Lyapunov exponents of the Hodge bundle and diffusion in billiards with periodic obstacles

Anton Zorich

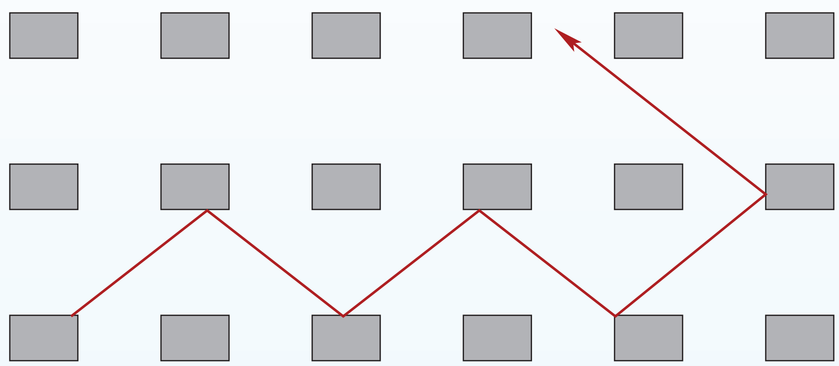
LEGACY OF VLADIMIR ARNOLD

Fields Institute, November 28, 2014

0. Model problem: diffusion in a periodic billiard

Diffusion in a billiard with periodic obstacles ("Windtree model" of P. and T. Ehrenfest; 1912)

Consider a billiard on the plane with $\mathbb{Z}^2$-periodic rectangular obstacles.



Old Theorem (V. Delecroix, P. Hubert, S. Lelièvre, 2011). *For all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory escapes to infinity with the rate $t^{2/3}$. That is,*

$$\max_{0\le\tau\le t}(\text{distance to the starting point at time } \tau) \sim t^{2/3}.$$

Here "$\frac{2}{3}$" is the Lyapunov exponent of certain "renormalizing" dynamical system associated to the initial one.

Remark. Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

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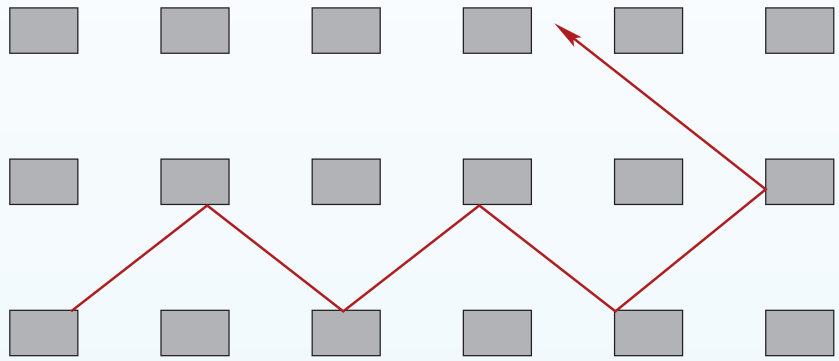

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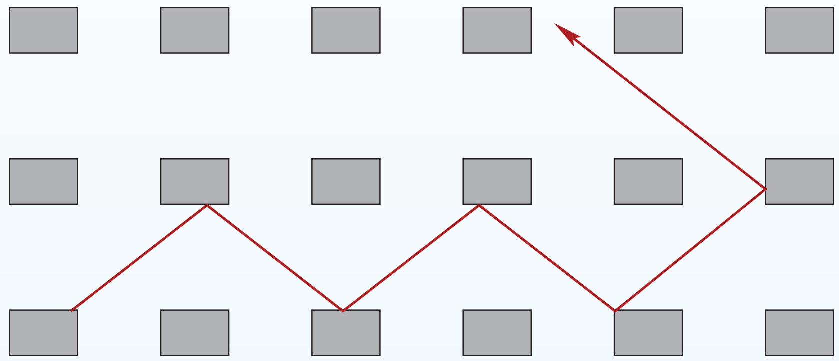

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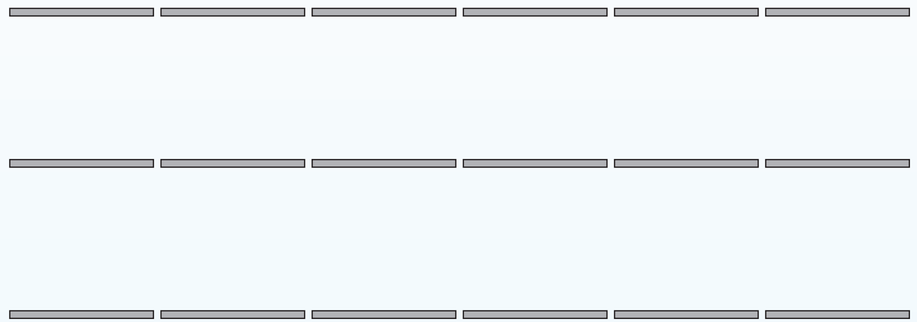

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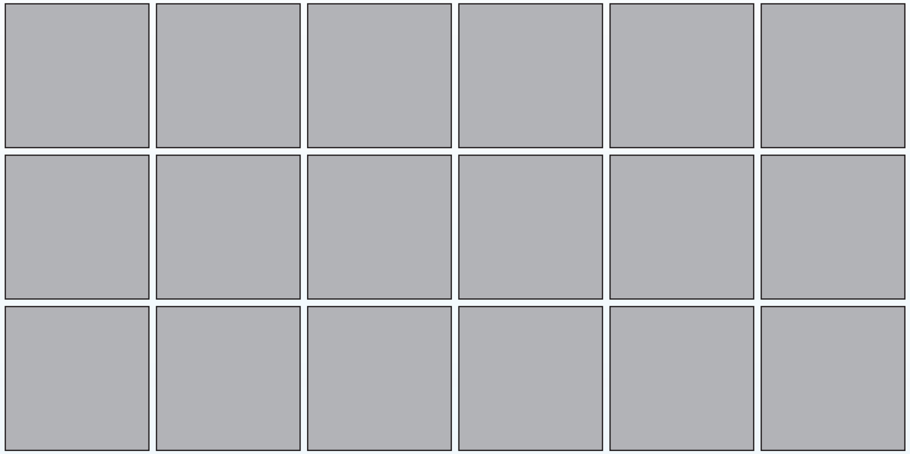

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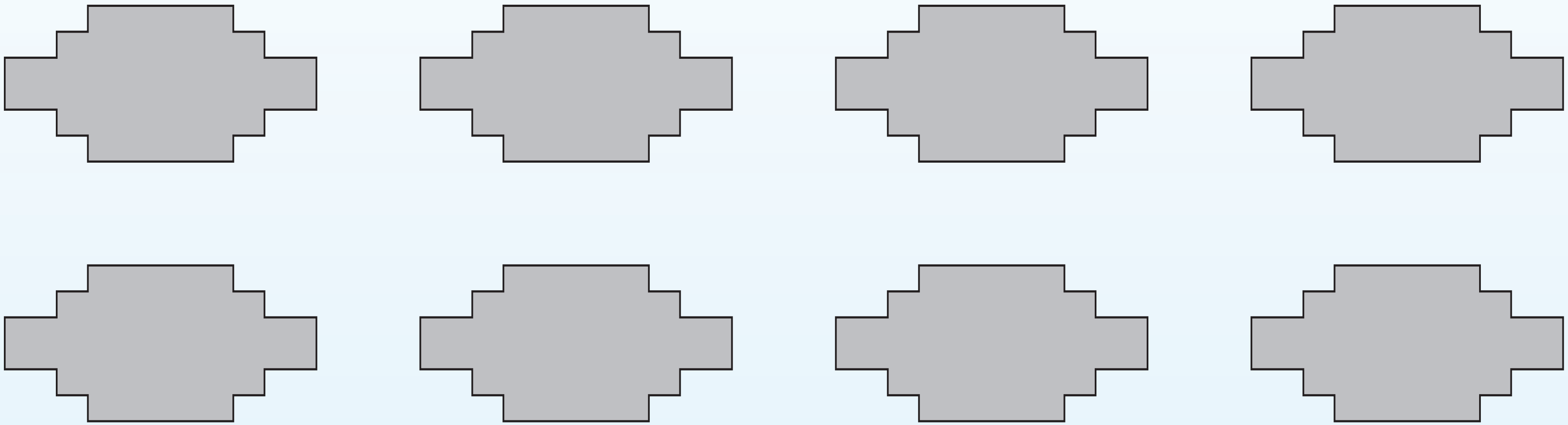
Changing the shape of the obstacle

Almost Old Theorem (V. Delecroix, A. Z., 2014). *Changing the shape of the obstacle we get a different diffusion rate. Say, for a symmetric obstacle with* $4m-4$ *angles* $3\pi/2$ *and with* $4m$ *angles* $\pi/2$ *the diffusion rate is*

$$\frac{(2m)!!}{(2m+1)!!} \sim \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as } m \to \infty\,.$$



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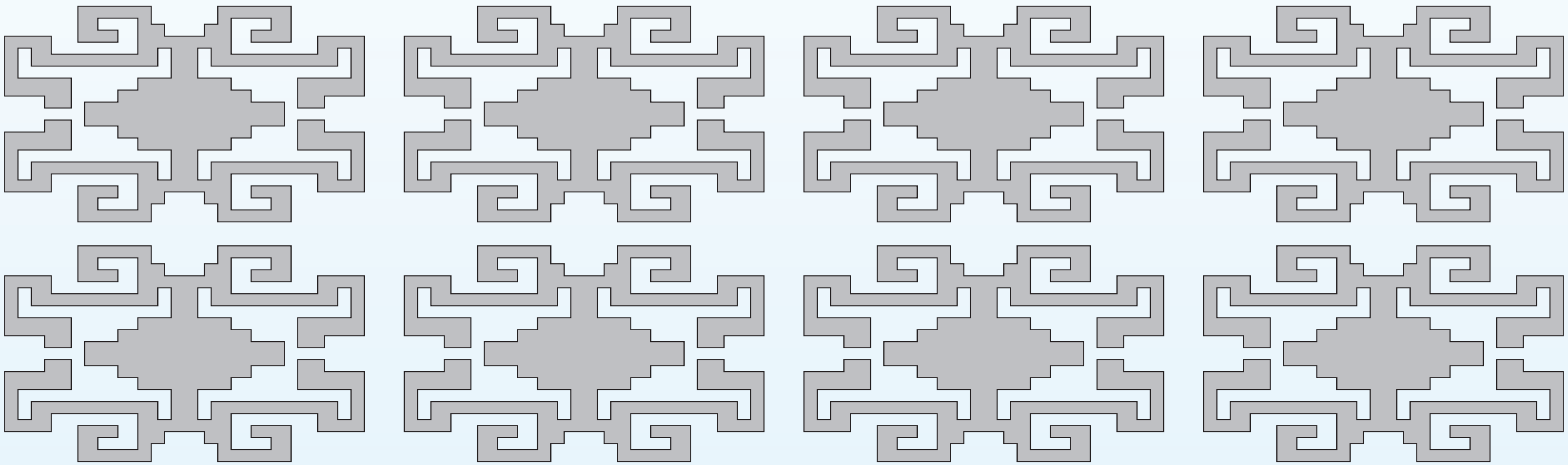

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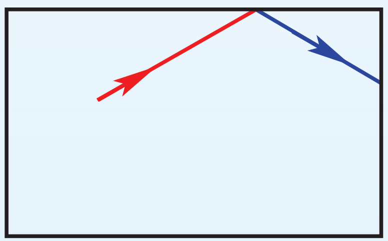
From a billiard to a surface foliation

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of four directions defining four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



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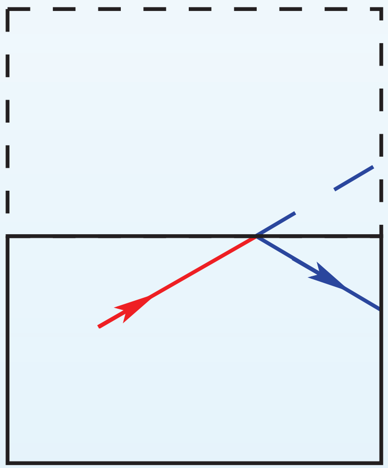

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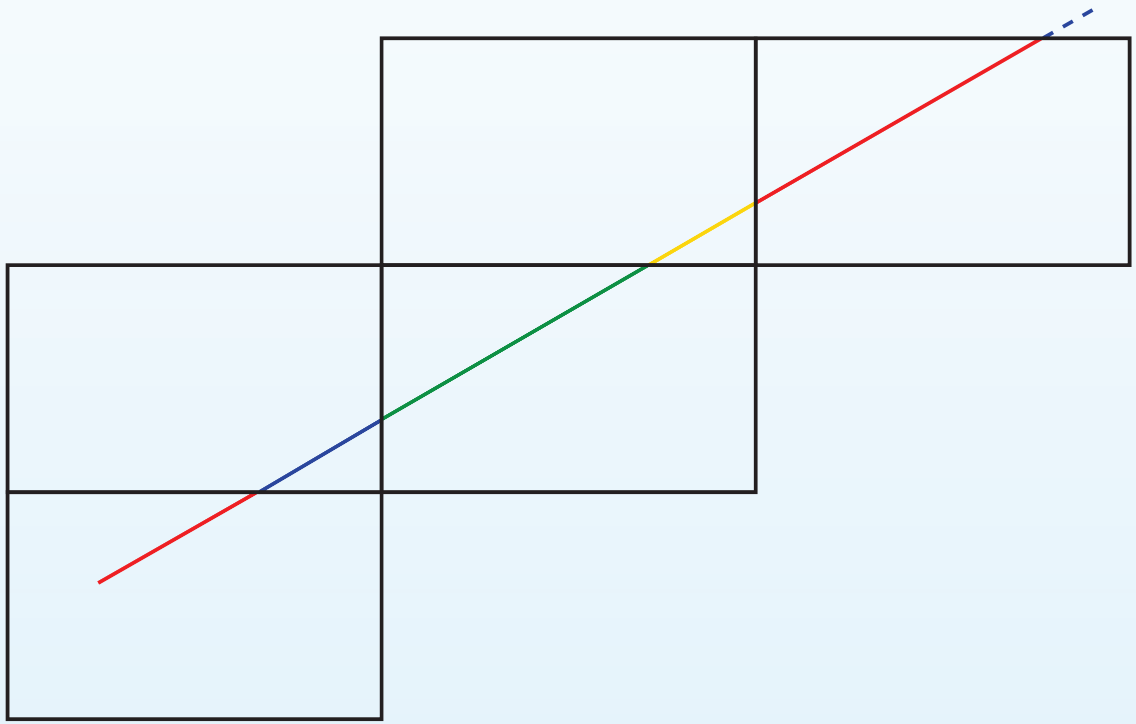

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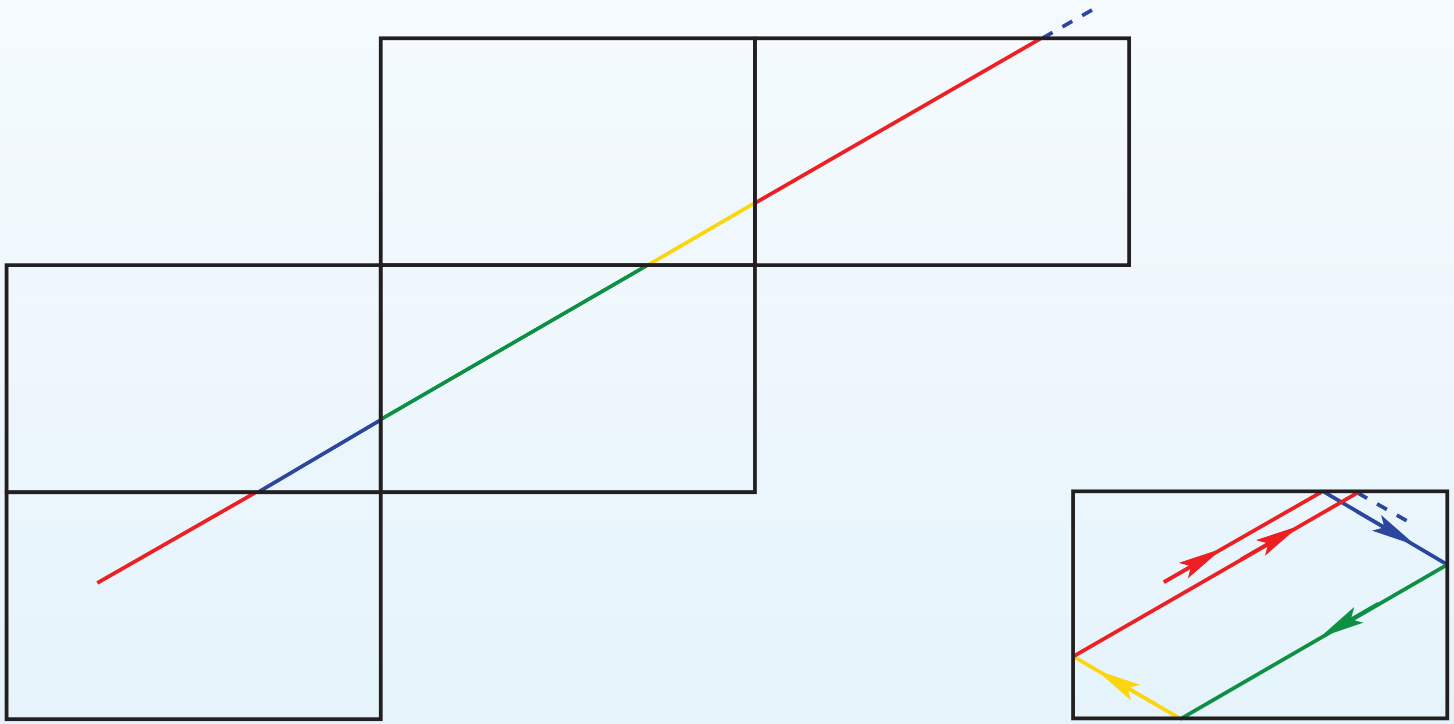

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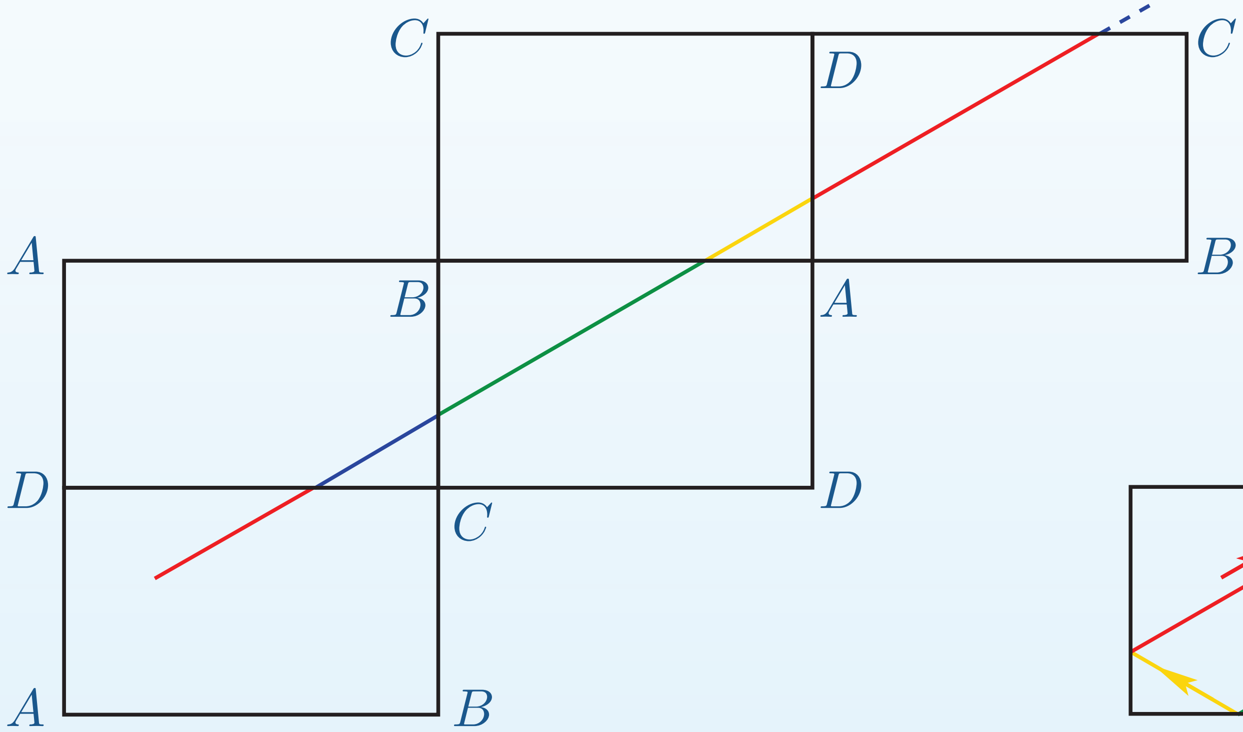

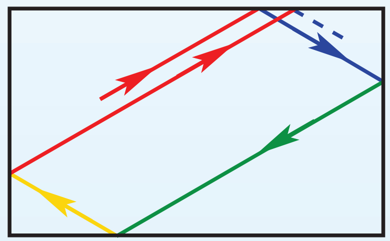

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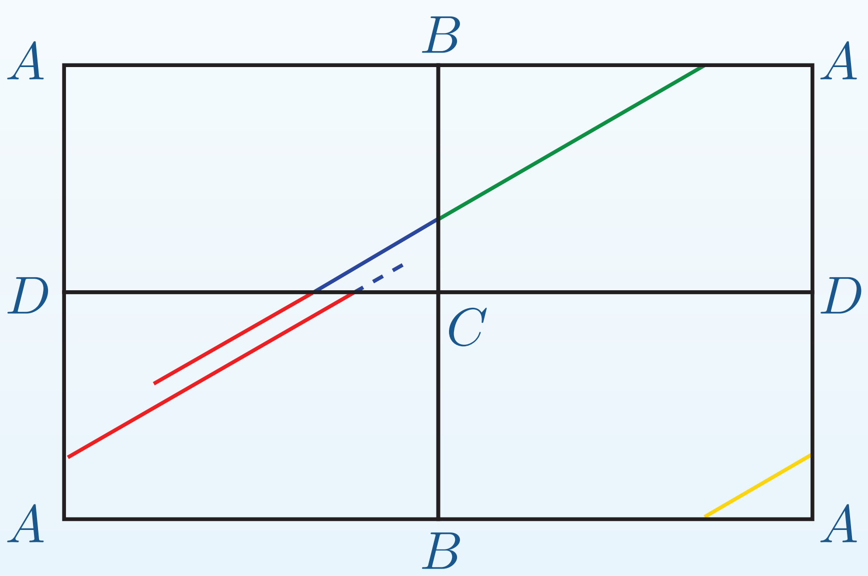
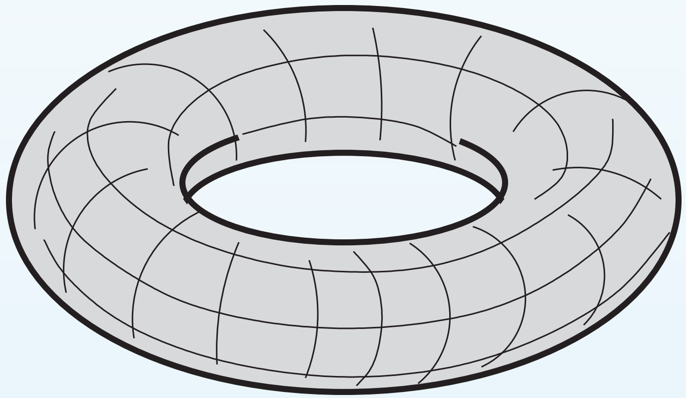


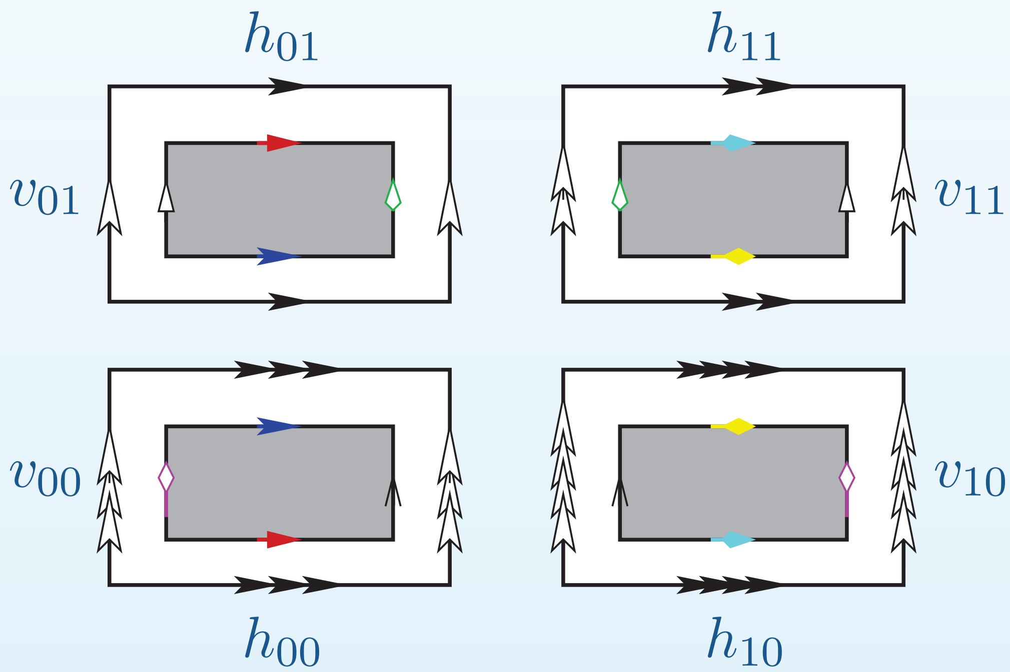


Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a “straight line” on the corresponding torus.

Displacement as intersection number

From the windtree billiard to a surface foliation

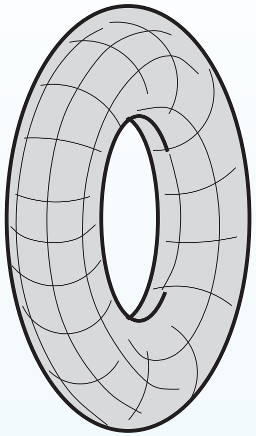
Similarly, taking four copies of our $\mathbb{Z}^2$-periodic windtree billiard we can unfold it to a foliation on a $\mathbb{Z}^2$-periodic surface. Taking a quotient over $\mathbb{Z}^2$ we get a compact flat surface endowed with a foliation in "straight lines". Vertical and horizontal displacement of the ball at time t is described by the intersection numbers $c(t) \circ v$ and $c(t) \circ h$ of the cycle $c(t)$ obtained by closing up the endpoints of the billiard trajectory after time t with the cycles $h = h_{00} + h_{10} - h_{01} - h_{11}$ and $v = v_{00} - v_{10} + v_{01} - v_{11}$.



1. Dynamics on the moduli space

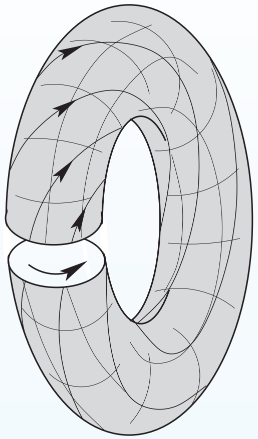
Dehn twist and deformations of a flat torus

Cut a torus along a horizontal circle.



Dehn twist and deformations of a flat torus

Twist progressively horizontal circles up to a complete turn on the opposite boundary component of the cylinder and then identify the boundary components.



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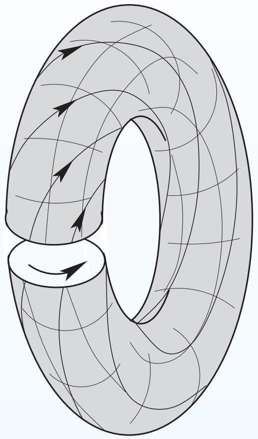

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\hat{f}_h} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 & \xrightarrow[f_h]{} & \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$

Dehn twist corresponds to the linear map $\hat{f}_h : \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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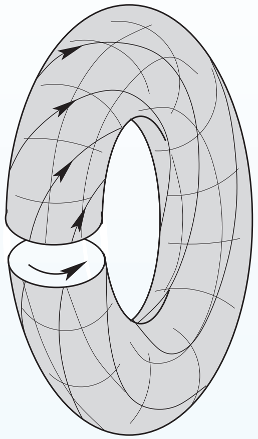

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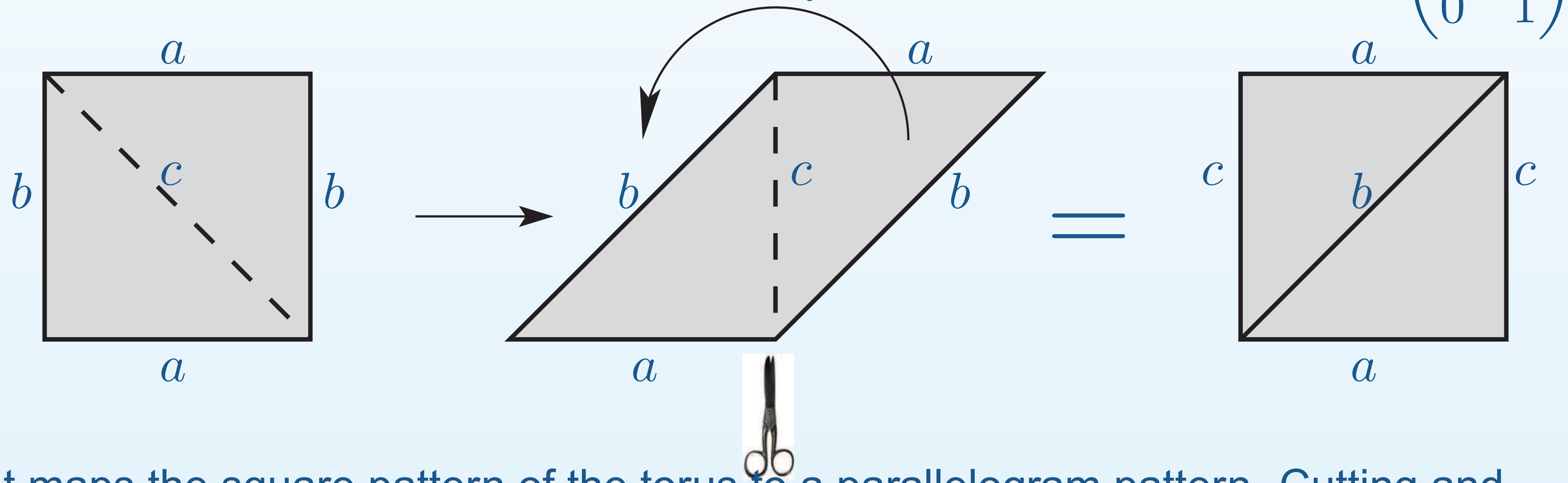
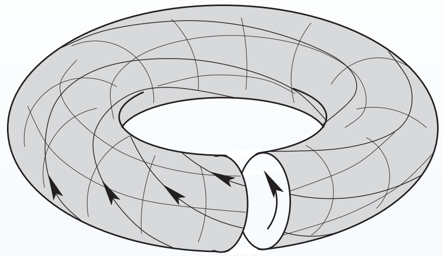
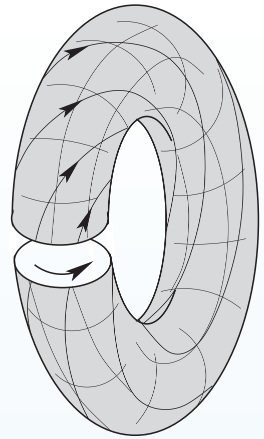


It maps the square pattern of the torus to a parallelogram pattern. Cutting and pasting appropriately we can transform the new pattern to the initial square one.

Arnold's cat (Fibonacci) diffeomorphism

Consider a composition
of two Dehn twists

$g = f_v \circ f_h =$  $\circ$ 

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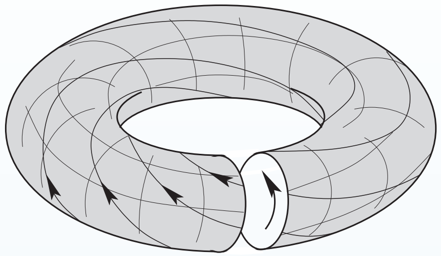
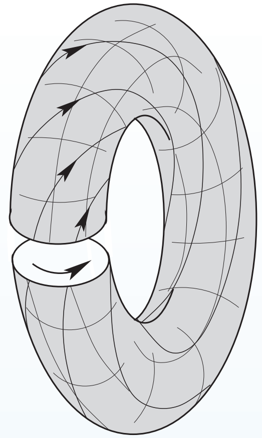
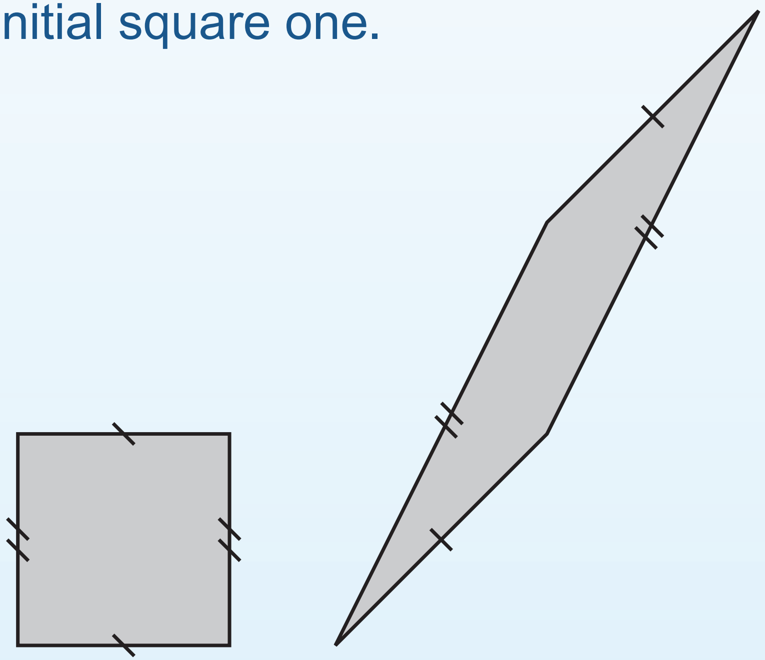
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

It corresponds to the integer linear map $\hat{g} : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Cutting and pasting appropriately the image parallelogram pattern we can check by hands that we can transform the new pattern to the initial square one.



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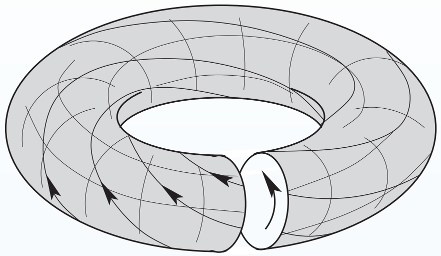
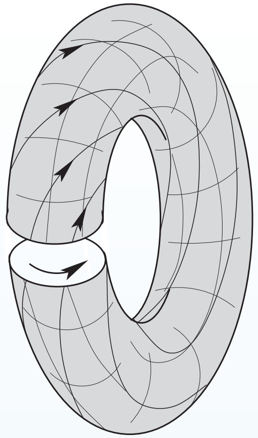
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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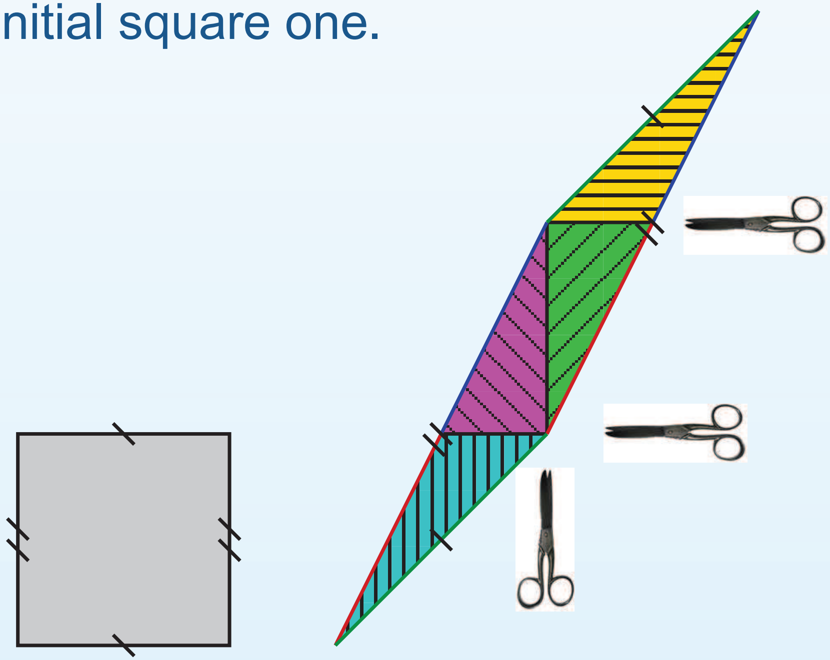

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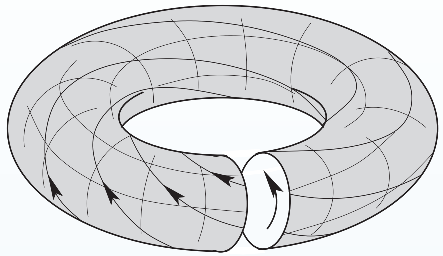
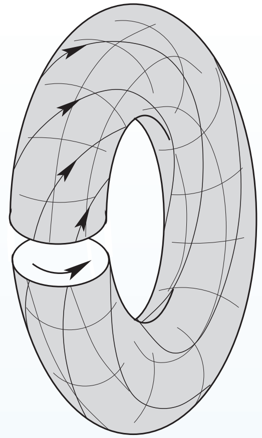
of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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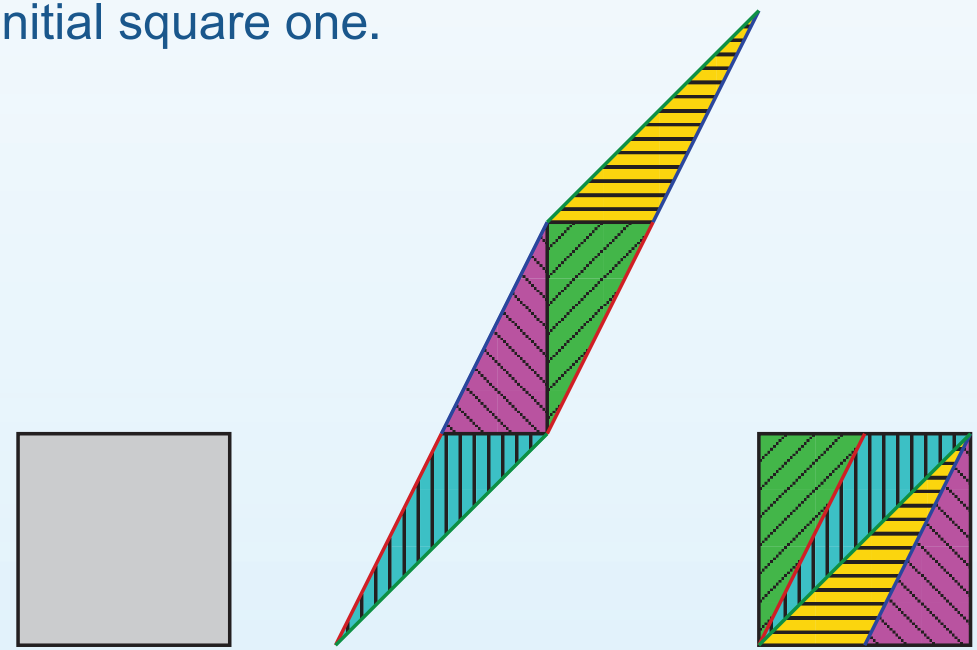

Pseudo-Anosov diffeomorphisms

Consider eigenvectors $\vec{v}_u$ and $\vec{v}_s$ of the linear transformation $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ with eigenvalues $\lambda = (3+\sqrt{5})/2 \approx 2.6$ and $1/\lambda = (3-\sqrt{5})/2 \approx 0.38$. Consider two transversal foliations on the original torus in directions $\vec{v}_u, \vec{v}_s$. We have just proved that expanding our torus $\mathbb{T}^2$ by factor λ in direction $\vec{v}_u$ and contracting it by the factor λ in direction $\vec{v}_s$ we get the original torus.

Definition. Surface automorphism homogeneously expanding in direction of one foliation and homogeneously contracting in direction of the transverse foliation is called a *pseudo-Anosov* diffeomorphism.

Consider a one-parameter family of flat tori obtained from the initial square torus by a continuous deformation expanding with a factor e^t in directions $\vec{v}_u$ and contracting with a factor e^t in direction $\vec{v}_s$. By construction such one-parameter family defines a closed curve in the space of flat tori: after the time $t_0 = \log \lambda_u$ it closes up and follows itself.

Observation. *Pseudo-Anosov diffeomorphisms define closed curves (actually, closed geodesics) in the moduli spaces of Riemann surfaces.*

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Space of lattices

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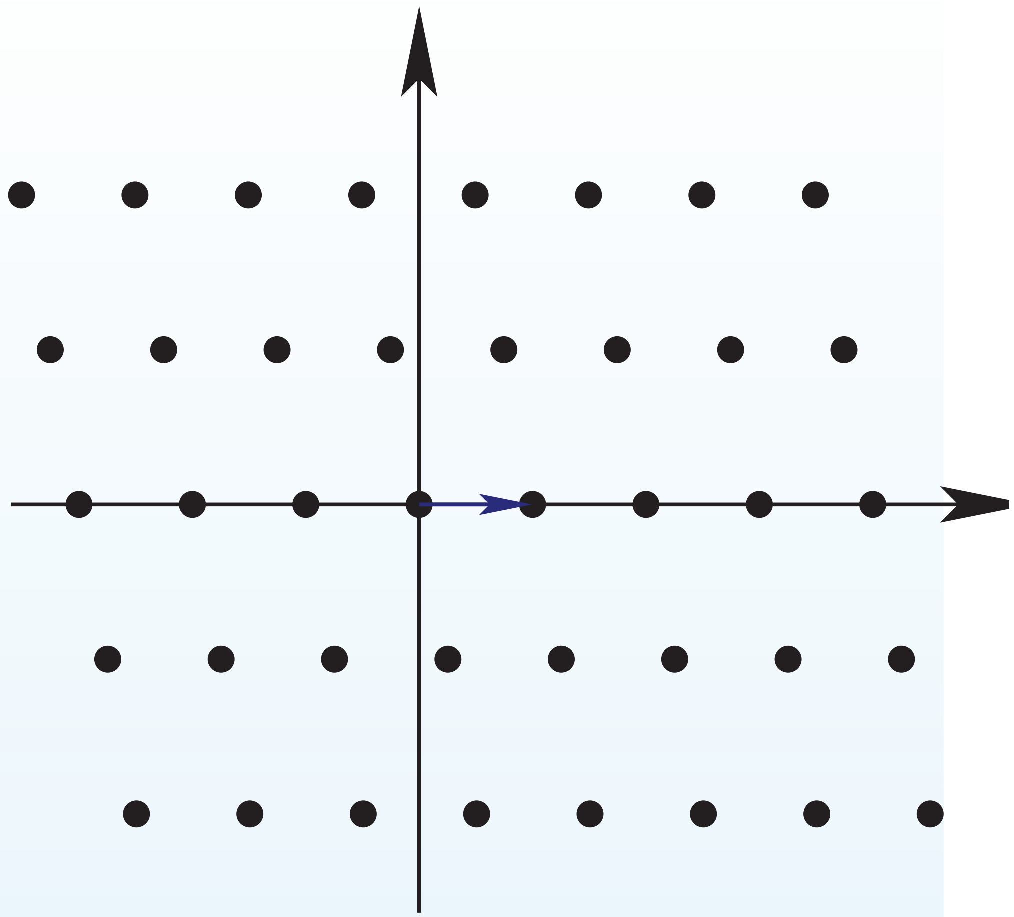

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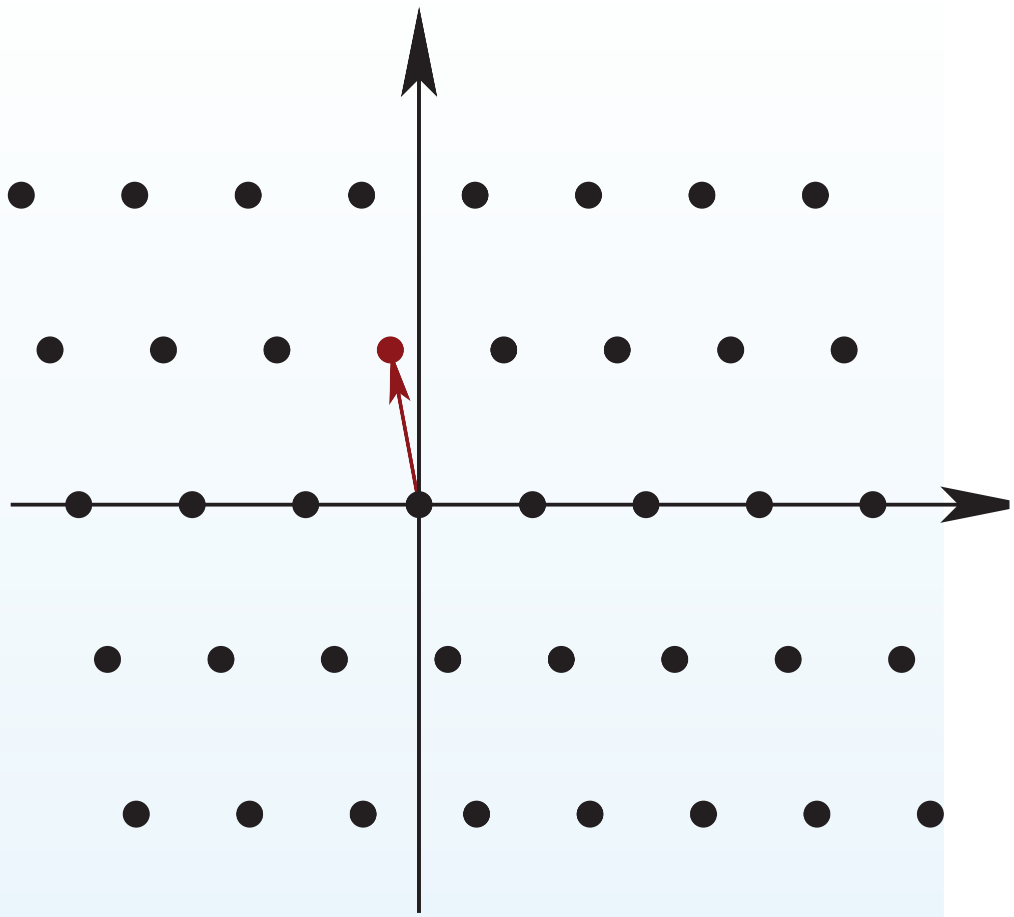
- Consider the lattice point closest to the origin and located in the upper half-plane.



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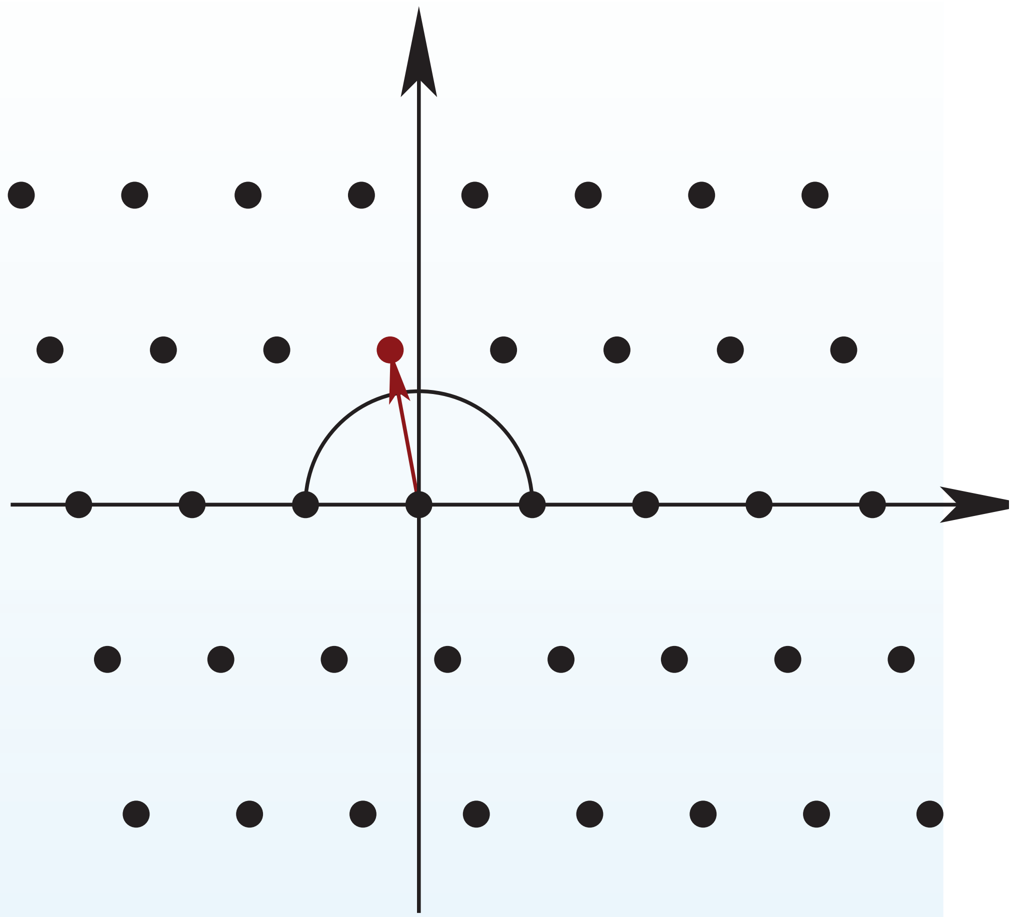
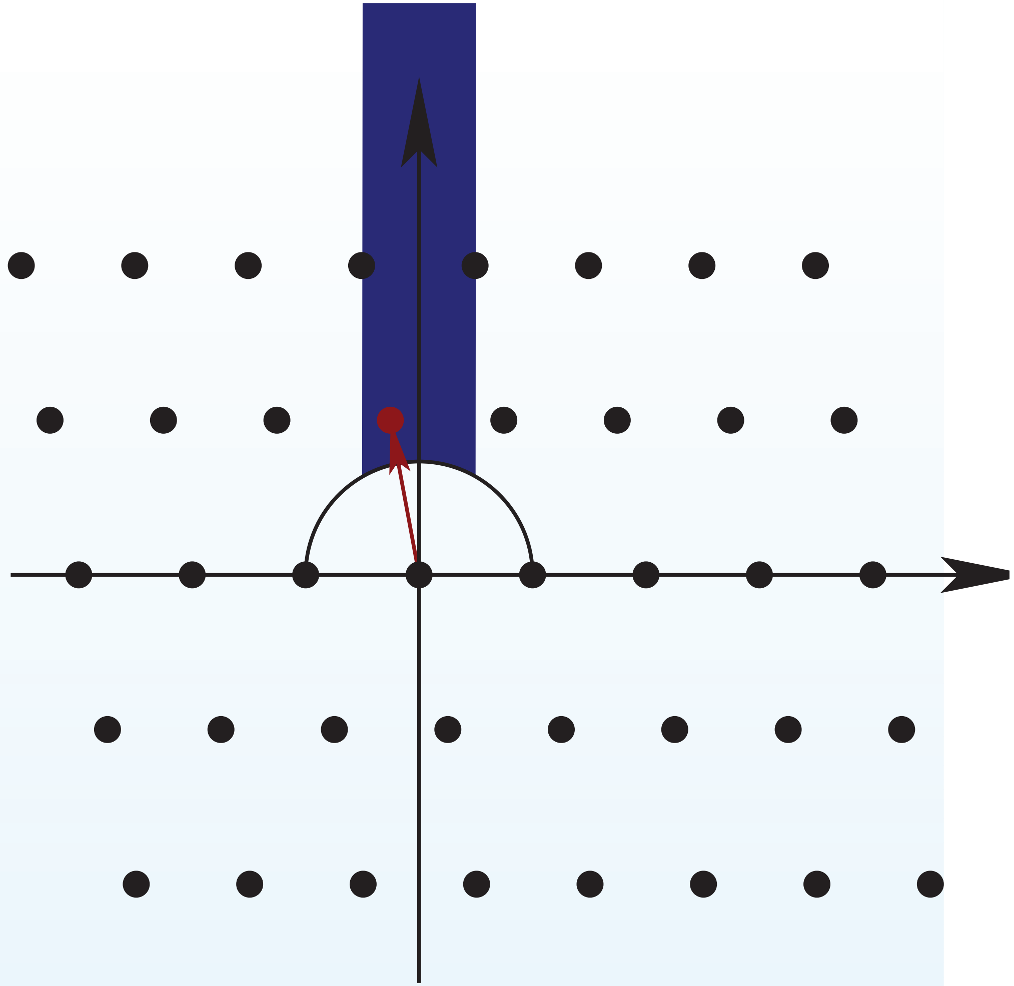


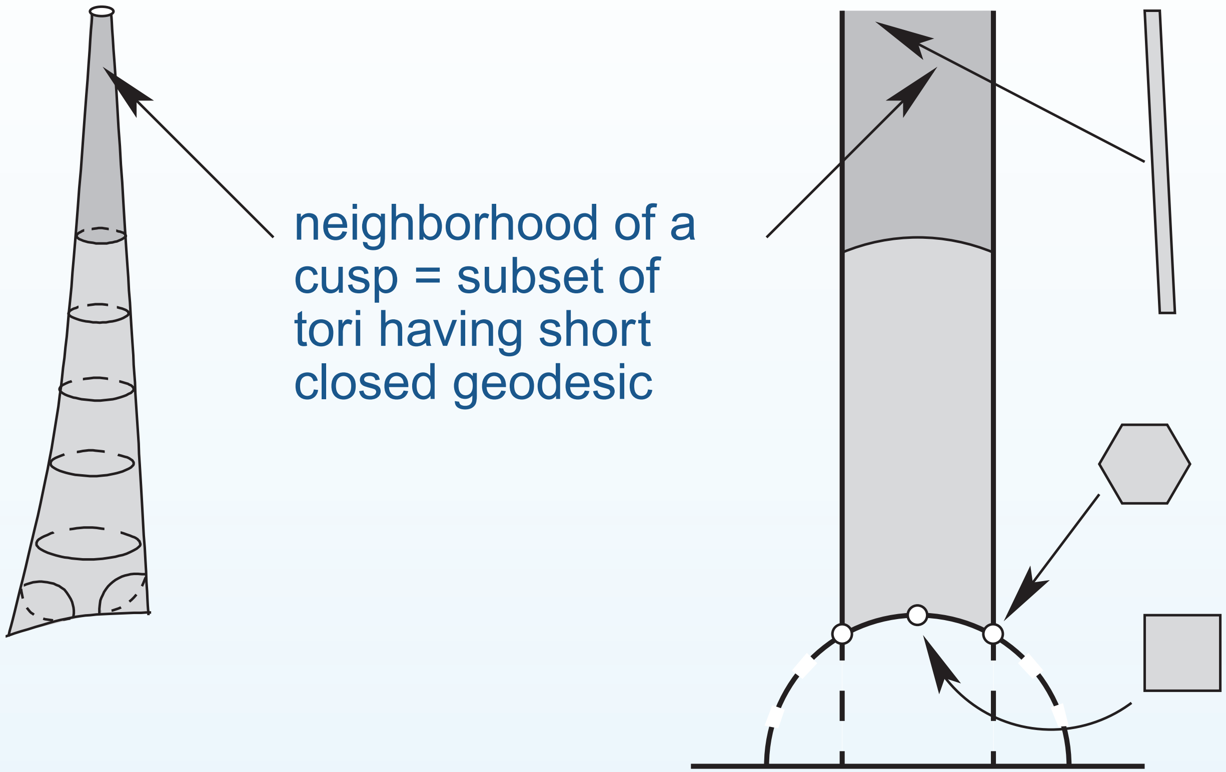
Space of lattices

- By a composition of homothety and rotation we can place the shortest vector of the lattice to the horizontal unit vector.
- Consider the lattice point closest to the origin and located in the upper half-plane.
- This point is located outside of the unit disc.
- It necessarily lives inside the strip $-1/2 \leq x \leq 1/2$.



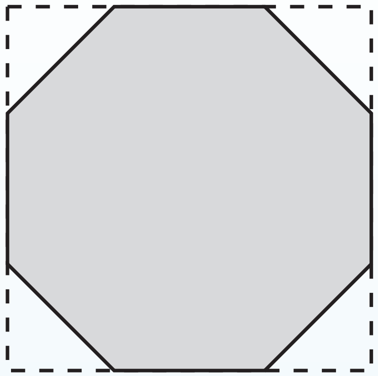
We get a fundamental domain in the space of lattices, or, in other words, in the moduli space of flat tori.

Moduli space of tori



The corresponding modular surface is not compact: flat tori representing points, which are close to the cusp, are almost degenerate: they have a very short closed geodesic. It also have orbifoldic points corresponding to tori with extra symmetries.

Very flat surface of genus 2



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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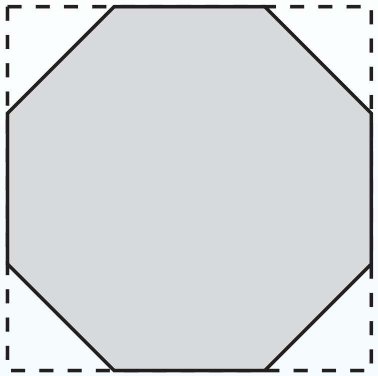

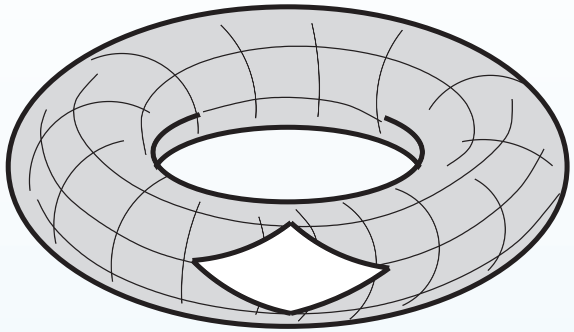

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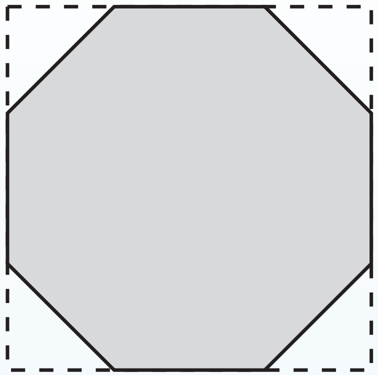

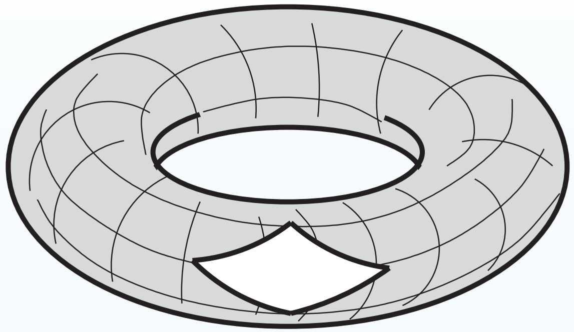

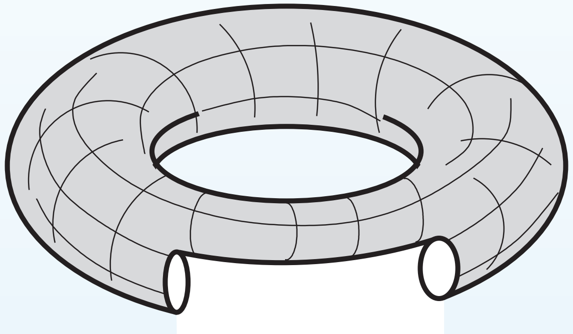

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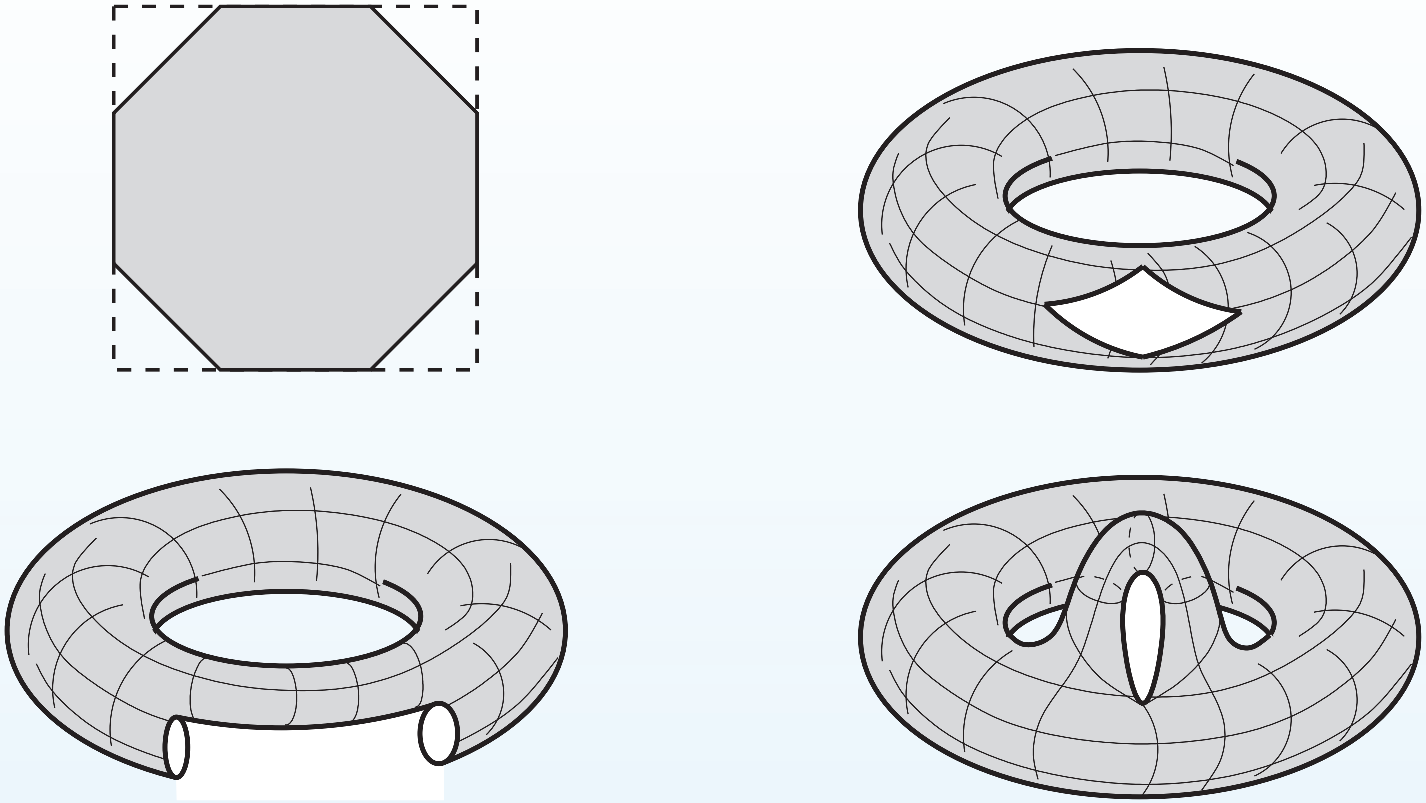

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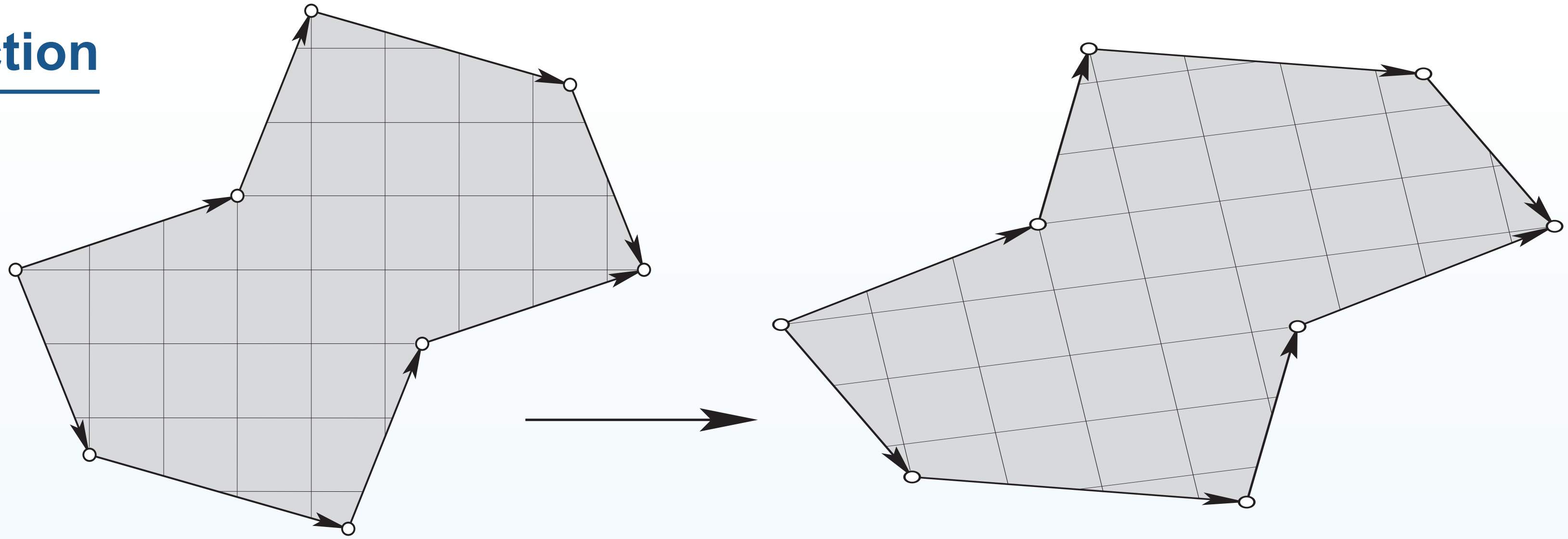
Group action



The group $\mathrm{SL}(2,\mathbb{R})$ acts on the each space $\mathcal{H}_1(d_1,\dots,d_n)$ of flat surfaces of unit area with conical singularities of prescribed cone angles $2\pi(d_i+1)$. This action preserves the natural measure on this space. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset \mathrm{SL}(2,\mathbb{R})$ induces a natural flow on $\mathcal{H}_1(d_1,\dots,d_n)$ called the *Teichmüller geodesic flow*.

Keystone Theorem (H. Masur; W. A. Veech, 1992). *The action of the groups* $\mathrm{SL}(2,\mathbb{R})$ *and* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *is ergodic with respect to the natural finite measure on each connected component of every space* $\mathcal{H}_1(d_1,\dots,d_n)$.

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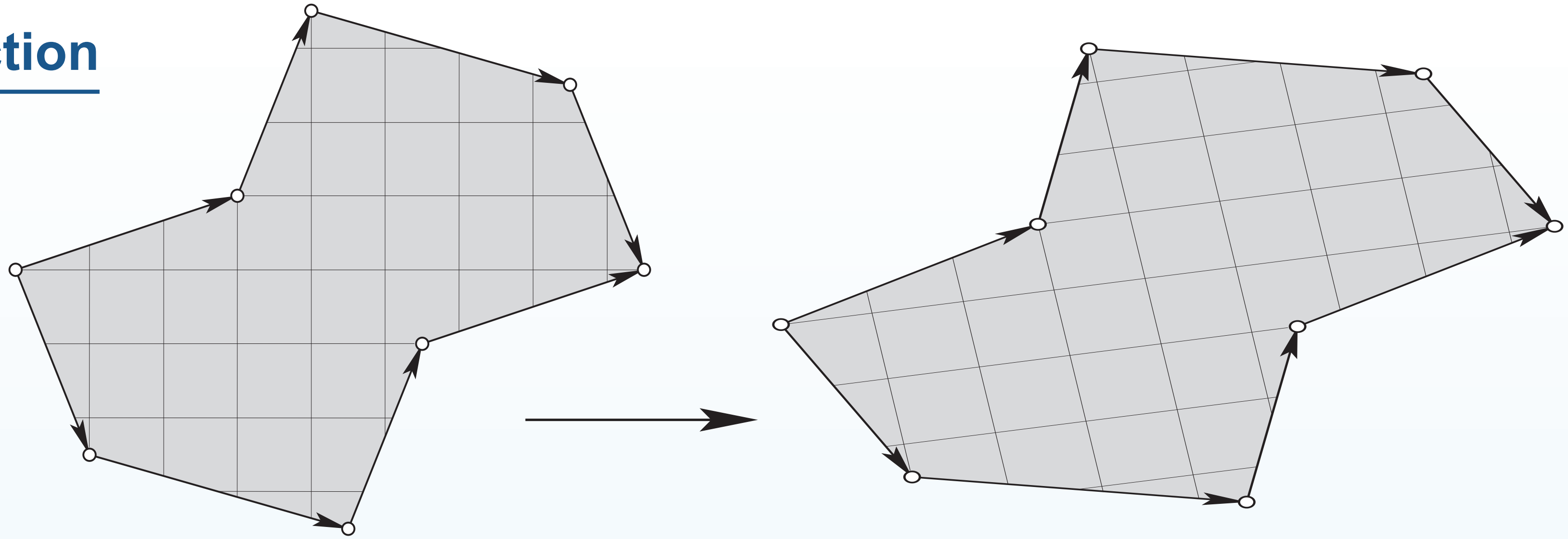

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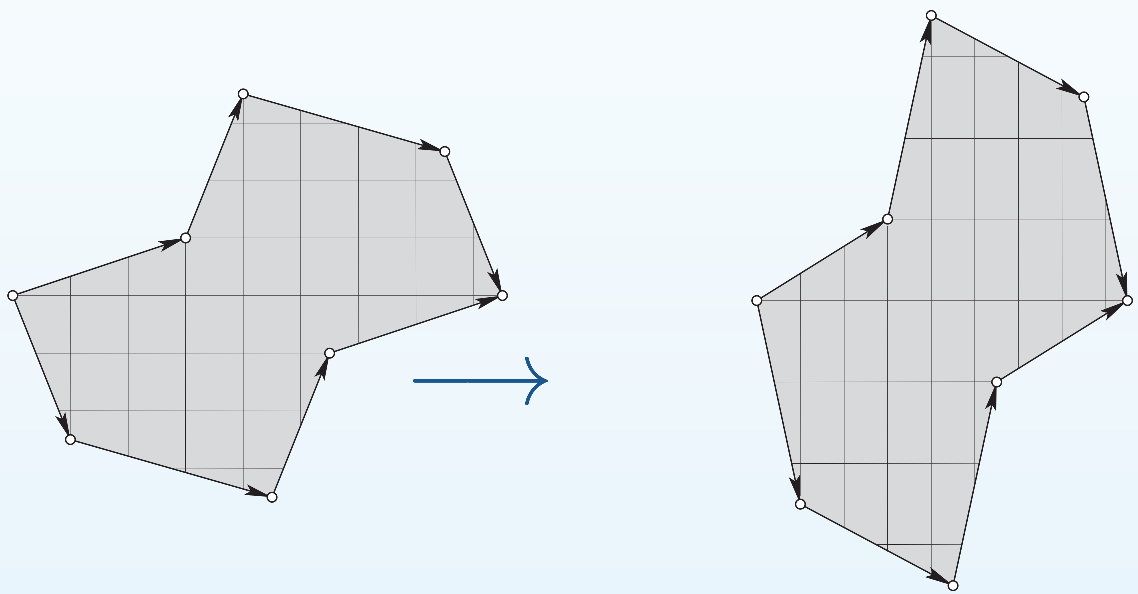
Magic of Masur—Veech Theorem

Theorem of Masur and Veech claims that taking at random an octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.



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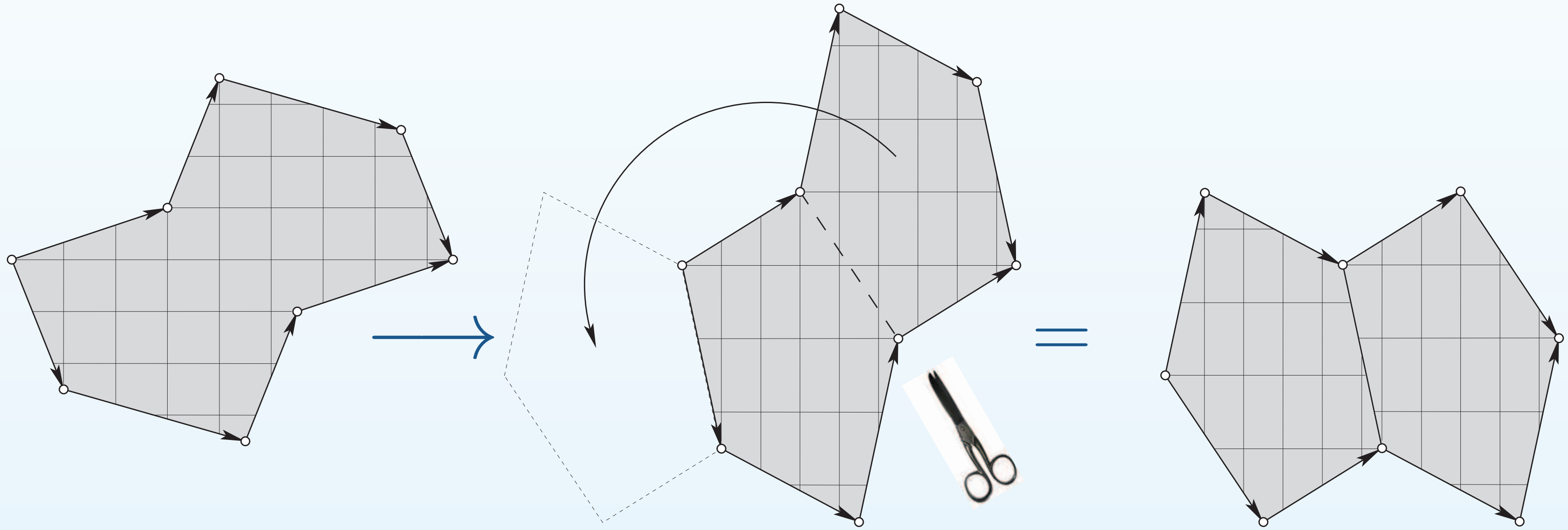
There is no paradox since we are allowed to cut-and-paste!



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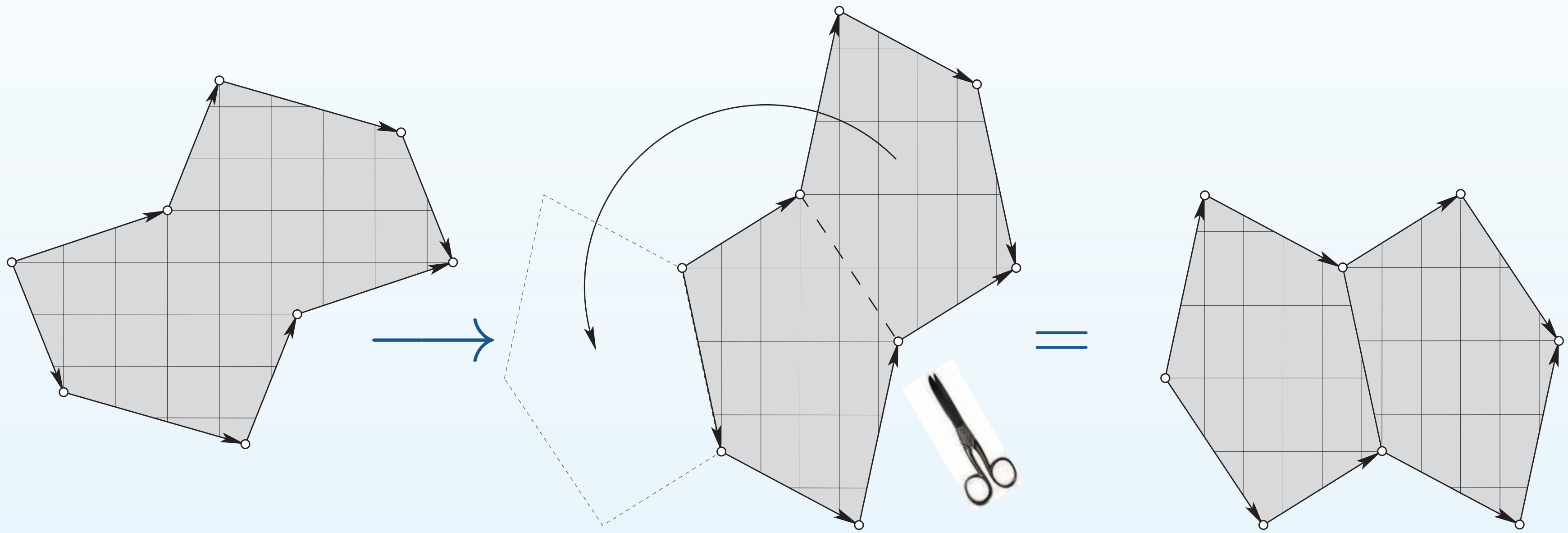
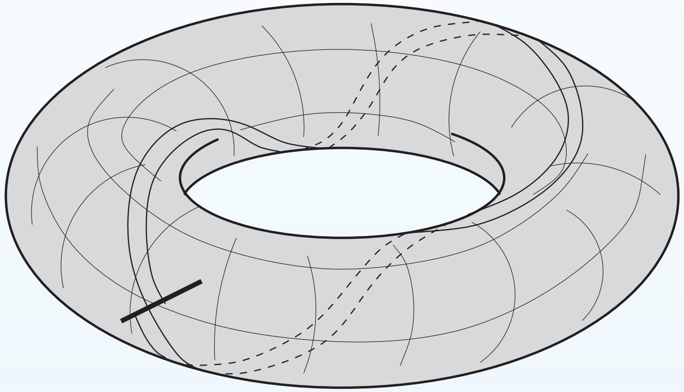


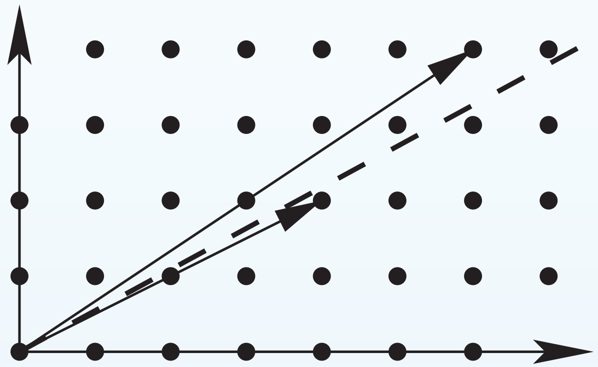
The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface.

2. Asymptotic flag of an orientable measured foliation

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \dots$ define a sequence of cycles $c_1, c_2, \dots$.





The *asymptotic cycle* is defined as $\lim_{n\to\infty} \dfrac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

Theorem (S. Kerckhoff, H. Masur, J. Smillie, 1986.) *For any flat surface directional flow in almost any direction is uniquely ergodic.*

This implies that for almost any direction the asymptotic cycle exists and is the same for all points of the surface.

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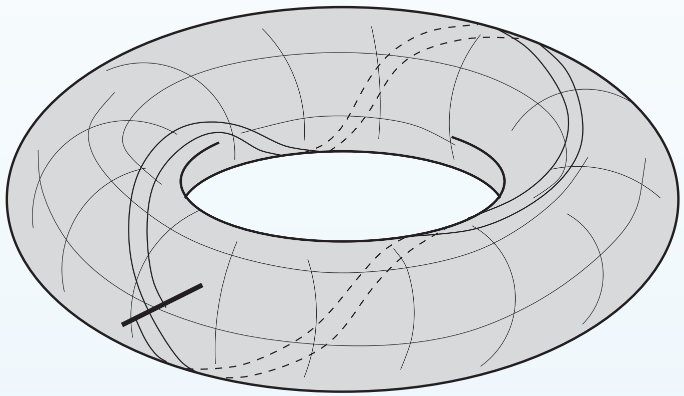

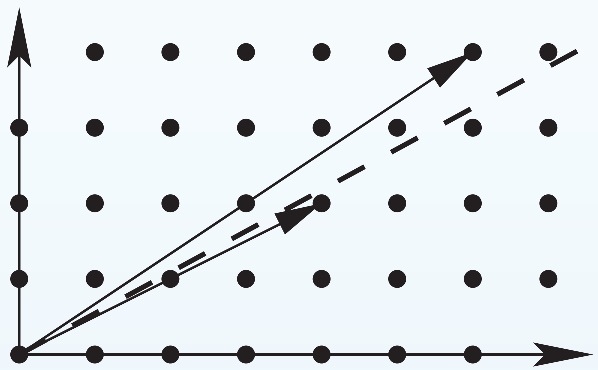

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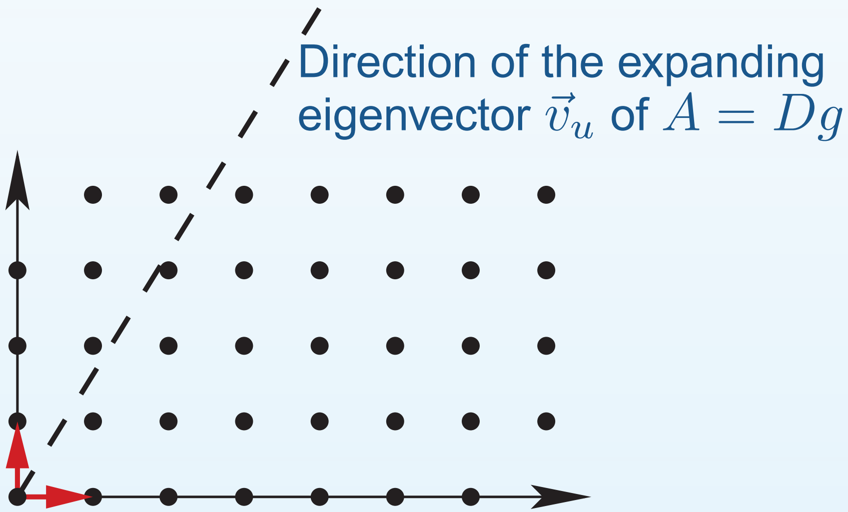
Asymptotic cycle in the pseudo-Anosov case

Consider a model case of the foliation in direction of the expanding eigenvector $\vec{v}_u$ of the Anosov map $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $Dg = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Take a closed curve γ and apply to it k iterations of g. The images $g_*^{(k)}(c)$ of the corresponding cycle $c = [\gamma]$ get almost collinear to the expanding eigenvector $\vec{v}_u$ of A, and the corresponding curve $g^{(k)}(\gamma)$ closely follows our foliation.



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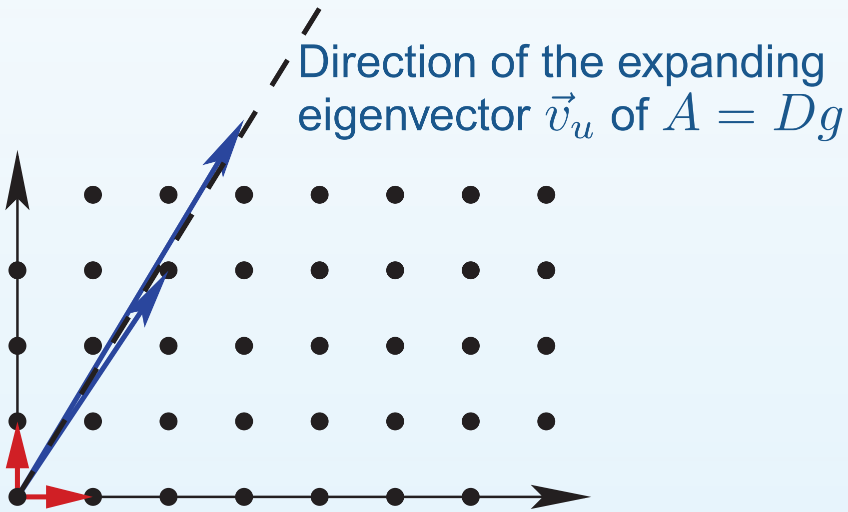

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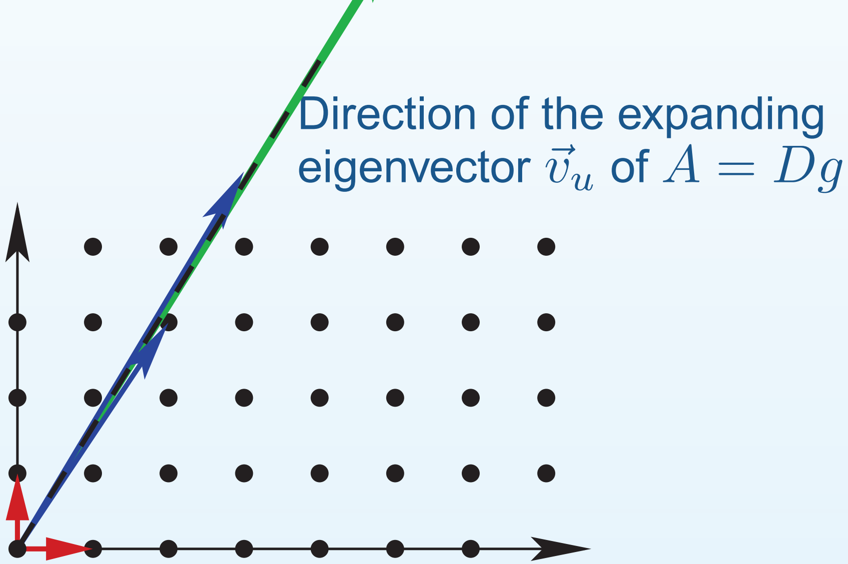
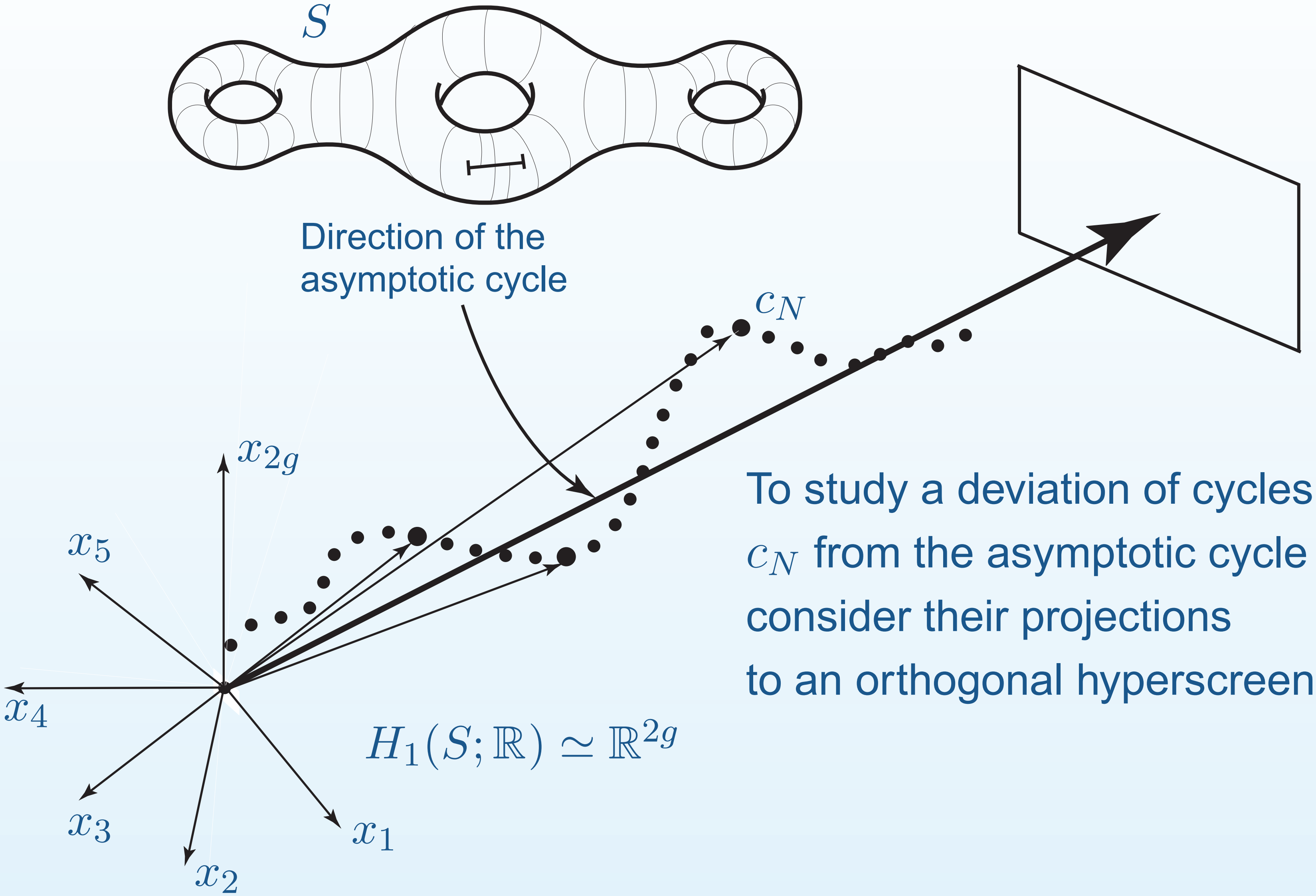


Asymptotic flag: empirical description



To study a deviation of cycles c_N from the asymptotic cycle consider their projections to an orthogonal hyperscreen

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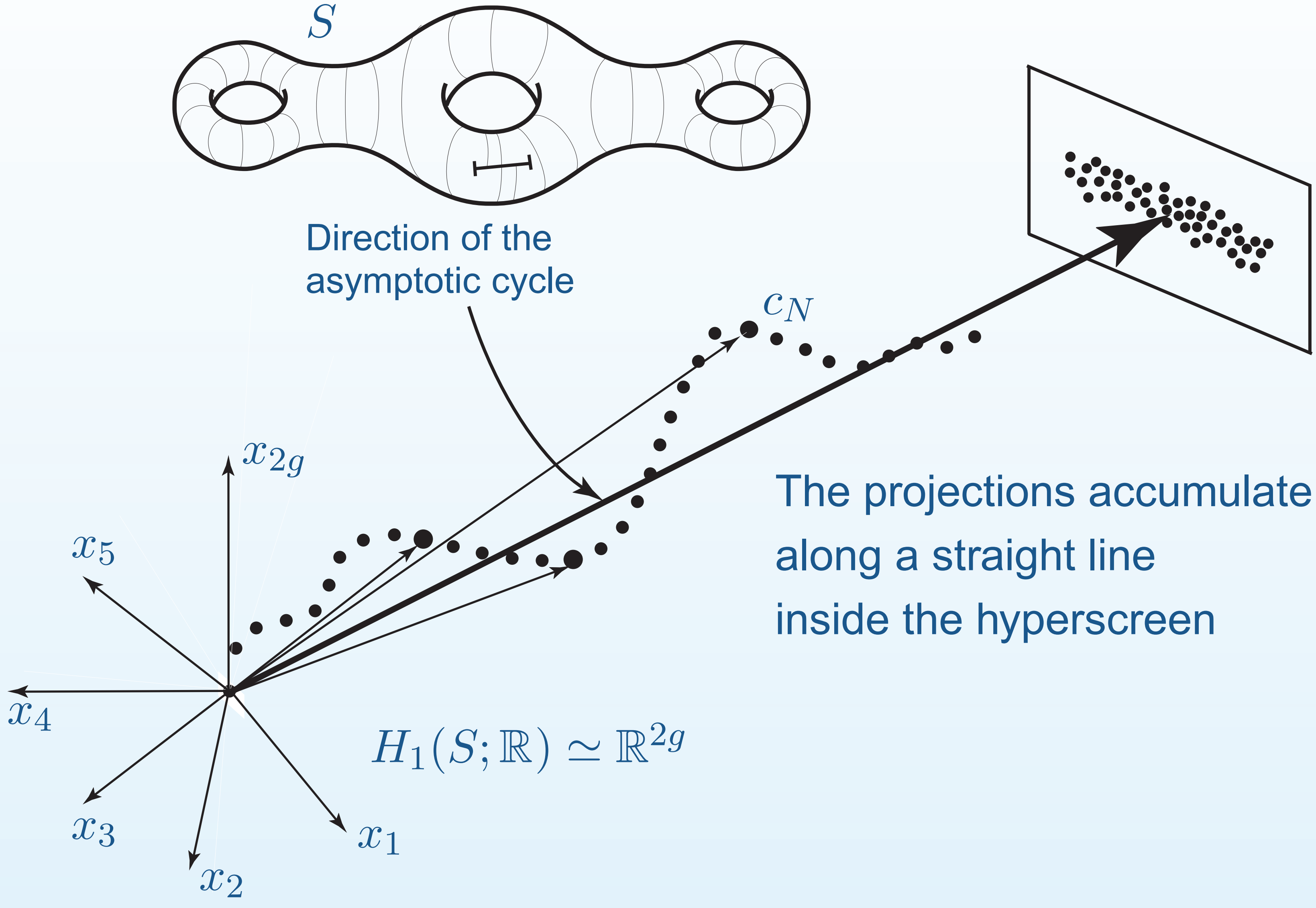

The projections accumulate
along a straight line
inside the hyperscreen

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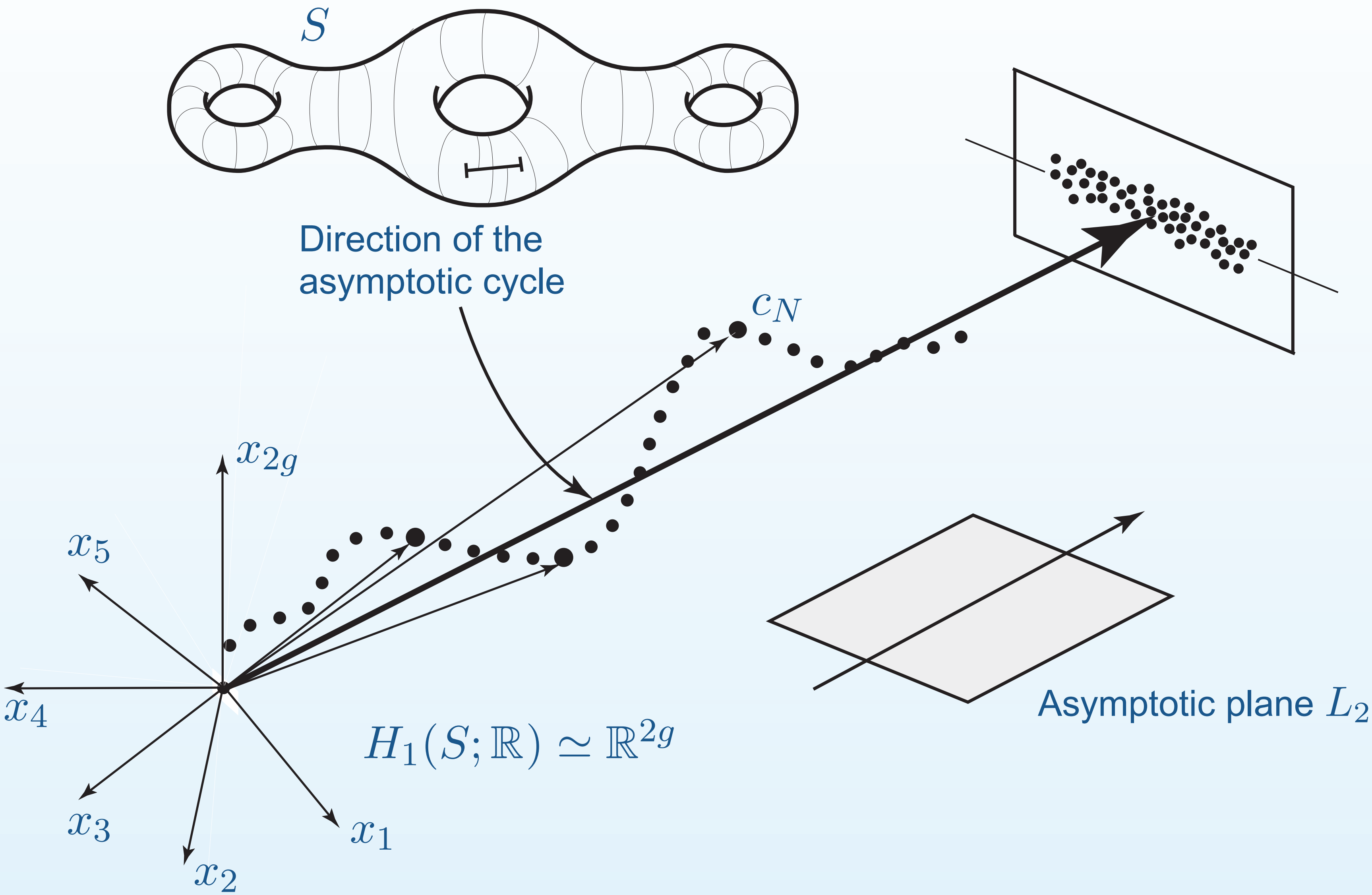

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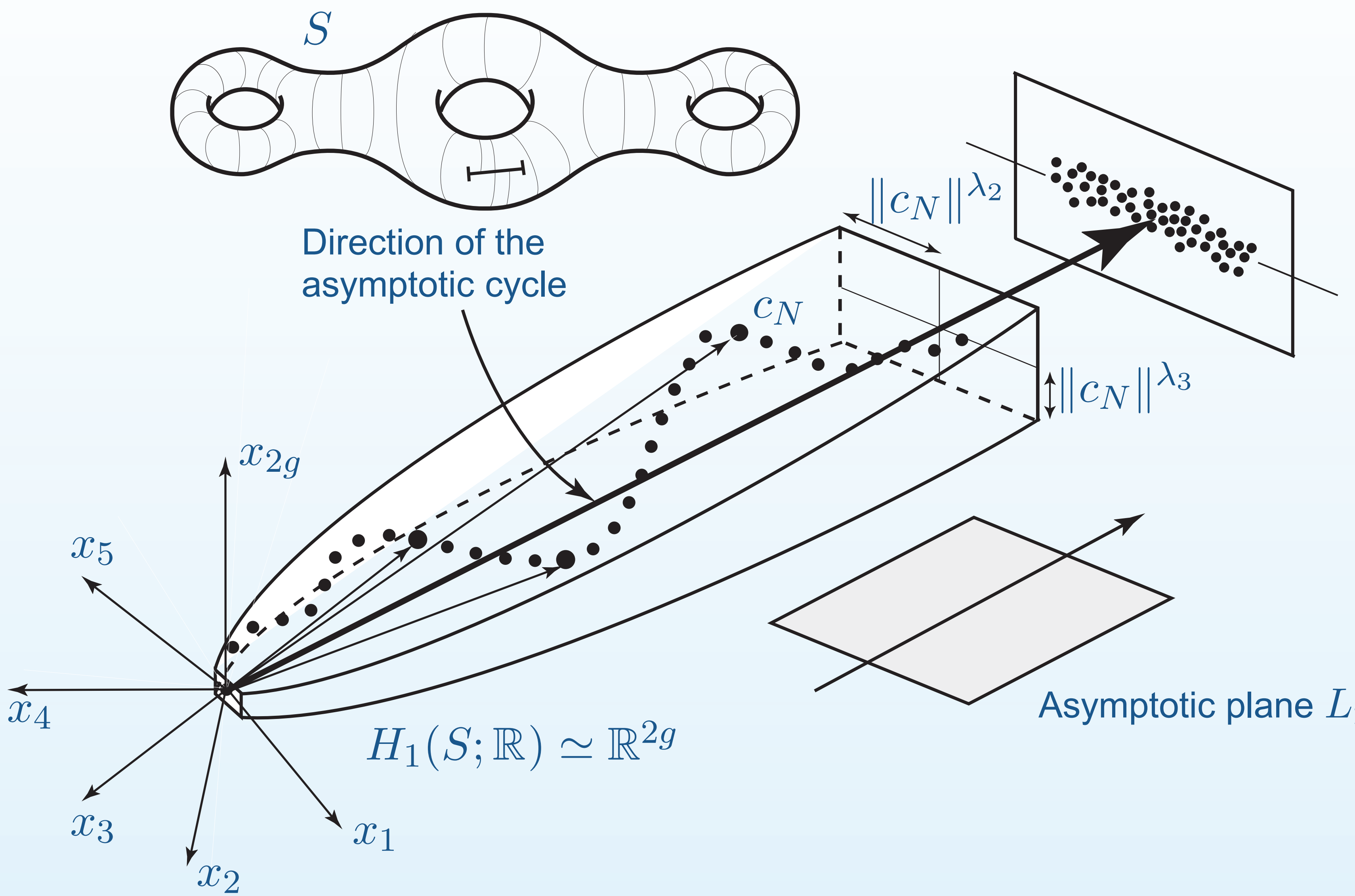

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Theorem (A. Z. , 1999) *For almost any surface S in any stratum $\mathcal{H}_1(d_1,\dots,d_n)$ there exists a flag of subspaces $L_1 \subset L_2 \subset \dots \subset L_g \subset H_1(S;\mathbb{R})$ such that for any $j = 1,\dots,g-1$*

$$\limsup_{N\to\infty} \frac{\log \operatorname{dist}(c_N, L_j)}{\log N} = \lambda_{j+1}$$

and

$$\operatorname{dist}(c_N, L_g) \le const,$$

where the constant depends only on S and on the choice of the Euclidean structure in the homology space.

The numbers $1 = \lambda_1 > \lambda_2 > \dots > \lambda_g$ are the top g Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on the corresponding connected component of the stratum $\mathcal{H}(d_1,\dots,d_n)$.

The strict inequalities $\lambda_g > 0$ and $\lambda_2 > \dots > \lambda_g$, and, as a corollary, strict inclusions of the subspaces of the flag, are difficult theorems proved later by G. Forni (2002) and by A. Avila–M. Viana (2007).

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Geometric interpretation of multiplicative ergodic theorem: spectrum of "mean monodromy"

Consider a vector bundle endowed with a flat connection over a manifold X^n. Having a flow on the base we can take a fiber of the vector bundle and transport it along a trajectory of the flow. When the trajectory comes close to the starting point we identify the fibers using the connection and we get a linear transformation $\mathcal{A}(x,1)$ of the fiber; the next time we get a matrix $\mathcal{A}(x,2)$, etc.

The multiplicative ergodic theorem says that when the flow is ergodic a *"matrix of mean monodromy"* along the flow

$$A_{mean} := \lim_{N\to\infty} \left(\mathcal{A}^*(x,N)\cdot\mathcal{A}(x,N)\right)^{\frac{1}{2N}}$$

is well-defined and constant for almost every starting point.

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Hodge bundle and Gauss–Manin connection

Consider a natural vector bundle over the stratum with a fiber $H^1(S;\mathbb{R})$ over a "point" (S,ω), called the *Hodge bundle*. It carries a canonical flat connection called *Gauss—Manin connection*: we have a lattice $H^1(S;\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, Teichmüller flow on $\mathcal{H}_1(d_1,\dots,d_n)$ defines a multiplicative cocycle acting on fibers of this bundle.

The monodromy matrices of this cocycle are symplectic which implies that the Lyapunov exponents are symmetric:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq -\lambda_g \geq \dots \geq -\lambda_2 \geq -\lambda_1$$

Morally, one can pretend that instead of the Teichmüller geodesic flow on the stratum $\mathcal{H}_1(d_1,\dots,d_n)$ we have a single closed geodesic passing through almost every point. We pretend that it defines some universal pseudo-Anosov diffeomorphism one and the same for almost all flat surfaces in $\mathcal{H}_1(d_1,\dots,d_n)$, and that the Lyapunov exponents are the logarithms of the eigenvalues of this universal pseudo-Anosov diffeomorphism.

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3. State of the art

Formula for the Lyapunov exponents

Theorem (A. Eskin, M. Kontsevich, A. Z., 2014) *The Lyapunov exponents λ_i of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L} \subseteq \mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1 + \lambda_2 + \dots + \lambda_g \;=\; \frac{1}{12} \cdot \sum_{i=1}^{n} \frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3} \cdot c_{area}(\mathcal{L})\,.$$

The proof is based on the initial Kontsevich formula $+$ analytic Riemann-Roch theorem $+$ analysis of $\det \Delta_{flat}$ under degeneration of the flat metric.

Theorem (A. Eskin, H. Masur, A. Z., 2003) *For $\mathcal{L} = \mathcal{H}_1(d_1,\dots,d_n)$ one has*

$$c_{area}(\mathcal{H}_1(d_1,\dots,d_n)) = \sum_{\substack{Combinatorial\ types\\ of\ degenerations}} (\text{explicit combinatorial factor}) \cdot \frac{\prod_{j=1}^{k} \operatorname{Vol} \mathcal{H}_1(\text{adjacent simpler strata})}{\operatorname{Vol} \mathcal{H}_1(d_1,\dots,d_n)}\,.$$

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Invariant measures and orbit closures

Fantastic Theorem (A. Eskin, M. Mirzakhani, 2014). *The closure of any* $\mathrm{SL}(2,\mathbb{R})$*-orbit is a suborbifold. In period coordinates* $H^1(S,\{\text{zeroes}\};\mathbb{C})$ *any* $\mathrm{SL}(2,\mathbb{R})$*-suborbifold is represented by an affine subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$*-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.*

Developement (A. Wright, 2014) *Effective methods of construction of orbit closures.*

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Solution of the generalized windtree problem (V. Delecroix–A. Z., 2014). Notice that any "windtree flat surface" S is a cover of a surface S_0 in the hyperelliptic locus $\mathcal{L}$ in genus 1, and that the cycles h and v are induced from S_0. Prove that the orbit closure of S_0 is $\mathcal{L}$. Using the volumes of the strata in genus zero, compute $c_{area}(\mathcal{L})$. Using the formula for $\sum \lambda_i = \lambda_1$ compute λ_1.

Invariant measures and orbit closures

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Artistic image of a billiard in a polygon

Varvara Stepanova. Joueurs de billard. Thyssen Museum, Madrid

Д.Б.Фукс исследовал геодезические „кривые" на многогранниках, и это привело его с странным наблюдениям и задачам комбинаторики конечных групп, которые чем-то напомнили мне эргодическую теорию дифференциалов Бельтрами и многоугольных бильярдов.

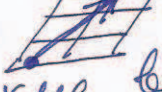
Пример: Геодезические на кубе. 

Рассмотрим (бесконечную) „развертку" куба в виде плоскости с решеткой, квадраты которой имеют размер граней куба.

На каждой грани геодезическая прямая, а при пересечении ребра она продолжается на соседней грани в виде такой прямой, что на развертке оба отрезка составляют одну прямую.

Каждое пересечение прямой на плоскости развертки с линией сетки (решетки) соответствует повороту куба на 90° вокруг нужного ребра.

Замкнутая геодезическая — это такая прямая на плоскости развертки, для которой последовательное перемножение нужных поворотов (соответствующих пересечениям с линиями сетки) поворачивает куб тождественно. Но можно следить и за его поступательными перемещениями при этих поворотах в $\mathbb{R}^3$, и тогда произведение будет движением аффинного евклидова пространства $\mathbb{R}^3$, сдвигающим его вдоль плоскости развертки: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+p \\ y+q \\ z \end{pmatrix}$.



И вот, Фукс решает вопрос: какие векторы $\binom{p}{q} \in \mathbb{Z}^2$ получаются из замкнутых геодезических (и сколько таких геодезических соответствует каждому реализуемому вектору: иногда это одно семейство, соответствующее определенной последовательности поворотов, но геодезической, а иногда — несколько, к. разным начальным точкам), тогда возникает интересная теоретико-числовая функция $N(p,q)$.

Я забыл вычисленные Фуксом ответы, но они были удивительные: удивительны и ограничения, на $\left(\frac{p}{q}\right)$, при которых $n(p,q) > 0$, и все случаи, когда $n(p,q)=1$, и наборы значений, реализуемые при коварном выборе пар (p,q).

Аналогичные вопросы он исследовал и для других многогранников (например, додекаэдра и тетраэдра, но и для некоторых неправильных — тоже).

Но, насколько я помню, все эти результаты оставались не полными решениями вопросов, а, скорее, удивительными примерами их разрешимости, тогда как в общем виде задача не поддается.

Я сразу же спросил Фукса (как раньше, Карпенкова): а по поводу цепных дробей, разрешима ли поставленная общая задача алгоритмически? Или может быть, удастся доказать её алгоритмическую неразрешимость? Остается только повторить этот оставшийся открытым вопрос.

Сходные алгоритмические вопросы вызывает и деятельность другого моего ученика, Жака-Оливье Муссафира, по комбинаторике групп кос и энтропиях кос — это он изобрел ради успешного перемешивания расплава из отработавших бутылок на фабрике по производству стекла. Перемешивают n стержней, вертикальных, с координатами $(x_i(t), y_i(t))$ для i-го стержня, — косу образует график этих движений в пространстве-времени $\mathbb{R}^3$. И вот, оказывается, некоторые косы перемешивают лучше, а другие хуже — и это связано с алгебраичностью некоторых чисел, описывающих эти динамические системы, и даже с некоторыми группами Галуа!

3. Статистика периодических решений хаотических динамических систем Фибоначчи.

Здесь (статья выйдет в Московском Математическом журнале, том имени Васильева) анализируются диаграммы Юнга перестановок конечных множеств, заданные разбиением множества на циклы перестановки.

Для случайных перестановок n объектов получающаяся своеобразные удивительные статистики (при усреднении по $n!$ перестановкам) инвариантов диаграмм выходит то же, что при экспериментах со случайными перестановками, вроде тасования колоды карт (например, я использовал в качестве датчиков случайных чисел номера телефонов из академических справочников разных стран, таблицы полей Галуа из p^2 элементов).

И вот, я сравниваю эти средние со статистикой таких же диаграмм для перестановок точек конечного тора $(\mathbb{Z}_n)^2$ (из n^2 точек) преобразованием $A(x,y)=(2x+y,\ x+y)$, которое я называю „преобразованием Фибоначчи", а физики — „кошкой Арнольда".

И статистики (при $n\to\infty$) получаются совсем не те, что для случайных перестановок. Например, иначе ведут себя такие параметры диаграммы Юнга: длина x, высота y, полнота $\lambda = s/(x,y)$ (где $s=\sum x_i$ — площадь диаграммы), $\mu = y/x$ я называю „асимметрией". И у всех этих величин интересные асимптотики при $s\to\infty$, разные для случайных перестановок и для циклов динамики кошки.

4. Упомяну ещё обширное исследование чисел Фробениуса $N(a_1,\dots,a_n)$ полугрупп натуральных чисел по сложению: если $(a_1,\dots,a_n)=1$, то целочисленные комбинации $x_1a_1+\dots+x_na_n$ ($x_i\geq 0$) выражают все числа $\ell \geq N$, и вот свойства этого числа N удивительны (см. таблицу ниже). Сильвестр нашел $N(a,b)=(a-1)(b-1)$ (скажем, $N(3,5)=8$), но ни формулы для $N(a,b,c)$, ни асимптотики при больших a,b,c нет.

Я доказываю оценки сверху и снизу вроде

$$C_1(\alpha)\sigma^{1+\frac{1}{n-1}} \leq N \leq C_2(\alpha)\sigma^{(2)}$$

где $\sigma(\vec a)=a_1+\dots+a_n$ и $\alpha(\vec a)=\vec a/\sigma(\vec a)$ — направление (вектора $\vec a$). Поразительно здесь то, что для некоторых направлений α достигается левая асимптотика, а для других — правая, и я, хотя и усреднял уже по симплексу направлений α, не нашел даже экспериментальной правдоподобной гипотезы об асимптотике средних при $\sigma\to\infty$.

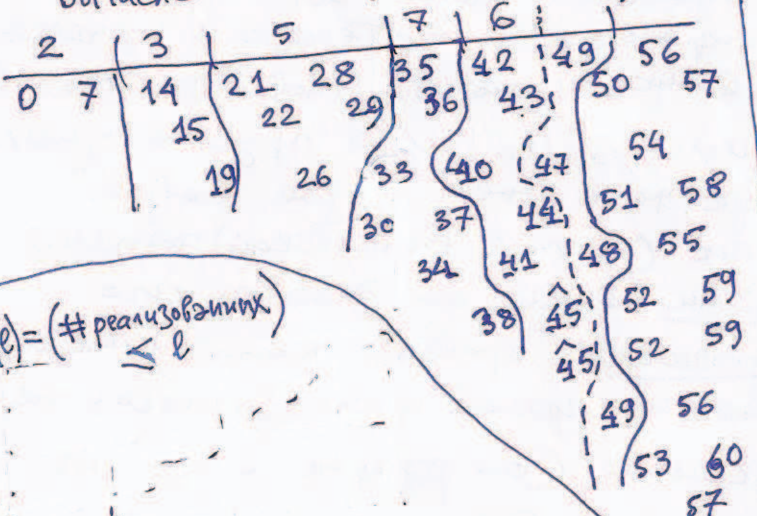
Отчет о моих собственных последних работах
(2005–2006): стр. 1–4.

сложности конечных последовательностей
1. Доклад о
нулей и единиц (ММО, 22 ноября 2005) есть
в Интернете на сайте общества и будет в
(новом) журнале „Функциональный анализ и другая
математика" (Шрингер в Москве).
В этом журнале и Вы приглашаетесь писать
(лучше по-английски), посылать рукописи нужно
электронно по адресу < phasis@AHA.RU >.

арифметические логарифмические "самая сложная" ...
Странный экспериментальный результат этой статьи — что
исследуемых
2. Статья „Статистика гладких функций
компонентами связности многообразий
графы, образованные
уровня гладкой функции Морса, как топологическое
пространство (с учетом упорядочения значений функции
в критических точках). Например, горы Эльбрус и
Везувий имеют графы (упорядоченные) Y и ⅄.
Нерешенный вопрос: сколько из этих графов
реализуется многочленами (степени, при которой получается
столько критических точек, сколько вершин у графа)?
Например, для многочленов степени 4 от двух
переменных, стремящихся на бесконечности к бесконечности
(и рассматриваемых как функции на S^2) графы —
деревья из 4 точек ветвления и 6 концевых вершин,
соединенных 9 ребрами). Таких (упорядоченных)
графов всего 17746. А сколько из них реализуется
многочленами степени 4 я не знаю (думаю, меньше
сотни).
Для тригонометрических многочленов (с 4 тройными
точками и 4 концевыми вершинами, соединенными
9 ребрами) $A\sin x + B\sin y + C\sin(x+y) + D\cos(x+y)$ число
графов гладких функций Морса 550, а число реализуемых
многочленами — не более 12 (думаю, ровно 12) — это
вариант 16й проблемы Гильберта.
Все эти
к сожалению, были им забыты, а потому остались не
исследованными (ни Гудковым, ни Петровским, ни Виро, ни Харламовым, ни
Никулиным, ни Шустиным, ни Оревковым и т.д.) →

Числа Фробениуса $N(a,b,c)$ с $a+b+c=41$

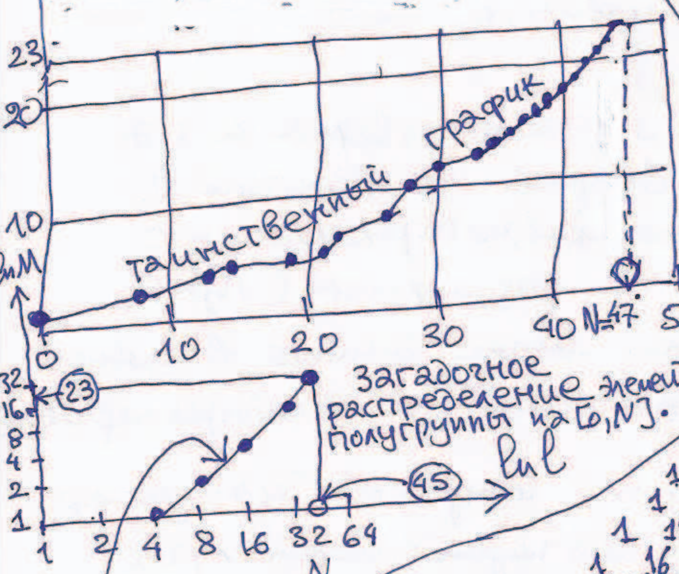
Вычисление $N(7,15,19) = 47$, реализованы 23 из 47 $\frac{23}{47} \approx \frac{1}{2}$ „загадочная постоянная".



Доказательство

$47 = 19 + 7\cdot 4$
$48 = 19 + 15 + 7\cdot 2$
$49 = 7\cdot 7$
$50 = 15 + 7\cdot 5$
$51 = 15\cdot 2 + 7\cdot 3$
$52 = 19\cdot 2 + 7\cdot 2$
$53 = 19\cdot 2 + 15$
$59 = 19 + 7\cdot 5$

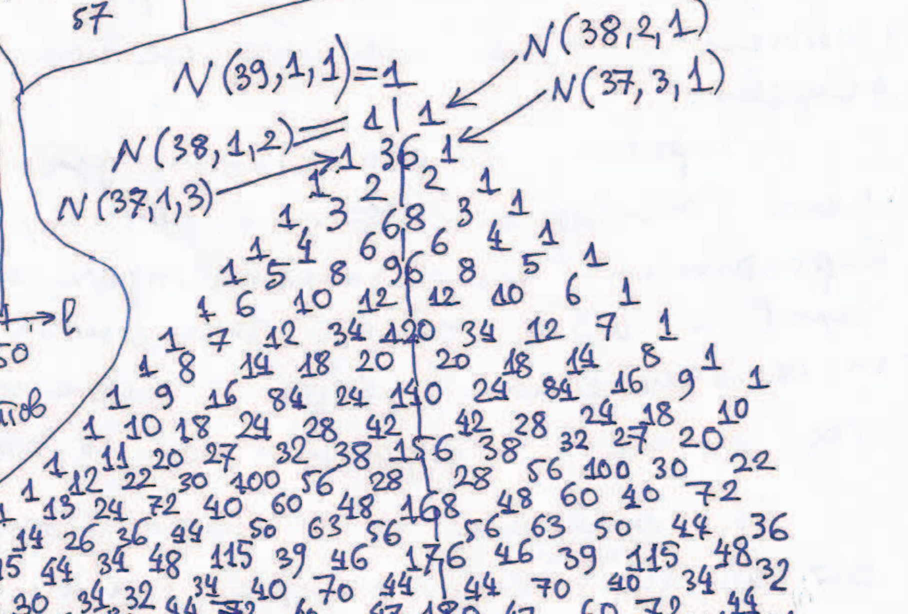
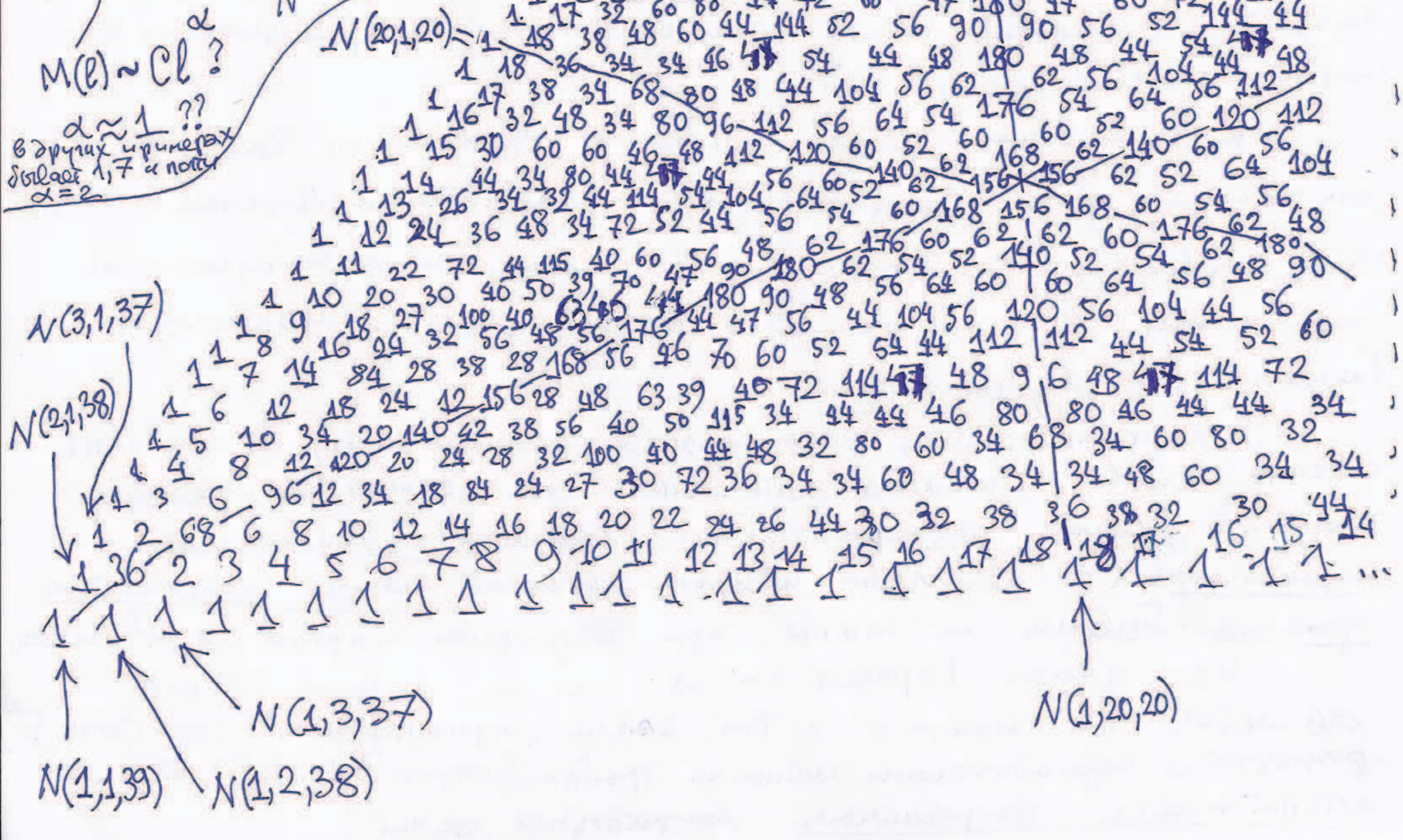
$N=47: 46 \neq 19x + 15y + 7z$



$M(l) \sim Cl^{\alpha}$?

$\alpha \simeq 1$??

в других примерах бывает 1,7 и почти $\alpha = 2$

В московском метро, около Академической, где мне выходить, меня взял за рукав пассажир, и сказал: „Вы меня не узнаёте, Владимир Игоревич? А ведь лет тридцать назад Вы читали нам в Хабаровске лекции, очень интересные. И вот, теперь я приехал на пару дней из Хабаровска в Москву, но в Институте Стеклова мне сказали, что Вы уже ушли в ИКИ, и я грустно ушел оттуда — а Вы как раз едете обратно в моем поезде метро!"

Оказывается, этот математик (Володя Быковский) занялся многомерными цепными дробями. Сейчас он доказал, что и для конечных, и для периодических дробей асимптотика средних (при увеличении периода и граней паруса) такая же, как у Вас с Максимом, универсальная (он раньше уже это опубликовал для обычных цепных дробей, со своей ученицей, Авдеевой).

Причем, по его словам, удается даже ответить и на мои (поставленные Вами в сторонке) вопросы о величинах вероятностей (треугольников, пятиугольников, четырехгранных углов и т.п.) и об их зависимости от размерности (бывают ли целочисленные ребра в трехмерном случае длиннее в среднем или короче в среднем, чем в двумерном или в четырехмерном).

Он ответил и на много других вопросов (например, всякий ли малый кусок паруса реализуется в периодическом случае, всякий ли большой кусок аппроксимируется периодическими с любой точностью и т.д.).

К сожалению, свое обещание прислать из Хабаровска (или теперь уже Владивостока) письменные сообщения о всех этих новостях он пока не выполнил, но я подумал, что это может быть интересно также и Вам, и Максиму, и Коркиной, и Муссафиру.

У Коркиной, кстати, дочка кончила среднюю школу, от чего мать смогла снова заниматься математикой. Она звонила, что доказала также: в больших размерностях (≥5?) возможна топологическая периодичность (с неполным набором периодов) паруса трансцендентного ортанта. Бывает ли такое при двумерном парусе в $\mathbb{R}^3$ — неясно.

Из задач Карпенкова я все не возьму в толк, является ли вопрос о том, какие „триангуляции" тора (даже T^2) реализуются периодическими цепными дробями (матрица $A \in SL(3, \mathbb{Z})$), алгоритмически разрешимым вопросом, или же нет?