Arnold diffusion for convex nearly integrable systems

V. Kaloshin

November 24, 2014

Plan of the talk

- Motivation: Ergodic and quasiergodic hypothesis.
- Nearly integrable systems and the problem of Arnold diffusion
- Results in 3, 4, and more degrees of freedom
- Indication of Arnold diffusion in the Solar system
- Stochastic aspects of Arnold diffusion

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Motivation: Ergodic Hypothesis

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Let X_H be the Hamiltonian flow associated to H.

$$\begin{cases} \dot{q} = \partial_p H \\ \dot{p} = -\partial_q H \end{cases} \tag{1}$$

Let $S_E = \{(q, p) \in T^*M : H(q, p) = E\}$ be an energy surface.

Ergodic Hypothesis (Boltzmann, Maxwell) Is a generic Hamiltonian flow X_H on a generic energy surface S_E ergodic?

Numerical doubts (**Fermi-Pasta-Ulam**) Chains of nonlinear springs

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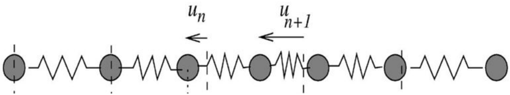

$$\ddot{u}_n = k(u_{n+1} - u_n) - k(u_n - u_{n-1}) + \alpha(u_{n+1} - u_n)^2 + \alpha(u_n - u_{n-1})^2$$

the α-term — nonlinearity. **Most** "small" solutions are **almost periodic**!

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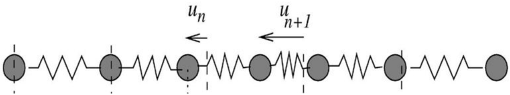

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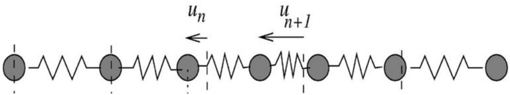

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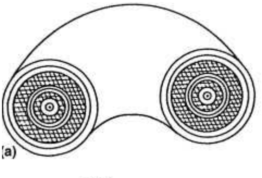
KAM theory Each *nearly integrable* systems has collections of invariant tori of positive measure $\Longrightarrow$ no ergodicity!



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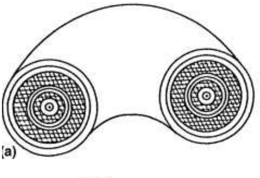

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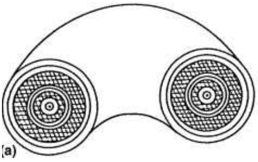

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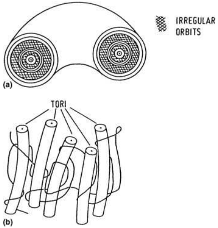

Integrable systems & action-angles coordinates

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian, $\varphi \in \mathbb{T}^n$ be angle, $I \in \mathbb{R}^n$ be action.

A Hamiltonian system is **Arnold-Liouville integrable** if for an open set $U \subset \mathbb{R}^n$ there exists a symplectic map $\Phi : \mathbb{T}^n \times U \to \mathbb{R}^{2n}$ s. t. $H \circ \Phi(\varphi, I)$ depends only on I and

$$\begin{cases} \dot{\varphi} = \partial_I (H \circ \Phi)(I) = \omega(I), \\ \dot{I} = 0. \end{cases}$$

(φ, I)**–action-angle coordinates**

In particular, $\Phi(\mathbb{T}^n \times U)$ is foliated by invariant n-dim'l tori & on each torus $\mathbb{T}^n$ the flow is linear.

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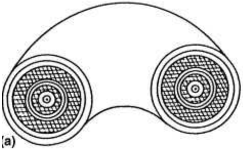

Integrable systems

- Newtonial two body problem.
- Pendulum $H = \frac{I^2}{2} - \cos 2\pi\varphi,\ (\varphi, I) \in T^*\mathbb{T} = \mathbb{T} \times \mathbb{R}$.
- Harmonic oscillator $\ddot{q} = -kq$ or $H = \frac{p^2}{2} + \frac{kq^2}{2}$.
- Motion in a central force field $\ddot{q} = F(\|q\|)q$.
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- Lagrange's top, Kovaleskaya's top, Euler top.
- Toda lattice: chain $\dots < x_0 < x_1 < \dots$ with the neighbor interaction $\sum_i \exp(x_i - x_{i+1})$
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Arnold diffusion

Arnold, 63: Let $(\varphi, I) \in T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n,\ t \in \mathbb{T}$.

- (weak form) Does there exist a real instability in many-dimensional problems of perturbation theory when the invariant tori do not divide the phase space? More precisely, for a generic perturbation $\varepsilon H_1(\varphi, I, t)$ the Hamiltonian

$$H_\varepsilon(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t)$$

has an orbit whose action component "travels" in action space, in particular, $\max_t \|I(t) - I(0)\| = O(1)$.

- (strong form) For any two open sets $U, U' \subset B^n$ the Hamiltonian $H_\varepsilon(\varphi, I, t)$ has an orbit whose action component "travels" from U to U', i.e. $I(0) \in U$ and $I(T) \in U'$ for some $T > 0$.

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Nearly integrable systems in dimension 2

Let $H_0(I) = \frac{I^2}{2}$. Time one map $(\varphi, I) \to (\varphi + I, I)$ (mod 1).

Let $H_\varepsilon(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t)$. The model time one map

$$f_\varepsilon : (\varphi, I) \to (\varphi', I') = (\varphi + I', I + \varepsilon \sin 2\pi\varphi) \text{ (mod 1)}.$$

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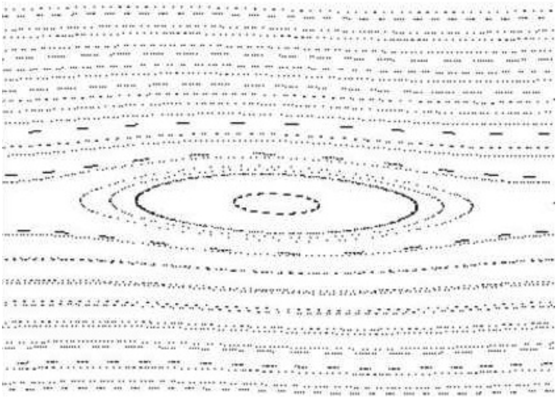
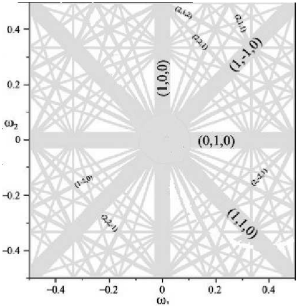


KAM Theorem, obstacles to instability

Let $H_0(I)$ have non-degenerate Hessian, e.g. $H_0(I) = \sum I_j^2/2$.



Width $\sim \sqrt{\varepsilon}$

KAM Theorem Let $H_\varepsilon(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t)$ be a smooth perturbation. Then with probability $1 - O(\sqrt{\varepsilon})$ has an initial condition in $\mathbb{T}^n \times B^n \times \mathbb{T}$ having a quasiperiodic orbit. Moreover, $\mathbb{T}^n \times B^n \times \mathbb{T}$ **with certain neighborhood of rational lines deleted** is laminated by invariant $(n+1)$-dimensional tori, one for each diophantine ω.

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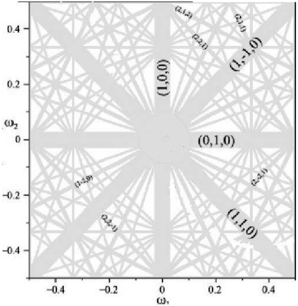

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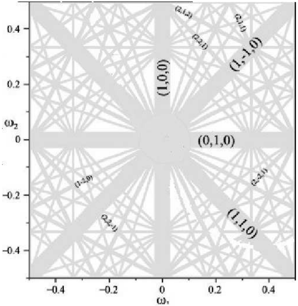

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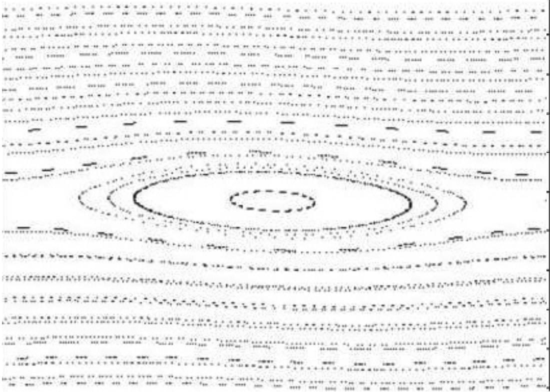

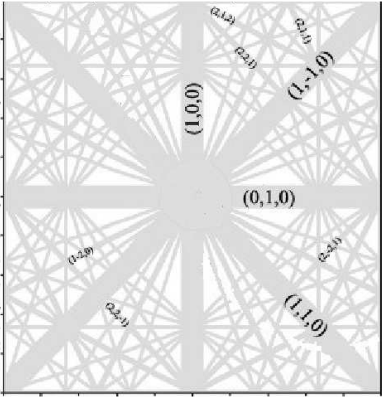

In $(2n+1)$-dimensional space there are $(n+1)$-dimensional tori.

For $n = 1$ they confine orbits!

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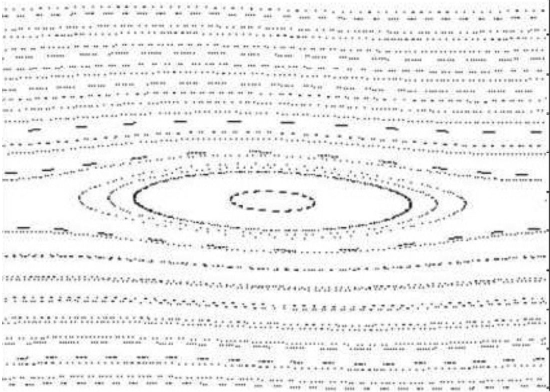

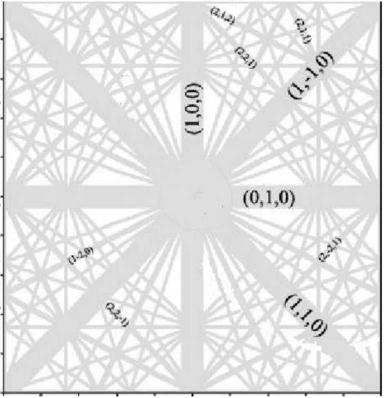

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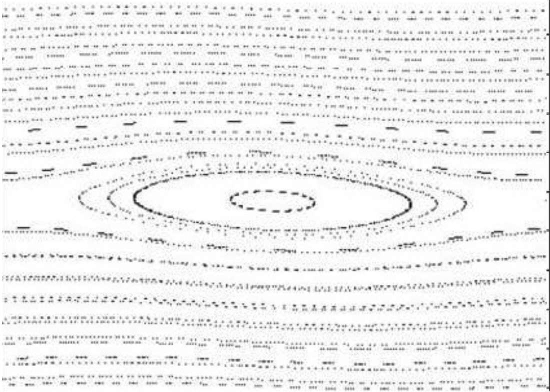

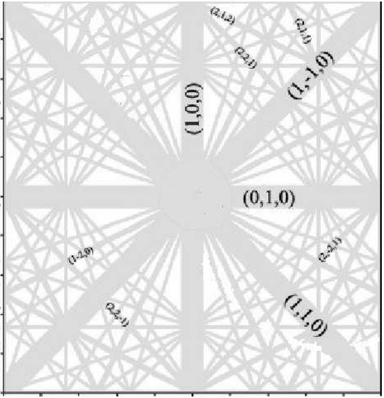

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A strong from of Arnold diffusion

Let $H_0(I)$ be smooth and strictly convex, $I \in B^n$.

The First Main Result For any $\gamma > 0$ & a generic smooth perturbation $\varepsilon H_1(\phi, I, t)$ the Hamiltonian

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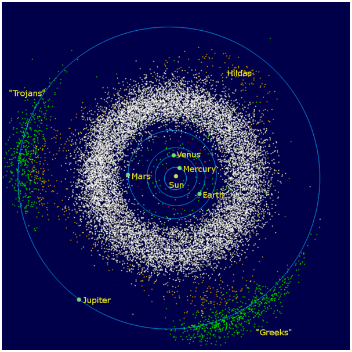
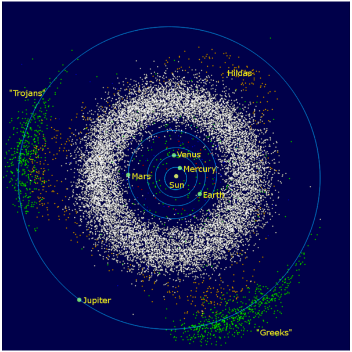
Instabilities in the Asteroid Belt





Moser: Is the Solar System Stable?
The Math Intelligencer, 78

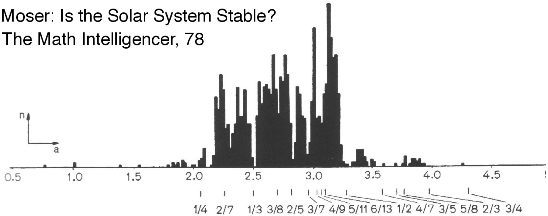


Fig. 5. The number of asteroids as a function of the semi-major axis a. The a-values corresponding to certain fractions of Jupiter's period are marked below. Some of these 'resonances' have produced gaps in the asteroid distribution.

J. Wisdom,83, Chaotic Behavior & the Origin of the 3/1 Kirkwood Gaps

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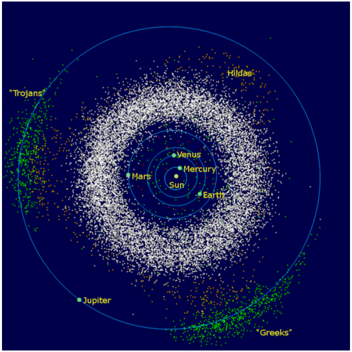

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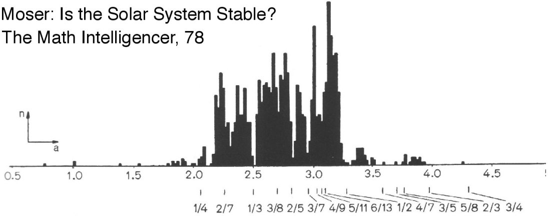

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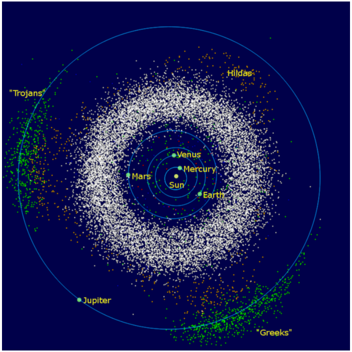

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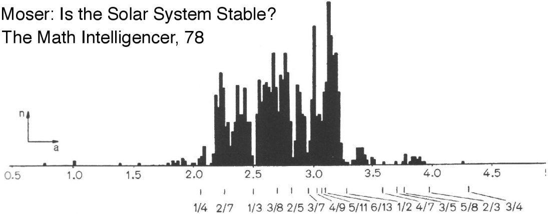

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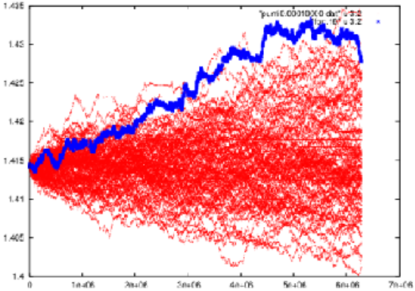
Diffusion conjecture Let

$$H_\varepsilon = \underbrace{\frac{I^2}{2}}_{\text{harm oscillator}} + \underbrace{\left(\frac{p^2}{2} + \cos q\right)}_{\text{pendulum}} + \varepsilon H_1(\varphi, I, q, p, t),\ \ \varphi, q, t \in \mathbb{T},\ \ I, p \in \mathbb{R},$$

where H_1 is a generic perturbation. Let Leb_ε be the norm Lebesgue measure on the $\sqrt{\varepsilon}$-ball around 0. Then $I(\frac{-t \cdot \ln \varepsilon}{\varepsilon^2})$ converges to a diffusion process wrt $Leb_{\sqrt{\varepsilon}}$.



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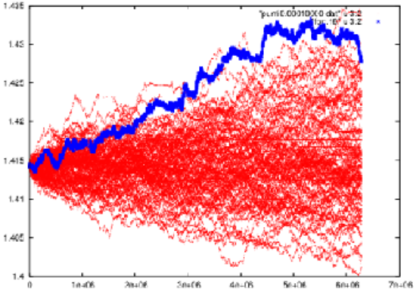

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$$f_0 : (\varphi, I) \to (\varphi + I + \varepsilon \cos \varphi, I + \varepsilon \cos \varphi),$$

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Theorem (joint work with O. Castejon) For $n \sim \varepsilon^{-2}$ such compositions satisfy the Central Limit Theorem, i.e.

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Stochastic Aspects of Arnold diffusion

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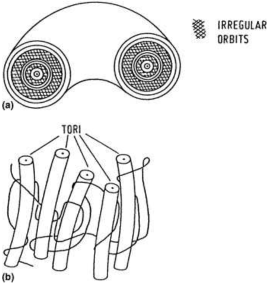

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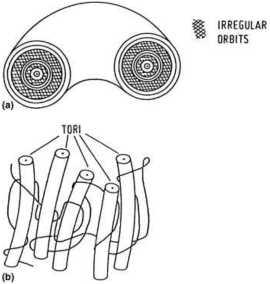
Diffusion mechanism



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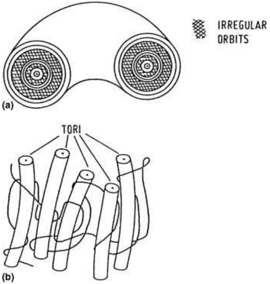

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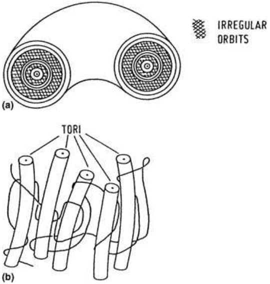

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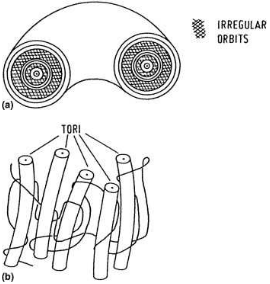

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Preprints contributing to the talk

- P. Bernard., V. Kaloshin, K. Zhang, Arnold diffusion in arbitrary degrees of freedom and crumpled 3-dimensional normally hyperbolic invariant cylinders, arXiv:1112.2773v Dec 2011, 58pp.
- V. Kaloshin, K. Zhang, A strong form of Arnold diffusion for two and a half degrees of freedom, arXiv:1212.1150 [math.DS] Dec 2012, 208pp.;
- J. Fejoz, M. Guardia, V. Kaloshin, P. Roldan, Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem, to appear in J of European Math Soc.
- V. Kaloshin, K. Zhang, Arnold diffusion for three and a half degrees of freedom, April 2014, 25pp.;
- V. Kaloshin, K. Zhang, Dynamics of the dominant Hamiltonian, with applications to Arnold diffusion, October 2014, 75pp.;
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