Chaos in semiconductor laser optical injection at fractional-order

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June 17-19, 2015 ***at the*** **Fields Institute, Stewart Library**

2015 Summer Solstice

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Topics

- Background
- Chaos and complexity
- Mathematical model
- Theory of Fractional Calculus
- Poincare-Bendixson Theorem
- Modified Trapezoidal Rule
- Numerical Results
- Conclusion

Overview

- Laser is nonlinear dynamical system with continuous variables in which the minimal condition for the onset of deterministic chaos is presence of at least three degrees of freedom.
- In this research, we have present the numerical results which show that chaos exists in mathematical model of semiconductor laser optical injection system with order less than three.
- This research focus on numerical analysis of fractional model of semiconductor laser subject to optical injection model proposed by S.Wieczorek

Chaos and complexity

- Deterministic chaos arises in certain nonlinear dynamical system at certain sets of parameters
- Chaos theory is proliferate after the study of weather model by Lorenz
- Mathematically, arbitrarily small variations in initial conditions produce differences which vary exponentially over time.
- The behaviour of the chaotic systems are sensitive to initial conditions the behaviour ranges from periodic, periodic doubling and route to chaos.
- Chaotic system can described by three differential equations

Nature of Laser

- The laser is the light that have been amplification by stimulated emission of radiation.
- The laser light emits almost monochromatic and coherent beam of electromagnetic radiation.
- The output of the laser may be continuous, constant-amplitude output or continuous wave or pulsed.
- The mathematical model of laser are described by differential equations of electromagnetic fields and population inversion

Semiconductor Laser

- Semiconductor lasers are nonlinear optical device that have many applications in optical telecommunications.
- The instability and irregularity in semiconductor can rise in certain type of these optical devices
- Semiconductor lasers subject to external optical injection have received a lot of attention, mainly due to applications such as injection locking, frequency stabilization and chirp reduction and complex dynamical behavior such as undamped relaxation oscillations, quasiperiodicity, routes to chaos via period doubling .

Equivalent of chaos and laser model

- Chaos theory is stem from the study of weather model by Lorenz
- Haken have proven the equivalence between the single mode laser equations and atmospheric chaotic dynamics model of Lorenz
- The differential equations that describe the Lorenz chaos are similar to the Haken laser model

Poincare-Bendixson Theorem

- Theorem state that condition that deterministic chaos can exists with at least three degrees of independent variables or at least order three.
- Recent studies by shows that chaos can exists in systems with order less than three include the chaotic dynamics of fractional-order Arneodo's systems, fractional Chen system, chaos in the Newton–Leipnik system with fractional order, chaos in a fractional order modified Duffing system, fractional order Chua' system, fractional-order Volta's system, chaos in a fractional-order Rössler system, fractional-order Lorenz chaotic, and fractional order logistic.

Theory of Fractional Calculus

- Theory originate from the query on the meaning of half-order derivative from L'Hopital to Leibniz in year 1695
- Theory have many form of definitions for different purpose
- Recently, theory have apply to study in many fields include Rheology, Electrical Network, Viscoelasticity, Diffusive Transport, Probability and Statistics, Control Theory, Rheology, Signal Processing, Chaotic Dynamic etc.

Fractional order Chaos

- Recently, the study of dynamical behavior of the fractional order systems has started to attract increasing attention
- For example, chaotic dynamics of fractional-order Arneodo's systems, chaos in the Newton–Leipnik system with fractional order ,Chaos in a fractional-order Rössler system, chaos in a fractional order modified Duffing system, the fractional order Chua’ system , fractional Chen system, fractional order Lorenz , fractional order van der Pol system , fractional-order Volta's system and fractional order Logistic model.

Numerical methods

- There are many numerical method for fractional differential equation i.e., Adomian Decomposition, Galerkin Approximation, Predictor Corrector Scheme, Modified Trapezoidal Rule etc.
- The numerical approach in this paper is based on modified trapezoidal rule proposed by Odibat

Mathematical model

- Mathematical model of single mode semiconductor laser with monochromatic optical injection described by S.Wieczorek are model by three-dimensional rate equations that consist of the complex electric field and the normalized population inversion which described by the following equations

$$E' = K + (\frac{1}{2}(1+i)n - i\omega)E$$

$$n' = -2\Gamma n - (1+2Bn)(|E|^2 - 1)$$

Mathematical model (cont)

- The model can be rewritten as three ordinary differential equations that can be used for direct integration by separating the imaginary and real part of the complex electric field

$$\frac{\partial E_x}{\partial t} = K + \frac{1}{2}E_x n + (\omega - \frac{1}{2}\alpha n)E_y$$

$$\frac{\partial E_y}{\partial t} = -(\omega - \frac{1}{2}\alpha n)E_x + \frac{1}{2}E_y n$$

$$\frac{\partial n}{\partial t} = -2\Gamma n - (1 + 2Bn)(E_x^2 + E_y^2 - 1)$$

Nomenclature

E_x electric field in x direction

E_y electric field in y direction

n population inversion

B rescaled cavity lifetime of photons

α linewidth enhancement

K injected field strength

Γ rescaled damping rate of the relaxation oscillations

ω detuning of the injected field from the solitary laser frequency

Fractional-order model

- The fractional-order semiconductor laser subject to optical injection model can explain by the following equations (where *q* are fractional-order)

$$\frac{\partial^q E_x}{\partial t^q} = K + \frac{1}{2}E_x n + (\omega - \frac{1}{2}\alpha n)E_y$$

$$\frac{\partial^q E_y}{\partial t^q} = -(\omega - \frac{1}{2}\alpha n)E_x + \frac{1}{2}E_y n$$

$$\frac{\partial^q n}{\partial t^q} = -2\Gamma n - (1 + 2Bn)(E_x^2 + E_y^2 - 1)$$

Numerical method

- In this paper, numerical calculation of the fractional differential equations are carried out by modified trapezoidal rule.
- The existence of chaos explain by power spectrum and phase space diagram

Modified Trapezoidal rule

Consider $y = f(x)$ over $[a,b]$ suppose that the interval $[a,b]$ is subdivided into m Subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of equal width $h = \dfrac{b-a}{m}$ by using the equally spaced nodes $x_k = x_0 + kh$ for $k = 1, 2, \ldots, m$

The composite trapezoidal rule for m subinterval is

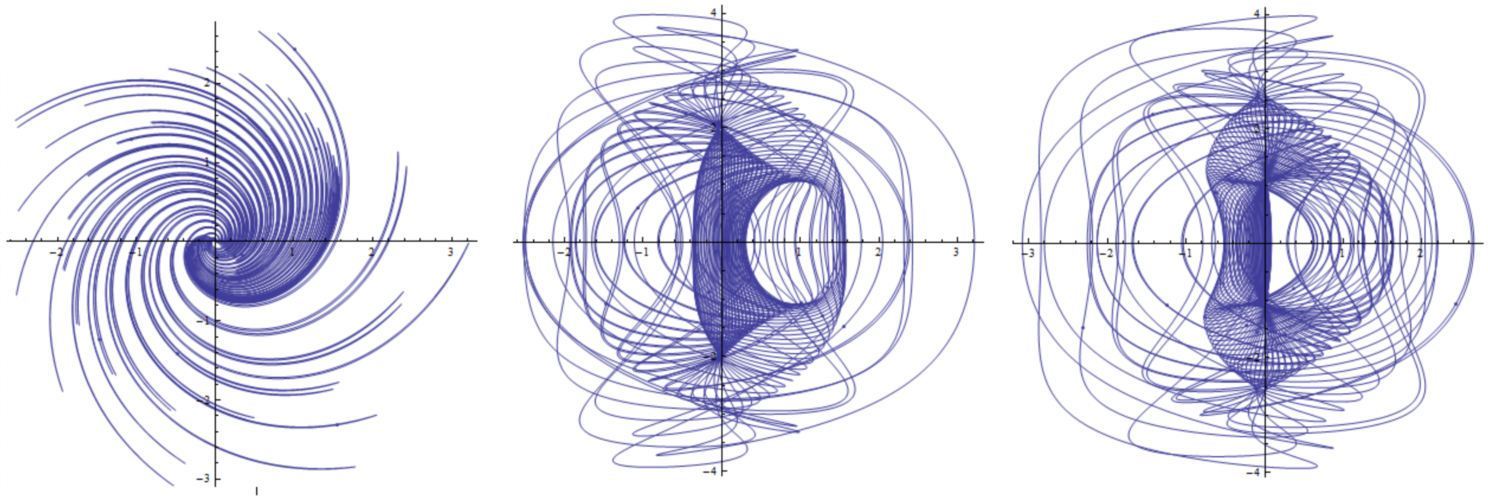
$$T(f,h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{k=1}^{m} f(x_k)$$

The formula can extended to use with fractional order differential as follow

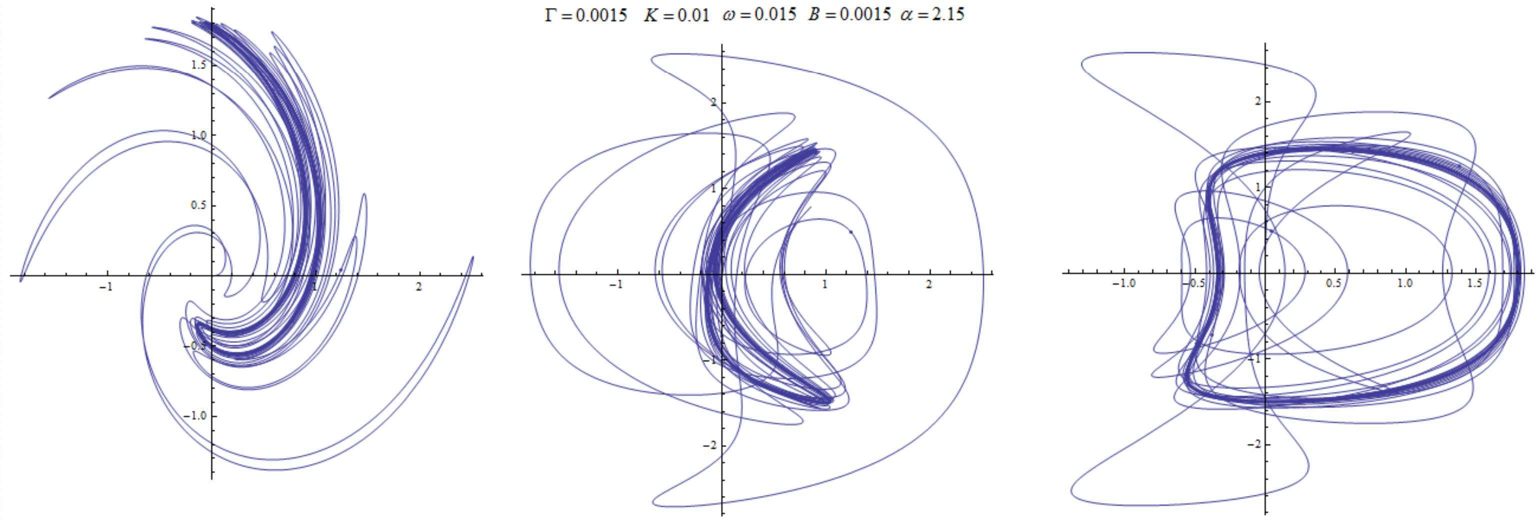
$$T(f,h,q) = ((k-1)^{q+1} - (k-q-1)k^{q})\frac{h^q f(0)}{\Gamma(q+2)} +$$

$$\sum_{j=1}^{k-1}((k-j+1)^{q+1} - 2(k-j)^{q+1} + (k-j-1)^{q+1})\frac{h^q f(x_j)}{\Gamma(q+2)}$$

Numerical Results



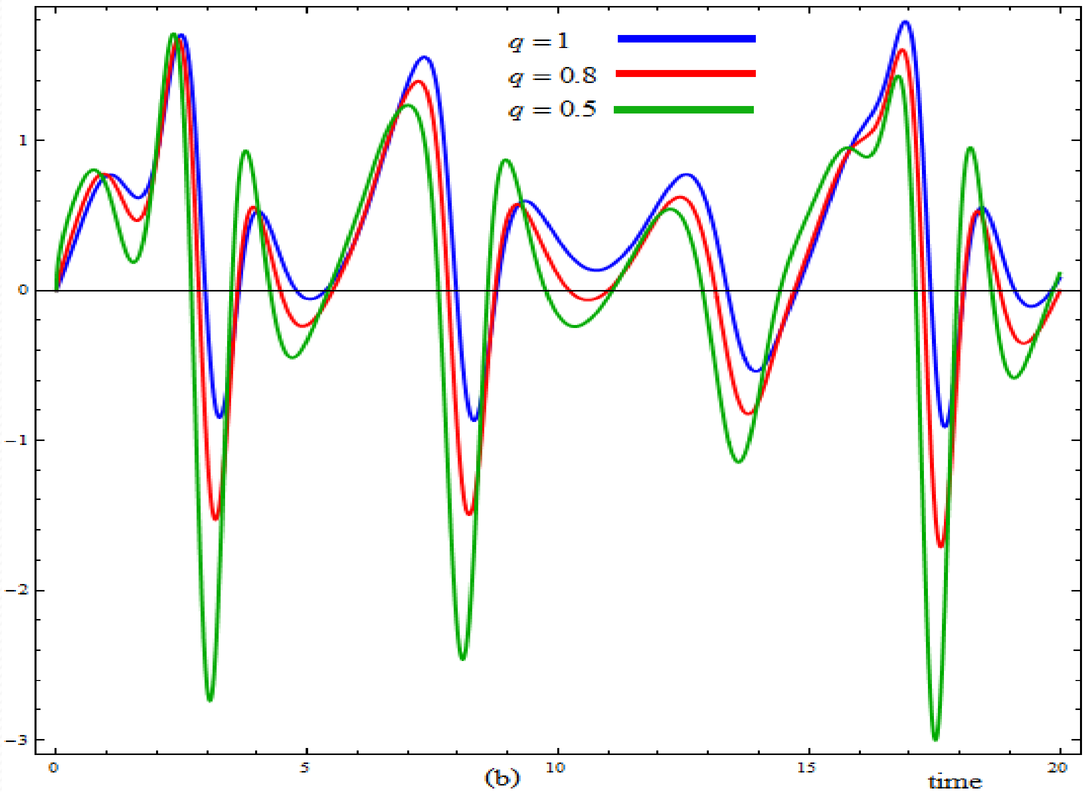
$\Gamma = 0.0015 \quad K = 0.01 \quad \omega = 0.015 \quad B = 0.0015 \quad \alpha = 2.15$



$\Gamma = 0.0015 \quad K = 0.1 \quad \omega = 0.015 \quad B = 0.0015 \quad \alpha = 2.25$

Time series

- time series of electric field at order q=0.5,q=0.8 and q=1

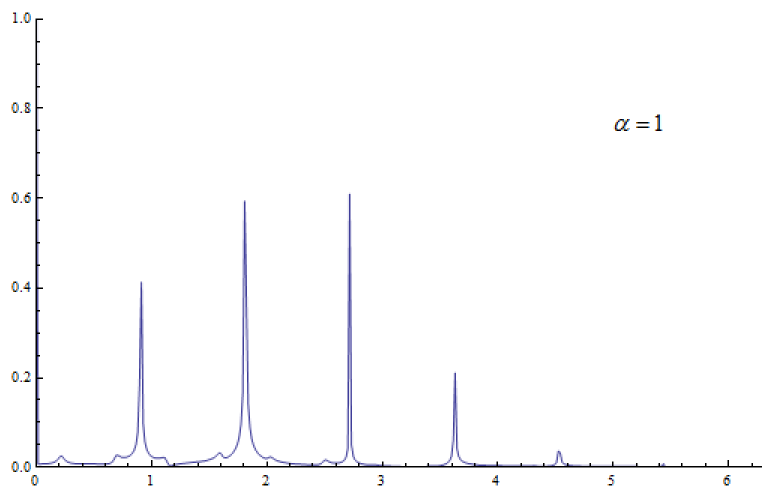


Power spectrum

- The power spectrum of a signal is defined as the square of its Fourier amplitude per unit time.
- For the limit cycle, the power spectrum show as single peak for single periodic orbit
- For the chaotic signal, the power spectrum show as continuous spectrum

Power spectrum

- Single peak on the left correspond limit cycle solution of integer-order differential equation
- Continuous spectrum on the right correspond with chaotic solution of fraction-order differential equation



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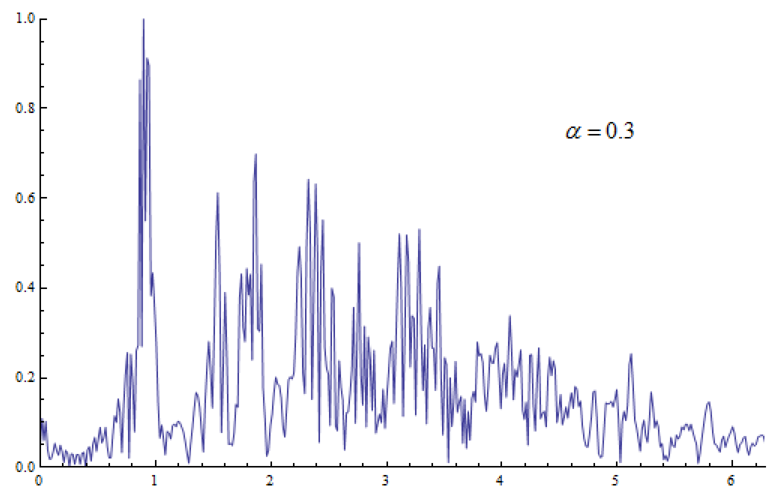

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Conclusion

- Chaos can exists in system with order less than three
- The fractional order integral are calculate by modify trapezoidal rule.
- Numerical results show that chaos exists in fractional-order model of semiconductor laser subject to optical injection
- The advantages in modeling dynamical system with fractional order are feasible to explain the transient mechanism.

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