Stable bases for moduli of sheaves

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Moduli of sheaves on $\mathbb{P}^2$

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- By the ADHM presentation, the space $\mathcal{M}_{v,w}$ is isomorphic to the space of quadruples (X, Y, A, B) of linear maps:

$$X, Y : \mathbb{C}^v \to \mathbb{C}^v, \qquad A : \mathbb{C}^w \to \mathbb{C}^v, \qquad B : \mathbb{C}^v \to \mathbb{C}^w$$

such that $[X, Y] + AB = 0$, modulo the action of $GL(v)$.

Fixed points and $K-$theory

- The torus $T = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^w$ acts on $\mathcal{M}_{v,w}$ via:

$$(q, t, U) \cdot (X, Y, A, B) \longrightarrow q \left(Xt, Yt^{-1}, AU^{-1}, UB\right)$$

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- These fixed points form a basis of localized K−theory groups:

$$K(w) = \bigoplus_{v \in \mathbb{N}} K_{v,w}, \qquad \text{where} \quad K_{v,w} = K_T^0(\mathcal{M}_{v,w})$$

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- As a vector space, $K(w)$ is isomorphic to $(\text{Fock space})^{\otimes w}$

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- For pairs of integers such that $v^+ = v^- + 1$, define:

$$\mathfrak{Z}_{v^+,v^-,w} = \{(\mathcal{F}_+ \subset \mathcal{F}_-)\}$$

$$\mathcal{M}_{v^+,w} \xleftarrow{\pi^+} \mathfrak{Z}_{v^+,v^-,w} \xrightarrow{\pi^-} \mathcal{M}_{v^-,w}$$

and the line bundle $\mathcal{L}|_{(\mathcal{F}^+ \subset \mathcal{F}_-)} = R\Gamma(\mathbb{P}^2, \mathcal{F}_-/\mathcal{F}_+)$ on $\mathfrak{Z}_{v^+,v^-,w}$

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- **Theorem** (Schiffmann-Vasserot, Nakajima): The operators

$$p_{\pm 1,m} = \pi_*^{\pm}\left(\mathcal{L}^m \cdot \pi^{\mp *}\right) \quad : \quad K_{\bullet,w} \longrightarrow K_{\bullet \pm 1,w}$$

give rise to an action of $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ on $K(w)$, for any given w.

Subalgebras of $U_{q,t}(\ddot{\mathfrak{gl}}_1)$

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$$\mathcal{B}_{\frac{m}{n}} = \langle p_{kn,km} \rangle_{k \in \mathbb{Z}} \subset U_{q,t}(\ddot{\mathfrak{gl}}_1)$$

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- $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ is quasitriangular Hopf w.r.t. the coproduct:

$$\Delta(p_{k,0}) = p_{k,0} \otimes 1 + c^k \otimes p_{k,0}$$

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- and with it one associates a universal $R-$matrix:

$$\mathcal{R} \in U_{q,t}(\ddot{\mathfrak{gl}}_1) \widehat{\otimes} U_{q,t}(\ddot{\mathfrak{gl}}_1)$$

$R-$matrices according to Maulik-Okounkov

- $\mathcal{R}$ factors in terms of the $R-$matrices of the subalgebras:

$$\mathcal{R} = \prod_{\frac{m}{n} \in [0;-0)}^{\rightarrow} \mathcal{R}(\mathcal{B}_{\frac{m}{n}}) = \prod_{\frac{m}{n} \in [0;-0)}^{\rightarrow} \exp \left(\sum_{k=1}^{\infty} \frac{p_{kn,km} \otimes p_{-kn,-km}}{k} \right)$$

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- as a **uniquely** determined class supported on the stable leaf:

$$s^{\sigma,\mathcal{L}}_{\boldsymbol{\lambda}} \in K_T(\text{attracting set of } \boldsymbol{\lambda}), \qquad \forall \boldsymbol{\lambda} \in X^{T_{\text{symp}}}$$

The geometric $R-$matrix

- When $X = \mathcal{M}_{v,w}$, Maulik-Okounkov define their $R-$matrices as the change of stable basis $\mathcal{R}^{0\to -0}_{\sigma\to-\sigma}$, where we define:

$$\mathcal{R}^{l\to l'}_{\sigma\to\sigma'} \ : \ K(w)\otimes K(w') \stackrel{\text{Stab}^l_\sigma}{\longrightarrow} K(w+w') \stackrel{\text{Stab}^{l'}_{\sigma'}}{\longleftarrow} K(w)\otimes K(w')$$

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$$\mathcal{R}^{0 \to -0}_{\sigma \to -\sigma} = \prod_{l=-\infty}^{0} \mathcal{R}^{l-\varepsilon \to l+\varepsilon}_{-\sigma \to -\sigma} \prod_{l=0}^{\infty} \mathcal{R}^{l-\varepsilon \to l+\varepsilon}_{\sigma \to \sigma}$$

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- These match the $R-$matrices from the $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ factorization:

$$\mathcal{R} = \mathcal{R}^{0 \rightarrow -0}_{\sigma \rightarrow -\sigma}, \qquad \qquad \mathcal{R}(\mathcal{B}_l) = \mathcal{R}^{l-\varepsilon \rightarrow l+\varepsilon}_{\sigma \rightarrow \sigma}$$

Explicit formulas in the stable basis

- Fix σ and write $l = \frac{m}{n}$. The above predicts that the map:

$$\text{Stab}^{\frac{m}{n}} \ : \ s_{\boldsymbol{\lambda}}^{\frac{m}{n}} \otimes s_{\boldsymbol{\mu}}^{\frac{m}{n}} \longrightarrow s_{\boldsymbol{\lambda} \oplus \boldsymbol{\mu}}^{\frac{m}{n}}$$

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- With respect to the above coproduct, the generators $p_{kn,km}$ are primitive. Hence we expect them to act component-wise:

$$p_{kn,km} \cdot s^{\frac{m}{n}}_{\boldsymbol{\lambda}} = \sum_{\text{by changing only one constituent partition}}^{\boldsymbol{\mu} \text{ obtained from } \boldsymbol{\lambda}} s^{\frac{m}{n}}_{\boldsymbol{\mu}} \cdot \text{ coefficient}$$

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- In fact, when $\gcd(m, n) = 1$ we have **(N)**:

$$p_{n,m} \cdot s^{\frac{m}{n}}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} = \boldsymbol{\lambda} + \text{added } n-\text{ribbon } R} s^{\frac{m}{n}}_{\boldsymbol{\mu}} \cdot (-1)^{\text{ht } R} \chi_m(R)$$

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$$e_{kn,km} \cdot s^{\frac{m}{n}}_{\boldsymbol{\lambda}} = \sum_{\text{no two next to each other}}^{\boldsymbol{\mu}=\boldsymbol{\lambda}+k \text{ added } n-\text{ribbons}} s^{\frac{m}{n}}_{\boldsymbol{\mu}} \cdot \prod_{i=1}^{k} (-1)^{\text{ht } R_i} \chi_m(R_i)$$

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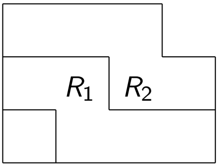
- 

$w = 1, \; n = 5, \; k = 2$

$\lambda = (1)$

$\mu = (4, 4, 3)$