (Latrémolière, 2014)

For all Leibniz quantum compact metric spaces $(\mathfrak{A}_1, \mathsf{L}_1)$, $(\mathfrak{A}_2, \mathsf{L}_2)$ *and* $(\mathfrak{A}_3, \mathsf{L}_3)$, *we have:*

$$\Lambda^*((\mathfrak{A}_1, \mathsf{L}_1), (\mathfrak{A}_3, \mathsf{L}_3)) \leqslant \Lambda^*((\mathfrak{A}_1, \mathsf{L}_1), (\mathfrak{A}_2, \mathsf{L}_2)) + \Lambda^*((\mathfrak{A}_2, \mathsf{L}_2), (\mathfrak{A}_3, \mathsf{L}_3)).$$

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Proof.

Let $\tau_{12} = (D_{12}, \mathsf{L}_{12}, \pi_1, \pi_2)$ be a tunnel from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$ and $\tau_{23} = (\mathfrak{D}_{23}, \mathsf{L}_{23}, \rho_2, \rho_3)$ be a tunnel from $(\mathfrak{A}_2, \mathsf{L}_2)$ to $(\mathfrak{A}_3, \mathsf{L}_3)$.

∎

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For all Leibniz quantum compact metric spaces $(\mathfrak{A}_1, \mathsf{L}_1)$, $(\mathfrak{A}_2, \mathsf{L}_2)$ and $(\mathfrak{A}_3, \mathsf{L}_3)$, we have:

$$\begin{aligned}\Lambda^*((\mathfrak{A}_1, \mathsf{L}_1), (\mathfrak{A}_3, \mathsf{L}_3)) \leqslant\ & \Lambda^*((\mathfrak{A}_1, \mathsf{L}_1), (\mathfrak{A}_2, \mathsf{L}_2)) \\ & + \Lambda^*((\mathfrak{A}_2, \mathsf{L}_2), (\mathfrak{A}_3, \mathsf{L}_3)).\end{aligned}$$

Proof.

Let $\mathfrak{D} = \mathfrak{D}_{12} \oplus \mathfrak{D}_{23}$. For all $\varepsilon > 0$, set $\mathsf{L}_\varepsilon(d_{12}, d_{23})$ as:

$$\max\left\{\mathsf{L}_{12}(d_{12}), \mathsf{L}_{23}(d_{23}), \frac{1}{\varepsilon}\|\pi_2(d_{12}) - \rho_2(d_{23})\|_{\mathfrak{A}_3}\right\}$$

for all $d_{12} \in \mathfrak{sa}(\mathfrak{D}_{12}), d_{23} \in \mathfrak{sa}(\mathfrak{D}_{23})$.

□

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Proof.

For all $\varepsilon > 0$, we check that $\tau_\varepsilon = (\mathfrak{D}_{12} \oplus \mathfrak{D}_{23}, \mathsf{L}_\varepsilon, \pi_1, \rho_3)$ is a tunnel from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_3, \mathsf{L}_3)$ with:

$$\chi(\tau_\varepsilon) \leqslant \chi(\tau_{12}) + \chi(\tau_{23}) + \varepsilon.$$

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Proof.

We conclude by choosing τ_{12} and τ_{23} such that

$$\chi(\tau_{12}) \leqslant \Lambda^*((\mathfrak{A}_1, \mathsf{L}_1), (\mathfrak{A}_2, \mathsf{L}_2)) + \varepsilon$$

and $\chi(\tau_{23}) \leqslant \Lambda^*((\mathfrak{A}_2, \mathsf{L}_2), (\mathfrak{A}_3, \mathsf{L}_3)) + \varepsilon$, then take the infimum over ε. □

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Proof.

Comment: the tunnels $\mathfrak{D}_\varepsilon$ are not in general of the form $(\mathfrak{A}_1 \oplus \mathfrak{A}_3, \ldots)$. To form such a tunnel would require taking a quotient, and this is why triangle inequality fails, for instance, with Rieffel's proximity, or the quantum Gromov-Hausdorff distance involves non-Leibniz seminorms.

□

Distance Zero

Theorem (Latrémolière, 2013)

For any two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$*:*

$$\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

*if and only if there exists a *-isomorphism* $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ *such that* $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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Proof.

Fix $\varepsilon > 0$ and let $\tau_\varepsilon = (\mathfrak{D}_\varepsilon, \mathsf{L}_\varepsilon, \pi_{\mathfrak{A}}^\varepsilon, \pi_{\mathfrak{B}}^\varepsilon)$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ of extent ε or less.

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Proof.

For any $a \in \mathfrak{sa}(\mathfrak{A})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, introduce the sets:

$$\mathfrak{l}_{\tau_\varepsilon}(a|l) = \{d \in \mathfrak{sa}(\mathfrak{D}_\varepsilon) : \pi_{\mathfrak{A}}^{\varepsilon}(d) = a, \mathsf{L}_\varepsilon(d) \leqslant l\},$$

and

$$\mathfrak{t}_{\tau_\varepsilon}(a|l) = \pi_{\mathfrak{B}}^{\varepsilon}(\mathfrak{l}_{\tau_\varepsilon}(a|l)).$$

□

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Proof.

The target sets $\mathfrak{t}_{\tau_\varepsilon}(a|l)$ are sort of an image of a for τ_ε. If $\varphi \in \mathscr{S}(\mathfrak{D}_\varepsilon)$ and $d \in \mathfrak{t}_{\tau_\varepsilon}(a|l)$ then there exists $\psi \in \mathscr{S}(\mathfrak{A})$ such that $\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}(\varphi, \psi \circ \pi_{\mathfrak{A}}) \leqslant \chi(\tau)$. Then:

$$|\varphi(d)| \leqslant |\varphi(d) + \psi \circ \pi_{\mathfrak{A}}(d)| + |\psi(a)| \leqslant l\chi(\tau_\varepsilon) + \|a\|_{\mathfrak{A}}.$$

□

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*if and only if there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.*

Proof.

One then deduces that:

$$\operatorname{diam}\left(\mathfrak{t}_{\tau_\varepsilon}(a|l), \|\cdot\|_{\mathfrak{B}}\right) \leqslant l\chi\left(\tau_\varepsilon\right) \leqslant l\varepsilon.$$

and $\mathfrak{t}_\tau(a|l)$ is a compact subset of the norm compact set $\{b \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}(b) \leqslant 1, \|b\| \leqslant \|a\| + 1\}$. □

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Theorem (Latrémolière, 2013)

For any two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$:

$$\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

*if and only if there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.*

Proof.

Thus $(\mathrm{t}_{\tau_\varepsilon}(a|l))_{\varepsilon>0}$ admits a converging subnet for the Hausdorff distance induced by $\|\cdot\|_{\mathfrak{B}}$, whose limit is a singleton. We can use a diagonal argument and our norm estimates to remove the dependence of the subnet on a and l. This defines a map π from $\mathfrak{A}$ to $\mathfrak{B}$.

□

Distance Zero

Theorem (Latrémolière, 2013)

For any two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$*:*

$$\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

*if and only if there exists a *-isomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ *such that* $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Proof.

The multiplicative property of π requires the norm estimate for $\mathfrak{l}_a\,(l|r)$, while the linearity does not.

∎

Comparison with the quantum Gromov-Hausdorff distance

We established:

Theorem (Latrémolière, 2013)

For any two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$*:*

$$\mathsf{dist}_q((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})).$$

Moreover, if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) = (C(X), \mathsf{L}_X)$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) = (C(Y), \mathsf{L}_Y)$ *where* X, Y *are compact metric spaces and* L_X *and* L_Y *are Lipschitz seminorms, then:*

$$\Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \mathrm{GH}(X, Y).$$

Thus the dual propinquity is an analogue of the Gromov-Hausdorff distance.

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Completeness

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Proof.

It is sufficient to work with a sequence $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ of Leibniz quantum compact metric spaces such that for all $n \in \mathbb{N}$ there exists $\tau_n = (\mathfrak{D}_n, \mathsf{L}^n, \pi_n, \rho_n)$ with:

$$\sum_{n=0}^{\infty} \lambda(\tau_n) < \infty.$$

For any $d = (d_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{sa}(\mathfrak{D}_n)$, we set:

$$\mathsf{S}(d) = \sup\{\mathsf{L}^n(d_n) : n \in \mathbb{N}\}.$$

□

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Proof.

Let

$$\mathfrak{L} = \left\{ (d_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{sa}\left(\mathfrak{D}_n\right) : \begin{array}{l} \forall n \in \mathbb{N} \\ \pi_{n+1}(d_n) = \rho_n(d_{n+1}) \\ \mathsf{S}\left((d_n)_{n\in\mathbb{N}}\right) < \infty \end{array} \right\}.$$

Let $\mathfrak{F}$ be the C*-algebra spanned by $\mathfrak{L}$ in $\prod_{n\in\mathbb{N}} \mathfrak{D}_n$ and:

$$\mathfrak{I} = \{(d_n)_{n\in\mathbb{N}} \in \mathfrak{F} : \lim_{n\to\infty} \|d_n\|_{\mathfrak{D}_n} = 0\}.$$

Our candidate for a limit to $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ is $\mathfrak{F}/\mathfrak{I}$. □

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Proof.

If $\varepsilon > 0$ and $d_n \in \mathfrak{sa}\left(\mathfrak{D}_n\right)$ for some $n \in \mathbb{N}$ with $\mathsf{L}^n(d_n) < \infty$ then we can find $d = (d_m)_{,\in\mathbb{N}}$ with $\mathsf{L}^n(d_n) \leqslant \mathsf{S}(d) \leqslant \mathsf{L}^n(d_n) + \frac{1}{2}\varepsilon$ and

$$\|d\|_{\mathfrak{F}} \leqslant \|d_n\|_{\mathfrak{D}_n} + 2\left(\mathsf{L}^n(d_n) + \varepsilon\right) \sum_{n=0}^{\infty} \lambda\left(\tau_n\right).$$

If $a_{n+1} = \omega_n(d_n)$, then there exists d_{n+1} in $\mathfrak{D}_{n+1}$ with $\mathsf{L}^{n+1}(d_{n+1}) \leqslant \mathsf{L}_{n+1}(a_{n+1}) + \frac{1}{2}\varepsilon$ and $\|d_{n+1}\|_{\mathfrak{D}_{n+1}} \leqslant \|a_{n+1}\|_{\mathfrak{A}_{n+1}} + 2(\mathsf{L}_{n+1}(a_{n+1}) + \varepsilon)$. Now $\mathsf{L}_{n+1}(a_{n+1}) \leqslant \mathsf{L}^n(d_n)$. □

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Proof.

We may use our lifting lemma to show for $m \in \mathbb{N}$:

- the map $(d_n)_{n\in\mathbb{N}} \in \mathfrak{F} \mapsto d_m \in \mathfrak{D}_m$ is a *-epimorphism,
- the Lip-norms L^m are quotient of S.

We then get two estimates:

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_n}}\left(\mathscr{S}(\mathfrak{A}_{n+1}), \mathscr{S}(\mathfrak{D}_n)\right) \leqslant 2\lambda\left(\tau_n\right)$$

and

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_n}}\left(\mathscr{S}(\mathfrak{D}_n), \mathscr{S}(\mathfrak{D}_{n+1})\right) \leqslant 2\max\left\{\lambda\left(\tau_n\right), \lambda\left(\tau_{n+1}\right)\right\}.$$

∎

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Proof.

We need a few technical lemmas to show that:

$$\operatorname{diam}\left(\mathscr{S}(\mathfrak{F}), \mathsf{mk}_{\mathsf{S}}\right) < \infty.$$

From this, we then can prove that $(\mathfrak{F}, \mathsf{S})$ is a Leibniz quantum compact metric space.

Using Blaschke selection theorem and our estimates, the sequences $(\mathscr{S}(\mathfrak{A}_n))_{n\in\mathbb{N}}$ and $(\mathscr{S}(\mathfrak{D}_n))_{n\in\mathbb{N}}$ converge to some weak* compact convex Z in $(\mathscr{S}(\mathfrak{F}), \mathsf{mk}_{\mathsf{S}})$.

□

Completeness

Theorem (Latrémolière, 2013)

The dual propinquity is complete.

Proof.

We now identify Z with the state space of $\mathfrak{F}/\mathfrak{J}$. Last, we endow $\mathfrak{F}/\mathfrak{J}$ with the quotient of S, which is a Lip-norm. However, *why* is it a Leibniz Lip-norm?

This is shown by truncating sequences in $\mathfrak{F}$ which all map to the same element in $\mathfrak{F}/\mathfrak{J}$. ∎

GPS

Bridges and a new distance

For any two Leibniz quantum compact metric spaces, a bridge $\gamma = (\mathfrak{D}, \omega, \rho_{\mathfrak{A}}, \rho_{\mathfrak{B}})$ provides the ingredients for a tunnel, if we can find $\lambda > 0$ such that:

$$a, b \mapsto \max\left\{ \mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda}\|\rho_1(a)\omega - \omega\rho_2(b)\|_{\mathfrak{D}} \right\}$$

is admissible, and in particular, defines a tunnel.

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Two Questions

1. How do we compute a possible $\lambda > 0$?

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is admissible, and in particular, defines a tunnel.

Two Questions

1. How do we compute a possible $\lambda > 0$?
2. What is the extent of the associated tunnel, as a function of $\lambda > 0$?

A distance from bridges: height

Let $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

Definition (F. Latrémolière, 2013)

The 1-level set $\mathscr{S}(\mathfrak{D}, \omega)$ of ω is:

$$\mathscr{S}(\mathfrak{D}, \omega) = \left\{ \varphi \in \mathscr{S}(\mathfrak{D}) \middle| \begin{array}{l} \varphi((1-\omega)^*(1-\omega)) = 0, \\ \varphi((1-\omega)(1-\omega)^*) = 0 \end{array} \right\}.$$

Our definition of bridge includes the requirement that this set is non-empty for the pivot of the bridge, to avoid trivialities.

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Our definition of bridge includes the requirement that this set is non-empty for the pivot of the bridge, to avoid trivialities.

The first quantity associated with bridges measure how much of an error we make by replacing the state space of $\mathfrak{A}$ or $\mathfrak{B}$ by the images of the 1-level set.

A distance from bridges: height

Let $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. We thus introduce:

Definition (Latrémolière, 2013)

The *height* of γ is the maximum of:

$$\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\{\varphi \circ \pi_{\mathfrak{A}} : \varphi \in \mathscr{S}(\mathfrak{D}, \omega)\}, \mathscr{S}(\mathfrak{A})\right)$$

and the same quantity for $\mathfrak{B}$ in place of $\mathfrak{A}$.

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and the same quantity for $\mathfrak{B}$ in place of $\mathfrak{A}$.

The next quantity we compute from bridges measure how far $\mathfrak{A}$ and $\mathfrak{B}$ are from the perspective of the bridge seminorm.

A distance from bridges: reach

Let $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

Definition (Latrémolière, 2013)

The *reach* of the bridge γ is the Hausdorff distance in $\mathfrak{D}$ between:

$$\{\pi_{\mathfrak{A}}(a)\omega \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\} \text{ and } \{\omega\pi_{\mathfrak{B}}(b) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}.$$

A distance from bridges: reach

Let $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

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The reach informs us, informally, on how far the images of the level set of ω in $\mathscr{S}(\mathfrak{A})$ and $\mathscr{S}(\mathfrak{B})$ are. It is, in some sense, the distance between the images of the Lip-balls for the bride seminorm:

$$\mathsf{bn}_{\gamma}(\cdot) : d_1, d_2 \in \mathfrak{D} \oplus \mathfrak{D} \mapsto \|d_1\omega - \omega d_2\|_{\mathfrak{D}}.$$

The Quantum Propinquity

We can use the reach and height of a bridge to define a new metric between Leibniz quantum compact metric spaces, or a tunnel.

Definition (Latrémolière, 2013)

The *length* of a bridge is the maximum of its reach and height.

The Quantum Propinquity

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We could try to define the distance between two Leibniz quantum compact metric spaces as the infimum of the lengths of all bridges between them. Yet this fails to satisfies the triangle inequality.

The Quantum Propinquity

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Definition (Latrémolière, 2013)

The *length* of a bridge is the maximum of its reach and height.

Instead, we define a *trek* between two Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a finite path of bridges $\tau_1, \tau_2, \ldots, \tau_n$ where τ_j ends where τ_{j+1} starts, and τ_1 starts at $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ while τ_n ends at $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. The *length* of a trek is the sum of the lengths of its paths.

The Quantum Propinquity

We can use the reach and height of a bridge to define a new metric between Leibniz quantum compact metric spaces, or a tunnel.

Definition (Latrémolière, 2013)

The *length* of a bridge is the maximum of its reach and height.

Definition (Latrémolière, 2013)

The infimum of the length of all treks from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a called the *quantum propinquity* between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

The Quantum Propinquity as a distance

Theorem (Latrémolière, 2013)

The quantum propinquity is a metric on the class of Leibniz quantum compact metric spaces which dominates the dual propinquity, and its restriction to the classical compact metric spaces is dominated by the Gromov-Hausdorff distance.

The Quantum Propinquity as a distance

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The quantum propinquity is a metric on the class of Leibniz quantum compact metric spaces which dominates the dual propinquity, and its restriction to the classical compact metric spaces is dominated by the Gromov-Hausdorff distance.

Proof of the comparison to the dual propinquity.

Given a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ of nonzero length $\lambda(\gamma) > 0$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, if:

$$\mathsf{L} : (a, b) \mapsto \max\left\{ \mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\lambda(\gamma)} \|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}} \right\}$$

then $(\mathfrak{A} \oplus \mathfrak{B}, \mathsf{L}, \iota_{\mathfrak{A}}, \iota_{\mathfrak{B}})$ is a tunnel of length λ, where $\iota_{\mathfrak{A}}, \iota_{\mathfrak{B}}$ are the canonical surjections. □

Using Bridges for Quantum Tori

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}_*^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n\mathbb{Z}^d$ such that:

1. $\lim_{n\to\infty} k_n = (\infty, \ldots, \infty)$,
2. *the unique lifts of σ_n to $\mathbb{Z}^d$ as multipliers converge pointwise to σ.*

Then:

$$\lim_{n\to\infty} \Lambda^* \left(C^*\left(\mathbb{Z}^d, \sigma\right), C^*\left(\mathbb{Z}^d_{k_n}, \sigma_n\right)\right) = 0.$$

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Notes on the proof.

This result strengthens our result for dist_q.

□

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Notes on the proof.

One approach is to use our old techniques and the unital nuclear distance (Kerr, Li). This relies on Blanchard's subtrivialization result — complicated. □

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Notes on the proof.

A somewhat more explicit approach uses the left regular representation, or sum of such, on $\ell^2(\mathbb{Z}^d)$. □

Using Bridges for Quantum Tori

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Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}_*^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n\mathbb{Z}^d$ such that:

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Notes on the proof.

We construct bridges $(\ell^2(\mathbb{Z}^d), T, \pi, \rho)$ between quantum or fuzzy tori, with π and ρ left regular representations (or sums) and T trace class, diagonal in the canonical basis. □

Using Bridges for Quantum Tori

Theorem (Latrémolière, 2013)

Let $d \in \mathbb{N} \setminus \{0,1\}$, σ a multiplier of $\mathbb{Z}^d$. For each $n \in \mathbb{N}$, let $k_n \in \overline{\mathbb{N}}_*^d$ and σ_n be a multiplier of $\mathbb{Z}_k^d = \mathbb{Z}^d / k_n \mathbb{Z}^d$ such that:

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Notes on the proof.

While we use estimates from our original work, we can not simply "truncate" elements using Fejer kernels, as we wish to stay within the C*-category.

□

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Notes on the proof.

Bridges, and in particular T, replaces, to a large extent, this truncation process. □

GPS

Escape at Infinity

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1. it is not a metric as it may be infinite,

Proof.

Let δ_x denote the Dirac measure at $x \in \mathbb{R}$. Let L be the Lipschitz seminorm associated with the usual metric on $\mathbb{R}$.

$$\mathsf{mk}_{\mathsf{L}}\left(\delta_0, \sum_{n\in\mathbb{N}} 2^{-n-1}\delta_{2^{2n}}\right) = \infty.$$

□

Escape at Infinity

For a non-compact locally compact metric space (X, m), the Monge-Kantorovich metric is less well-behaved:

1. it is not a metric as it may be infinite,
2. it does not metrize the weak* topology, even on closed balls,

Proof.

Working in $\mathbb{R}$ again, we have:

$$\forall n \in \mathbb{N} \quad \mathsf{mk}_{\mathsf{L}}\left(\delta_0, \frac{n}{n+1}\delta_0 + \frac{1}{n+1}\delta_{n+1}\right) = 1$$

yet $\left(\delta_0, \frac{n}{n+1}\delta_0 + \frac{1}{n+1}\delta_{n+1}\right)_{n\in\mathbb{N}}$ weak* converges to δ_0. □

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Problems 1,2,3 are attributable to one main feature of the non-compact case: probability measures can escape at infinity.

Moreover, the restriction of the Monge-Kantorovich metric to pure states is not the original metric in general. The natural context for the Monge-Kantorovich metric consists of the *proper metric spaces*.

A first approach

Definition (Latrémolière, 2007)

The *bounded-Lipschitz distance* bl_{L} associated with a Lipschitz pair $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ as:

$$\sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\right\}.$$

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Theorem (Latrémolière, 2007)

Let $(\mathfrak{A}, \mathsf{L})$ be a Lipschitz pair and let:

$$\mathfrak{B} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1 \text{ and } \|a\|_{\mathfrak{A}} \leqslant 1\}.$$

Then the following are equivalent:

1. bl_{L} *metrizes the weak* topology of* $\mathscr{S}(\mathfrak{A})$,
2. *For some* $h \in \mathfrak{A}, h > 0$ *the set* $h\mathfrak{B}h$ *is norm precompact,*
3. *For all* $h \in \mathfrak{A}, h > 0$, *the set* $h\mathfrak{B}h$ *is norm precompact.*

A first approach

Definition (Latrémolière, 2007)

The *bounded-Lipschitz distance* bl_{L} associated with a Lipschitz pair $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is defined for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ as:

$$\sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\right\}.$$

- This notion was used, for instance, by Bellissard, Marcolli, Reihani (2010) for the study of metric properties of spectral triples over C*-crossed-products by $\mathbb{Z}$.
- This notion was also used in mathematical physics (J. Wallet, Cagnache-d'Andrea-Martinetti)

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However...

The bounded-Lipschitz distance only sees the space "locally", i.e. balls of a radius above 1 are the whole space. We still wish to understand the Monge-Kantorovich metric. We are back to: *How do we control behavior at infinity?* This was unsolved for more than a decade!

Dobrushin's tightness

Dobrushin discovered a sufficient condition for metrizing the weak* topology on well-behaved sets of probability measures:

Theorem (Dobrushin, 1970)

Let (X,d) be a (locally compact) metric space. If a subset $\mathscr{T}$ of $\mathscr{S}(C_0(X))$ satisfies for some $x_0 \in X$:

$$\lim_{r\to\infty} \sup\left\{ \int_{x:d(x,x_0)>r} d(x_0,x)\, d\mathbb{P}(x) : \mathbb{P} \in \mathscr{T} \right\} = 0$$

then the weak topology restricted to $\mathscr{T}$ is metrized by the Monge-Kantorovich metric associated to the Lipschitz seminorm for d.*

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then the weak topology restricted to $\mathscr{T}$ is metrized by the Monge-Kantorovich metric associated to the Lipschitz seminorm for d.*

It is *very challenging* to extend this notion to the noncommutative setting.

Quantum Topographic Spaces

Definition (Latrémolière, 2012)

A *Lipschitz triple* $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a Lipschitz pair $(\mathfrak{A}, \mathsf{L})$ and an Abelian C*-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ containing an approximate unit for $\mathfrak{A}$.

Let $\mathcal{K}(\mathfrak{M})$ be the collection of all compact subsets of the Gel'fand spectrum of $\mathfrak{M}$ and χ_K be the indicator function of K in $\mathfrak{M}$.

Definition (Latrémolière, 2012)

A subset $\mathscr{T}$ of the state space $\mathscr{S}(\mathfrak{A})$ of a Lipschitz triple $(\mathfrak{A}, \mathfrak{M}, \mathsf{L})$ is *tame* when there exists $\mu \in \mathscr{S}(\mathfrak{A})$ and $C \in \mathcal{K}(\mathfrak{M})$ such that $\mu(\chi_C) = 1$ and:

$$\lim_{K \in \mathcal{K}(\mathfrak{M})} \sup \left\{ |\varphi(a - \chi_K a \chi_K)| : \begin{array}{l} \varphi \in \mathscr{T}, a \in \mathfrak{sa}\left(\mathfrak{u}\mathfrak{A}\right) \\ \mathsf{L}(a) \leqslant 1, \mu(a) = 0 \end{array} \right\} = 0.$$

Quantum Locally Compact Metric Spaces

Definition (Latrémolière, 2012)

A *quantum locally compact metric space* is a Lipschitz triple such that:

1. For all $K \in \mathcal{K}(\mathfrak{M})$, the set $\{\varphi \in \mathscr{S}(\mathfrak{A}) : \varphi(\chi_K) = 1\}$ has finite radius for mk_{L},
2. The topology induced on every tame subset of $\mathscr{S}(\mathfrak{A})$ by mk_{L} is the weak* topology.

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Example (Latrémolière, 2012)

If $(C(\mathbb{R}^2_\sigma), L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, D)$ is the Gayal, Gracia-Bondia, Iochum, Schücker, Varilly spectral triple over the Moyal plane $C(\mathbb{R}^2_\sigma)$, then $(C(R^2_\sigma), \mathsf{L}, \mathfrak{M}_\sigma)$ is a quantum locally compact metric space for $\mathfrak{M}_\sigma$ generated by the Harmonic oscillator basis projections and $\mathsf{L}(a) = \|[D,a]\|$ $(a \in C(\mathbb{R}^2_\sigma))$.

Characterization of quantum locally compact metric spaces

Theorem (Latrémolière, 2012)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ be a Lipschitz triple. The following are equivalent:

1. $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a quantum locally compact metric space,
2. There exists a state $\mu \in \mathscr{S}(\mathfrak{A})$, $K \in \mathcal{K}(\mathfrak{M})$ with $\mu(K) = 1$ such that for all compactly supported $a, b \in \mathfrak{M}$, the set:
$$\{acb : c \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A}), \mathsf{L}(c) \leqslant 1, \mu(c) = 0\}$$
is norm precompact,
3. For all states $\mu \in \mathscr{S}(\mathfrak{A})$ for which there exists $K \in \mathcal{K}(\mathfrak{M})$ with $\mu(K) = 1$, and for all compactly supported $a, b \in \mathfrak{M}$, the set $\{acb : c \in \mathfrak{sa}(\mathfrak{u}\mathfrak{A}), \mathsf{L}(c) \leqslant 1, \mu(c) = 0\}$ is norm precompact.

GPS

1. *Quantum Compact Metric Spaces*
   - The Monge Kantorovich distance
   - Compact Quantum Metric Spaces
2. *The Gromov-Hausdorff Propinquity*
   - The quantum Gromov-Hausdorff distance
   - The dual propinquity
   - The Quantum Propinquity
3. *Locally Compact Quantum Metric Spaces*
   - Topographies
   - Convergence for locally compact quantum metric spaces

Proper Quantum Metric Spaces

An analogue of proper quantum metric spaces is given by:

Definition (Latrémolière, 2014)

A quantum locally compact metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ is a *strong proper quantum metric space* when:

1. L is lower semi-continuous,
2. L is Leibniz,
3. there exists a compactly supported approximate unit $(e_n)_{n\in\mathbb{N}}$ for $\mathfrak{A}$ in $\mathfrak{M}$ such that $\lim_{n\rightarrow\infty} \mathsf{L}(e_n) = 0$,
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4. the restriction of L to $\mathfrak{M}$ has a dense domain.

A pointed proper quantum metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M}, \mu)$ is a proper quantum metric space $(\mathfrak{A}, \mathsf{L}, \mathfrak{M})$ and a state μ of $\mathfrak{A}$ whose restriction to $\mathfrak{M}$ is pure.

Gromov-Hausdorff Convergence

We wish to define a notion of convergence for pointed proper quantum metric space which extends the original Gromov-Hausdorff convergence for pointed proper metric spaces.

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Definition (Gromov, 1981)

Let (X, x) and (Y, y) be two pointed proper metric spaces. Let δ_r be the infimum of $\varepsilon > 0$ such that for some isometric embeddings of X, Y in some Z then:

$$\begin{cases} \mathscr{B}_X(x, r) \subseteq_\varepsilon Y, \mathscr{B}_Y(y, r) \subseteq_\varepsilon X, \\ x \text{ and } y \text{ are within } \varepsilon \text{ in } Z. \end{cases}$$

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The Gromov-Hausdorff distance between (X,x) and (Y,y) is the infimum of $r > 0$ such that $\delta_{\varepsilon^{-1}} \leqslant \varepsilon$.

The problem of lifting Lipschitz functions

A difficulty in the locally compact concerns (even though McShane's theorem still holds, of course):

Lipschitz Extensions

If f is a 1-Lipschitz function on a locally compact metric space which vanishes at infinity, then it may not have a 1-Lipschitz extension which vanishes at infinity. For instance, if $X = (0,1) \times [0,1]$, and $Y = (0,1) \times \left\{\frac{1}{2}\right\}$, and if f is 2 on Y, then no extension of f is both 1-Lipschitz and vanish at infinity.

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We need to rework our notion of a tunnel to accommodate difficulties in lifting Lipschitz functions. The situation is manageable when working with proper metric spaces, but is surprising.

Some notations

Definition (Latrémolière, 2014)

Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$ be two pointed proper quantum metric spaces. A passage $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a quantum locally compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}})$ with two *-morphisms $\pi_{\mathfrak{A}} : \mathfrak{D} \twoheadrightarrow \mathfrak{A}$ and $\pi_{\mathfrak{B}} : \mathfrak{D} \twoheadrightarrow \mathfrak{B}$ mapping $\mathfrak{M}_{\mathfrak{D}}$ to $\mathfrak{M}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{B}}$ respectively..

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Definition (Latrémolière, 2014)

Let $(\mathfrak{A}, \mathsf{L}, \mathfrak{M}, \mu)$ be a pointed proper quantum metric space. For any compact K in the Gel'fand spectrum $\sigma(\mathfrak{M})$ of $\mathfrak{M}$, let p_K be the indicator function of K in $\mathfrak{A}^{**}$. If K is the closed ball centered at μ and radius r in $\sigma(\mathfrak{M})$ then p_K is also denoted by p_r. The elements $a \in \mathfrak{sa}(\mathfrak{A})$ such that $p_K a p_K = a$ are said to be locally supported.

Left Admissibility

Definition (Latrémolière, 2014)

Let $r > 0$. An *left r-admissible* pair (K, ε) is a compact K in $\sigma(\mathfrak{M}_{\mathfrak{D}})$ and $\varepsilon > 0$ such that for any $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$ and $p_r a p_r = a$, there exists $d \in \mathfrak{sa}(\mathfrak{D})$:

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4. We have:

$$\{\varphi \circ \pi_{\mathfrak{A}} : \varphi \in \mathscr{S}(\mathfrak{A}) : \varphi(p_r)\} \subseteq \{\varphi \in \mathscr{S}(\mathfrak{D}) : \varphi(p_K) = 1\}$$
$$\subseteq_{\varepsilon} \{\varphi \circ \pi_{\mathfrak{B}} : \varphi \in \mathscr{S}(\mathfrak{B}) : \varphi(p_r)\}.$$

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The notion of right admissibility is defined identically.

Admissibility and Extent

The notions of admissibility and extent are interdependent in this context.

Definition (Latrémolière, 2014)

Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{M}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a passage from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{M}_{\mathfrak{A}}, \mu_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}}, \mu_{\mathfrak{B}})$. A pair (K, ε) is r-admissible when it is both left and right r-admissible, while $\mathsf{L}_{\mathfrak{D}}$ restricts to a Leibniz Lip-norm on the K-locally supported elements of $\mathfrak{D}$, and is lower semi-continuous.

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Definition (Informal, Latrémolière, 2014)

The r-extent of a passage is the smallest $\varepsilon > 0$ such that (K, ε) is r-admissible for some compact K. A passage with a finite r-extent is called a r-tunnel.

The topographic Propinquity

Definition (Latrémolière, 2014)

Let $\mathbb{A}, \mathbb{B}$ be two pointed proper quantum metric spaces. The r-local propinquity $\Lambda^*{}_r(\mathbb{A}, \mathbb{B})$, for $r > 0$, between $\mathbb{A}$ and $\mathbb{B}$ is the infimum of the r-extents of r-tunnels between $\mathbb{A}$ and $\mathbb{B}$.

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The topographic Gromov-Hausdorff propinquity is an infra-metric which generalizes the dual propinquity, up to equivalence.

The topographic Propinquity as Inframetric

Theorem (Latrémolière, 2014)

Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$ be three pointed proper quantum metric spaces. Then:

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Thus we have a generalized Gromov-Hausdorff convergence for noncommutative geometry.

Thank you!

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