Cohomology of Lie algebra of Hamiltonian vector fields: experimental data, conjectures, and theorems

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I. M. Gelfand, D. I. Kalinin, and D. B. Fuchs, *On the cohomology of the Lie algebra of Hamiltonian vector fields*, Functional Anal. Appl., **6**:3 (1972), 193–196.

J. Perchik, *Cohomology of Hamiltonian and related formal vector fields Lie algebras*, Topology, **15**:4 (1976), 395–404.

The Lie algebra:

$$\mathfrak{H}_2 = \mathbb{C}(x,y)/\{\text{const}\};\ [P,Q] = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}$$

or

$$\mathfrak{H}_2 = \text{Span}\{e_{pq} \mid p \geq -1, q \geq -1, p+q \geq -1\},$$

$$[e_{pq}, e_{rs}] = \begin{vmatrix} p+1 & q+1 \\ r+1 & s+1 \end{vmatrix} e_{p+r,q+s}.$$

We consider the Lie algebra cohomology of $\mathfrak{H}_2$ with trivial coefficients, $H^n(\mathfrak{H}_2)$.

[*Definition*: for a (complex) Lie algebra $\mathfrak{g}$,

$$C^n(\mathfrak{g}) = \text{Hom}_{cont}(\Lambda^n \mathfrak{g}, \mathbb{C}),$$

the differential $d^n \colon C^n(\mathfrak{g}) \to C^n(\mathfrak{g})$ acts as

$$d^n c(g_1, \dots, g_{n+1}) = \sum_{1 \leq r < s \leq n+1} (-1)^{r+s-1} c([g_r, g_s], g_1, \dots \widehat{g}_r \dots \widehat{g}_s \dots, g_{n+1}),$$

and

$$H^n(\mathfrak{g}) = \text{Ker}\, d_n / \text{Im}\, d_{n-1}.]$$

The Lie algebra $\mathfrak{H}_2$ is bigraded:

$$\mathfrak{H}_2 = \bigoplus_{p,q} (\mathfrak{H}_2)_{pq}, \ \ (\mathfrak{H}_2)_{pq} = \mathbb{C}e_{pq}.$$

Accordingly, the cohomology is bigraded:

$$H^n(\mathfrak{H}_2) = \bigoplus_{p,q} H^n_{pq}(\mathfrak{H}_2).$$

TRIVIAL THEOREM. *If $p \neq q$, then $H^n_{pq}(\mathfrak{H}_2) = 0$ for all n.*

Thus, we need to consider only the cohomology $H^n_{qq}(\mathfrak{H}_2)$ which we will denote briefly as $H^n_q(\mathfrak{H}_2)$.

The cohomology $H^n_q(\mathfrak{H}_2)$ for small values of q can be easily calculated. Here are the results:

$$\dim H^n_{-1}(\mathfrak{H}_2) = \begin{cases} 1, & \text{if } n = 2, 5, \\ 0, & \text{if } n \neq 2, 5; \end{cases}$$

$$\dim H^n_0(\mathfrak{H}_2) = \begin{cases} 1, & \text{if } n = 0, 7, \\ 0, & \text{if } n \neq 0, 7; \end{cases}$$

$$H^n_q(\mathfrak{H}_2) = 0 \text{ for } q = 1, 2, 3 \text{ and all } n.$$

A short-living conjecture that $H_q^n(\mathfrak{H}_2) = 0$ for $q > 0$ was disproved in 1972 by I. M. Gelfand, D. I. Kalinin and myself. With the help of a computer of that epoch, we were able to prove the following:

THEOREM.

$$H_4^n(\mathfrak{H}_2) = \begin{cases} \mathbb{C} & \textit{for } n = 7, 11, \\ 0 & \textit{for } n \neq 7, 11; \end{cases}$$
$$H_5^n(\mathfrak{H}_2) = 0 \ \textit{for all } n,$$

and neither of $H_7^n(\mathfrak{H}_2), H_{11}^n(\mathfrak{H}_2)$ *can be zero for all* n*.*

A breakthrough in computation of the cohomology of the Lie algebra $\mathfrak{H}_2$ was achieved in 1976 by J. Perchik. His idea was to replace the hard problem of computing cohomology by the easier problem of computing the Euler characteristics. This idea cannot be applied to our problem directly, since the Euler characteristic of the complex $\{C^n_{qq}(\mathfrak{H}_2), d\}$ is always zero. To avoid this zero, one can replace the cohomology of the Lie algebra $\mathfrak{H}_2$ by the relative cohomology of this Lie algebra modulo $\mathfrak{sl}(2) = \mathrm{Span}(e_{-1,1}, e_{00}, e_{1,-1}) \subset \mathfrak{H}_2$.

[*Definition.* Let $\mathfrak{h}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. We define $C^n(\mathfrak{g}, \mathfrak{h})$ as the space of those $c \in C^n(\mathfrak{g})$ which are annihilated by every element of $\mathfrak{h}$ and whose differential dc has the same property. The cohomology $H^n(\mathfrak{g}, \mathfrak{h})$ of the complex $\{C^n(\mathfrak{g}, \mathfrak{h}), d\}$ is the relative cohomology of $\mathfrak{g}$ modulo $\mathfrak{h}$.]

Since $\mathfrak{sl}(2) \not\subset (\mathfrak{H}_2)_{00}$, but $\mathfrak{sl}(2) \subset (\mathfrak{H}_2)_0 = \bigoplus_{p+q=0} (\mathfrak{H}_2)_{pq}$, the relative complex $\{C^n(\mathfrak{H}_2, \mathfrak{sl}(2)), d\}$ as well as the relative cohomology $H^n(\mathfrak{H}_2, \mathfrak{sl}(2))$ do not keep the bigrading, but do preserve a grading: $H^n(\mathfrak{H}_2, \mathfrak{sl}(2)) = \oplus_r H^n_r(\mathfrak{H}_2, \mathfrak{sl}(2))$.

The standard tools of the Lie algebra cohomology theory make the problems of computing $H^n_{qq}(\mathfrak{H}_2)$ and $H^n_{2q}(\mathfrak{H}_2, \mathfrak{sl}(2))$ equivalent.

THEOREM (Perchik) *Let*

$$\mathcal{P}(t,x) = \prod_{\substack{-1 \le \alpha < \infty \\ \beta \equiv \alpha \bmod 2 \\ -\alpha - 2 \le \beta \le \alpha + 2 \\ (\alpha,\beta) \ne (0,0)}} (1 - t^{\beta} x^{\alpha});$$

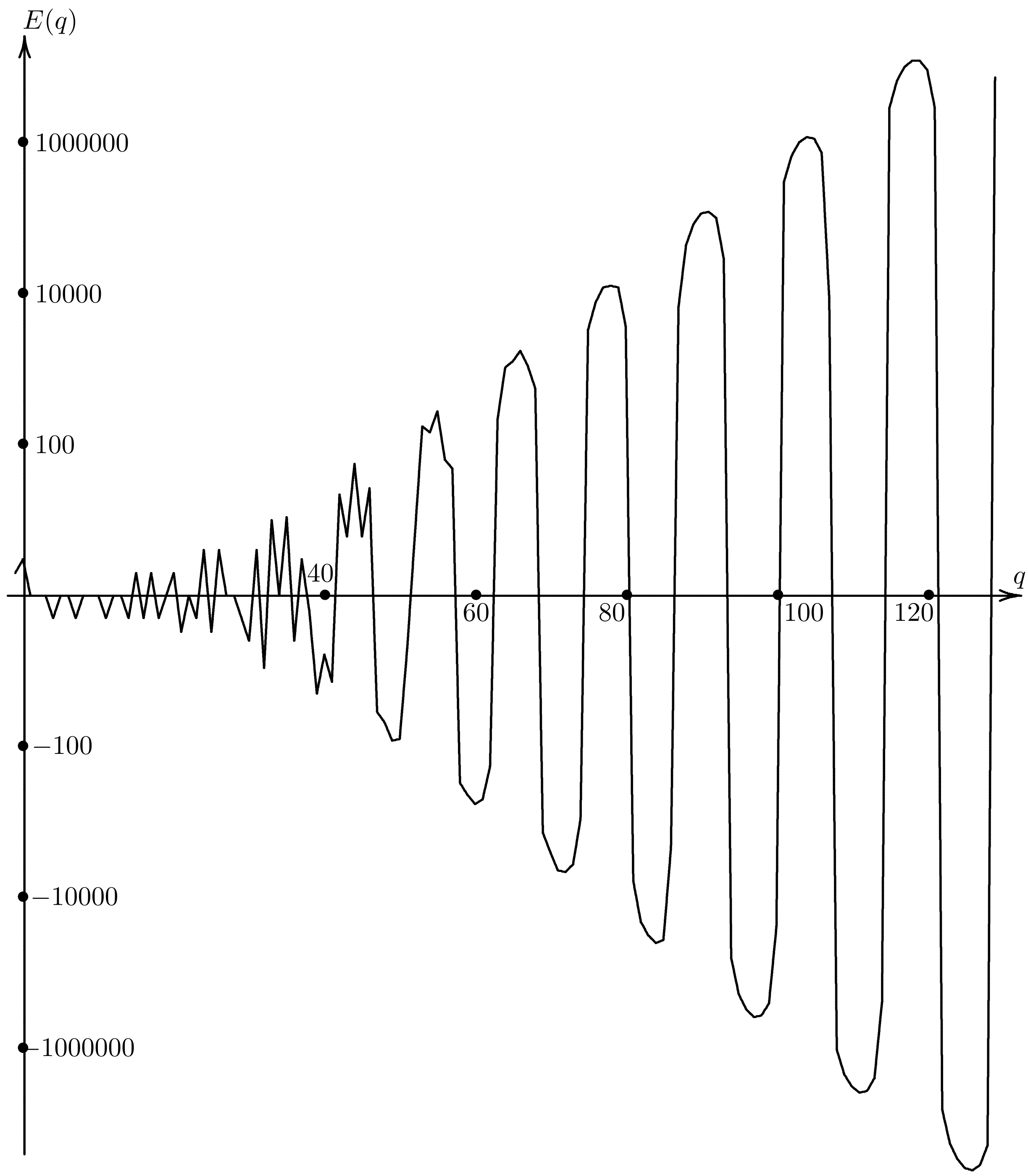
in other words, $\mathcal{P}(t,x)$ *is an infinite product*

$$\begin{array}{c} (1 - t^1 x^{-1})(1 - t x^{-1}) \\ (1 - t^{-2}) \qquad\qquad\qquad\qquad (1 - t^2) \\ (1 - t^{-3}x)\ (1 - t^{-1}x)\ (1 - tx)\ (1 - t^3 x) \\ (1 - t^{-4}x^2)(1 - t^{-2}x^2)(1 - x^2)(1 - t^2 x^2)(1 - t^4 x^2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

Then the Euler characteristic of the complex $\{C^n_{2q}(\mathfrak{H}_2, \mathfrak{sl}(2)), d\}$ *equals half the coefficient at* x^{2q} $(= t^0 x^{2q})$ *in the series* $\mathcal{P}(x,t)$.

q	$E(q)$	q	$E(q)$	q	$E(q)$	q	$E(q)$	q	$E(q)$
−1	1	26	3	53	173	80	3584	107	8541
0	2	27	0	54	145	81	−5935	108	−1028485
1	0	28	0	55	271	82	−21098	109	−2151079
2	0	29	−1	56	62	83	−31806	110	−3195064
3	0	30	−3	57	47	84	−40358	111	−3758464
4	−1	31	3	58	−303	85	−36595	112	−3619783
5	0	32	−8	59	−428	86	−2204	113	−2444842
6	0	33	9	60	−583	87	6624	114	−241084
7	−1	34	0	61	−508	88	44607	115	2909301
8	0	35	10	62	−182	89	82862	116	6509215
9	0	36	−3	63	217	90	114857	117	9863184
10	0	37	2	64	1027	91	119528	118	12052883
11	−1	38	−9	65	1252	92	97775	119	12061060
12	0	39	−20	66	1775	93	29235	120	9163196
13	0	40	−5	67	1079	94	−64387	121	2834605
14	−1	41	−13	68	560	95	−189286	122	−6466360
15	1	42	21	69	−1374	96	−295141	123	−17871392
16	−1	43	5	70	−2562	97	−372236	124	−29128258
17	1	44	54	71	−4303	98	−360596	125	−37864369
18	-1	45	5	72	−4480	99	−253826	126	−40696075
19	0	46	24	73	−3613	100	−23742	127	−35201663
20	1	47	−33	74	−920	101	296868	128	−19167388
21	−2	48	−47	75	3333	102	663644	129	7151485
22	0	49	−82	76	7579	103	995744		
23	−1	50	−79	77	12288	104	1175649		
24	3	51	−4	78	12866	105	1123744		
25	−2	52	5	79	11810	106	725381		

We see from this table that the absolute value of $E(q)$ becomes very large: within the table it reaches fourty million. But it becomes more surprising when one looks at the logarithmic graph of $E(q)$ shown in the next slide.



E(q)
1000000
10000
100
−100
−10000
−1000000
40
60
80
100
120
q

Conjecture. *For large values of q, the cohomology*

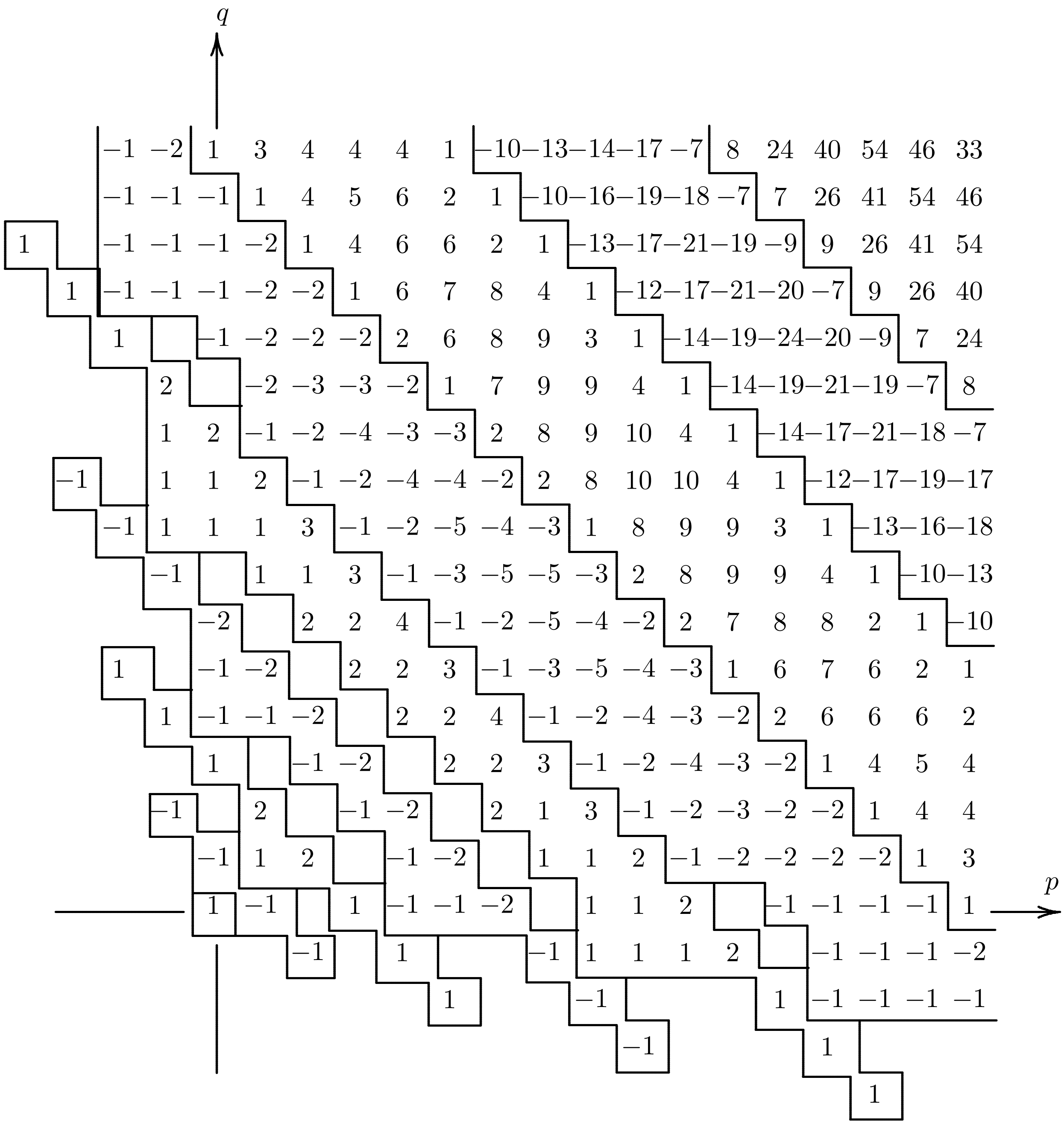
$$H^n_{2q}(\mathfrak{H}_2, \mathfrak{sl}(2))$$

is concentrated around one dimension, or to adjacent dimensions. With the growth of q, this dimension slowly grows. When it is even, the Euler characteristic is positive; when it becomes odd, the Euler characteristic becomes negative. In the intermediate zone, the Euler characteristic passes through zero and changes sign.

The total dimension of the cohomology grows exponentially with the growth of q.

For $q = 4$, the computations of Gelfand, Kalinin and myself show that the cohomology is concentrated in dimension 7. However, no other computations of $H^n_{2q}(\mathfrak{H}_2, \mathfrak{sl}(2))$ are known today.

Still, there are some results concerning the cohomology of the Lie algebra $L_1\mathfrak{H}_2 \subset \mathfrak{H}_2$ spanned by e_{pq} with $p+q \geq 1$. This Lie algebra is bigraded, so there are complexes $\{C^n_{pq}(L_1\mathfrak{H}_2), d\}$ and the cohomology $H^n_{pq}(L_1\mathfrak{H}_2)$. The table for the Euler characteristics $E(p, q)$ of these complexes is shown in the next slide.


q
−1 −2 1 3 4 4 4 1 −10 −13 −14 −17 −7 8 24 40 54 46 33
−1 −1 −1 1 4 5 6 2 1 −10 −16 −19 −18 −7 7 26 41 54 46
1 −1 −1 −1 −2 1 4 6 6 2 1 −13 −17 −21 −19 −9 9 26 41 54
1 −1 −1 −1 −2 −2 1 6 7 8 4 1 −12 −17 −21 −20 −7 9 26 40
1 −1 −2 −2 −2 2 6 8 9 3 1 −14 −19 −24 −20 −9 7 24
2 −2 −3 −3 −2 1 7 9 9 4 1 −14 −19 −21 −19 −7 8
1 2 −1 −2 −4 −3 −3 2 8 9 10 4 1 −14 −17 −21 −18 −7
−1 1 1 2 −1 −2 −4 −4 −2 2 8 10 10 4 1 −12 −17 −19 −17
−1 1 1 1 3 −1 −2 −5 −4 −3 1 8 9 9 3 1 −13 −16 −18
−1 1 1 3 −1 −3 −5 −5 −3 2 8 9 9 4 1 −10 −13
−2 2 2 4 −1 −2 −5 −4 −2 2 7 8 8 2 1 −10
1 −1 −2 2 2 3 −1 −3 −5 −4 −3 1 6 7 6 2 1
1 −1 −1 −2 2 2 4 −1 −2 −4 −3 −2 2 6 6 6 2
1 −1 −2 2 2 3 −1 −2 −4 −3 −2 1 4 5 4
−1 2 −1 −2 2 1 3 −1 −2 −3 −2 −2 1 4 4
−1 1 2 −1 −2 1 1 2 −1 −2 −2 −2 −2 1 3
1 −1 1 −1 −1 −2 1 1 2 −1 −1 −1 −1 1
p
−1 1 −1 1 1 1 2 −1 −1 −1 −2
1 −1 1 −1 −1 −1 −1
−1 1
1

There exists a computation of the cohomology $H^n_{pq}(L_1\mathfrak{H}_2)$ for $n = 0, 1, 2, 3$, and the results fairly correspond to the four alternating patches of positive and negative values of $E(p, q)$.

Conjecture. *This holds for all n.*

In conclusion, let me mention that the Euler characteristics $E(q)$ and $E(p, q)$ of the two tables displayed are closely related to each other. The actual formula is

$$\begin{aligned} E(q) = {} & E(q, q) + E(q + 1, q + 1) + E(q - 1, q + 2) \\ & - E(q - 1, q + 1) - E(q, q + 1) - E(q, q + 2). \end{aligned}$$