Convergence of Filtered Spherical Harmonic Equations for Radiation Transport

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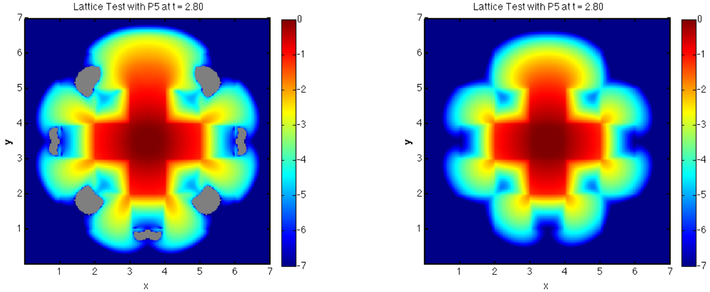
Outline & References

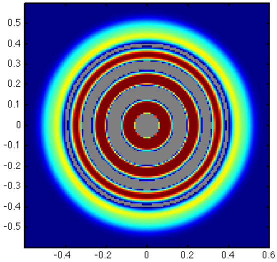
- Filtered P_N equations[1]
- Convergence analysis
  - Modified equation[2]
  - Galerkin estimate[3]
  - Convergence estimates
- Numerical experiments using StaRMAP[4]

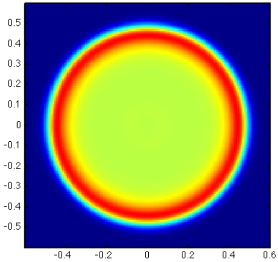
[1] McClarren, Hauck, JCP 2010
[2] Radice et al., JCP 2013
[3] Schmeiser, Zwirchmayr, SINUM 1999
[4] Seibold, Frank, TOMS 2014

Checkerboard: P_5 versus FP_5



Line Source: P_9 versus FP_9





Challenges

Challenges in radiation transport:

- Highly heterogeneous media
- Media/initial conditions/sources lead to non-smooth solutions
- Preserve realizability, rotational invariance
- Capture beams

Challenges for spectral methods:

- Spectral methods achieve fast convergence for smooth solutions
- But suffer from the Gibbs phenomenon
- Idea of filtering: dampen the coefficients in the expansion
- Con: Some adjustments of the filter strength may be required for different problems
- Pro: Speed, overall accuracy, and simplicity

FILTERED P_N

Radiation Transport

$$\partial_t\psi(t,x,\Omega)+\Omega\cdot\nabla_x\psi(t,x,\Omega)+\sigma_{\mathrm{a}}(x)\psi(t,x,\Omega)-(\mathcal{Q}\psi)(t,x,\Omega)=S(t,x,\Omega)$$

- $\psi(t,x,\Omega)$: density of particles, with respect to the measure $d\Omega dx$, which at time $t\in\mathbb{R}$ are located at position $x\in\mathbb{R}^3$ and move in the direction $\Omega\in\mathbb{S}^2$.
- Scattering operator
$$(\mathcal{Q}\psi)(t,x,\Omega)=\sigma_{\mathrm{s}}(x)\left[\int_{\mathbb{S}^2} g(x,\Omega\cdot\Omega')\psi(t,x,\Omega')d\Omega'-\psi(t,x,\Omega)\right]$$

$$\mathcal{T}\psi=S$$

Sphercial Harmononic P_N equations

Notation:

- Real-valued spherical harmonic m_ℓ^k, $\ell = 0, 1, \dots,$ $k = -\ell, \dots, \ell$
- Angular integration $\langle \cdot \rangle = \int_{\mathbb{S}^2} (\cdot) \, d\Omega$

Spectral Galerkin method:

- Expand unknown $\psi \approx \psi_{\mathrm{PN}} \equiv \mathbf{m}^T \mathbf{u}_{\mathrm{PN}}$
- Plug into equation and project residual

$$\langle \mathbf{m} \mathcal{T}(\mathbf{m}^T \mathbf{u}_{\mathrm{PN}}) \rangle = \langle \mathbf{m} S \rangle =: \mathbf{s}.$$

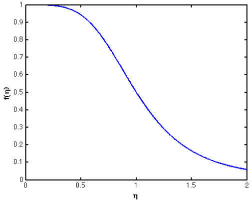
- Other combinations of ansatz and projection can be used!

P_N equations

$$\partial_t \mathbf{u}_{\mathrm{PN}} + \mathbf{A} \cdot \nabla_x \mathbf{u}_{\mathrm{PN}} + \sigma_{\mathrm{a}} \mathbf{u}_{\mathrm{PN}} - \sigma_{\mathrm{s}} \mathbf{G} \mathbf{u}_{\mathrm{PN}} = \mathbf{s},$$

where $\mathbf{A} := \langle \mathbf{m} \mathbf{m}^T \Omega \rangle$ and $\mathbf{G}$ is diagonal

Filtering



- Filtering well-known in spectral methods
- A filter of order α is a function $f \in C^{\alpha}(\mathbb{R}^{+})$, which fulfills
  $f(0) = 1,\ f^{(k)}(0) = 0,\ \text{for } k = 1, \ldots, \alpha - 1, \text{and } f^{(\alpha)}(0) \neq 0$
- Additional condition
$$f(\eta) \geq C(1-\eta)^{k}\,, \quad \eta \in [\eta_0, 1]$$
- Filtering the expansion after every time step
$$\sum_{\ell=0}^{N} \sum_{k=-\ell}^{\ell} \left(f\left(\frac{\ell}{N+1} \right) \right)^{\beta \Delta t} \mathrm{u}_{\ell}^{\mathrm{k}} \mathrm{m}_{\ell}^{\mathrm{k}}.$$

NUMERICAL ANALYSIS

Main Result

Galerkin estimate

$$\|\psi(t,\cdot,\cdot)-\psi_{\mathrm{FPN}}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \leq \|\psi(t,\cdot,\cdot)-\mathcal{P}_N\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} + t\Big(\|\mathbf{a}_{N+1}\cdot\nabla_x\langle\mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} + \beta\|\mathbf{G}_f\langle\mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}\Big)$$

Rates

$$\|\psi(t,\cdot,\cdot)-\mathcal{P}\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \leq CN^{-q}\|\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^q(\mathbb{S}^2)))}$$

$$\|\mathbf{a}_{N+1}\cdot\nabla_x\langle\mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \leq CN^{-r}\|\nabla_x\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^r(\mathbb{S}^2)))}$$

$$\|\mathbf{G}_f\langle\mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \leq \begin{cases} CN^{-q+1/2}, & \alpha > q-\frac{1}{2} \\ CN^{-\alpha+\varepsilon}, & \alpha \leq q-\frac{1}{2} \end{cases}$$

Sobolev Spaces

- $H^q(\mathbb{S}^2)$ Sobolev space on the unit sphere with norm

$$\|\Phi\|_{H^q(\mathbb{S}^2)} := \left(\sum_{|\alpha| \le q} \int_{\mathbb{S}^2} |D^\alpha \Phi(\Omega)|^2 d\Omega \right)^{1/2}$$

- Spherical harmonics are eigenfunctions of Laplace-Beltrami operator

$$\mathcal{L} m_\ell^k = -\lambda_\ell m_\ell^k, \quad \lambda_\ell = \ell(\ell+1)$$

- Expansion coefficients $\Phi_\ell^k := \langle m_\ell^k \Phi \rangle$ of any function $\Phi \in H^{2q}(\mathbb{S}^2)$ satisfy

$$\Phi_\ell^k = \langle m_\ell^k \Phi \rangle = \frac{1}{(-\lambda_\ell)^q} \langle (\mathcal{L}^q m_\ell^k) \Phi \rangle = \frac{1}{(-\lambda_\ell)^q} \langle m_\ell^k \mathcal{L}^q \Phi \rangle$$

Spectral Convergence

- L^2-orthogonal projection of a generic function $\Phi \in L^2(\mathbb{S}^2)$ onto $\mathbb{P}_N$
$$\mathcal{P}_N \Phi = \mathbf{m}^T \langle \mathbf{m}\mathbf{m}^T \rangle^{-1} \langle \mathbf{m}\Phi \rangle = \mathbf{m}^T \langle \mathbf{m}\Phi \rangle$$
- Projection onto polynomials of exact degree ℓ
$$(\mathcal{P}_\ell - \mathcal{P}_{\ell-1})\Phi = \mathbf{m}_\ell^T \langle \mathbf{m}_\ell \mathbf{m}_\ell^T \rangle^{-1} \langle \mathbf{m}_\ell \Phi \rangle = \mathbf{m}_\ell^T \langle \mathbf{m}_\ell \Phi \rangle$$
- Spectral convergence
$$\begin{aligned} \|\langle \mathbf{m}_\ell \Phi \rangle\|^2_{\mathbb{R}^{n_\ell}} &= \|(\mathcal{P}_\ell - \mathcal{P}_{\ell-1})\Phi\|^2_{L^2(\mathbb{S}^2)} \leq \|(\mathcal{I} - \mathcal{P}_\ell)\Phi\|^2_{L^2(\mathbb{S}^2)} \\ &= \sum_{k=\ell+1}^{\infty} |\phi_\ell|^2 = \sum_{k=\ell+1}^{\infty} \frac{1}{(-\lambda_\ell)^{2q}} |\langle \mathbf{m}_\ell \mathcal{L}^q \Phi \rangle|^2 \\ &\leq \frac{1}{(\ell(\ell+1))^{2q}} \|\phi\|^2_{H^{2q}(\mathbb{S}^2)} \end{aligned}$$

Step 1: Modified Equation

- Time step

$$\mathbf{u}_{\mathrm{FPN}}^{n+1,*} = \mathbf{u}_{\mathrm{FPN}}^{n} - \Delta t(\mathbf{A}\cdot\nabla_x \mathbf{u}_{\mathrm{FPN}}^{n} + \sigma_{\mathrm{a}}\mathbf{u}_{\mathrm{FPN}}^{n} - \sigma_{\mathrm{s}}\mathbf{G}\mathbf{u}_{\mathrm{FPN}}^{n} - \mathbf{s}^n)$$

- Filtering

$$\mathbf{u}_{\mathrm{FPN}}^{n+1} = \mathbf{f}^{\beta\Delta t}\mathbf{u}_{\mathrm{FPN}}^{n+1,*} = \mathbf{u}_{\mathrm{FPN}}^{n+1,*} + \Delta t\frac{\exp(\beta\log(\mathbf{f})\Delta t) - 1}{\Delta t}\mathbf{u}_{\mathrm{FPN}}^{n+1,*}$$

- Operator split discretization of

Modified equation

$$\partial_t\mathbf{u}_{\mathrm{FPN}} + \mathbf{A}\cdot\nabla_x\mathbf{u}_{\mathrm{FPN}} + \sigma_{\mathrm{a}}\mathbf{u}_{\mathrm{FPN}} - \sigma_{\mathrm{s}}\mathbf{G}\mathbf{u}_{\mathrm{FPN}} - \beta\mathbf{G}_f\mathbf{u}_{\mathrm{FPN}} = \mathbf{s},$$

where $\mathbf{G}_f$ is diagonal with entries $\log\left(f\left(\frac{\ell}{N+1}\right)\right)$, $\ell = 0,\dots,N$.

Step 2: Galerkin Estimate

- Residual
  $\psi - \psi_{\mathrm{FPN}} = (\psi - \mathcal{P}_N\psi) + \mathcal{P}_N\psi - \psi_{\mathrm{FPN}} = (\psi - \mathcal{P}_N\psi) + \mathbf{m}^T\mathbf{r}$
- Multiply by $\mathbf{m}^T\mathbf{r}$ and integrate in angle and space

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\mathbf{r}|^2 dx = -\int_{\mathbb{R}^3} \mathbf{r}_N^T \mathbf{a}_{N+1} \cdot \nabla_x \langle \mathbf{m}_{N+1}\psi\rangle dx$$
$$- \sigma_{\mathrm{f}} \int_{\mathbb{R}^3} \mathbf{r}^T \mathbf{G}_f \langle \mathbf{m}\psi\rangle dx - \int_{\mathbb{R}^3} \mathbf{r}^T \mathbf{M}\mathbf{r} dx \,.$$

- $\mathbf{M} := \sigma_{\mathrm{a}}\mathbf{I} - \sigma_{\mathrm{s}}\mathbf{G} - \sigma_{\mathrm{f}}\mathbf{G}_{\mathrm{f}}$ is positive definite
- This yields

$$\partial_t \|\mathbf{r}\|_{L^2(\mathbb{R}^3;\mathbb{R}^n)} \leq \|\mathbf{a}_{N+1} \cdot \nabla_x \langle \mathbf{m}_{N+1}\psi\rangle\|_{L^2(\mathbb{R}^3;\mathbb{R}^{2N+1})}$$
$$+ \sigma_{\mathrm{f}} \|\mathbf{G}_{\mathrm{f}} \langle \mathbf{m}\psi\rangle\|_{L^2(\mathbb{R}^3;\mathbb{R}^n)}$$

- Control error by projection error + residual $\mathbf{r}$

Step 3: Convergence Estimate

- Estimate filter term

$$
\begin{aligned}
&\|\mathbf{G}_\mathrm{f}\langle \mathbf{m}\psi(t,\cdot,\cdot)\rangle\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^n)} \\
=&\sum_{\ell=0}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\|\langle \mathbf{m}_\ell\psi(t,\cdot,\cdot)\rangle\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{n_\ell})} \\
=&\sum_{\ell=1}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\|(\mathcal{P}_\ell-\mathcal{P}_{\ell-1})\psi(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \\
=&C\sum_{\ell=1}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\|(\mathcal{I}-\mathcal{P}_{\ell-1})\psi(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \\
\le&C\sum_{\ell=1}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\frac{1}{\ell^{2q}}\|\psi(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^3;H^q(\mathbb{S}^2))}
\end{aligned}
$$

Step 3: Convergence Estimate

- For $\theta \leq 2q$

$$\sum_{\ell=1}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\frac{1}{\ell^{2q}}$$

$$\leq \frac{1}{(N+1)^{\theta-1}} \underbrace{\frac{1}{N+1}\sum_{\ell=1}^{N} \log^2\left(f\left(\tfrac{\ell}{N+1}\right)\right)\left(\tfrac{N+1}{\ell}\right)^{\theta}}_{=:\Sigma}$$

- Interpret as Riemann sum

$$\Sigma \sim \int_0^1 \log^2\left(f(\eta)\right)\eta^{-\theta}d\eta$$

- Around $\eta = 0$, $\log f(\eta) \leq C\eta^{\alpha}$
- Σ Integrable for $\theta < 2\alpha + 1$

Step 3: Convergence Estimate

Two cases:

Case 1: $\alpha > q - \frac{1}{2}$. Choose $\theta = 2q$, convergence limited by the regularity of ψ

$$\|\mathbf{G}_\mathrm{f}\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \leq CN^{-q+1/2}$$

Case 2: $\alpha \leq q - \frac{1}{2}$. Choose $\theta = 2\alpha + 1 - \delta$, where $\delta > 0$ is arbitrary, convergence limited by the filter order

$$\|\mathbf{G}_\mathrm{f}\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \leq CN^{-\alpha+\varepsilon},$$

where $\varepsilon = \delta/2$

Main Result

Galerkin estimate

$$
\begin{aligned}
\|\psi(t,\cdot,\cdot) - \psi_{\mathrm{FPN}}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))}
&\le \|\psi(t,\cdot,\cdot) - \mathcal{P}\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \\
&\quad + t\Big(\|\mathbf{a}_{N+1}\cdot\nabla_x\langle \mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \\
&\qquad + \beta\|\mathbf{G}_f\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}\Big)\,,
\end{aligned}
$$

Rates

$$
\|\psi(t,\cdot,\cdot) - \mathcal{P}\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \le CN^{-q}\|\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^q(\mathbb{S}^2)))}
$$

$$
\begin{aligned}
\|\mathbf{a}_{N+1}\cdot\nabla_x\langle \mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}
&\le CN^{-r}\|\nabla_x\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^r(\mathbb{S}^2)))}
\end{aligned}
$$

$$
\|\mathbf{G}_f\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \le \begin{cases} CN^{-q+1/2}, & \alpha > q - \frac{1}{2} \\ CN^{-\alpha+\varepsilon}, & \alpha \le q - \frac{1}{2} \end{cases}
$$

Sharper Estimate

Galerkin estimate

$$\|\psi(t,\cdot,\cdot) - \psi_{\mathrm{FPN}}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))}$$
$$\leq \|\psi(t,\cdot,\cdot) - \mathcal{P}\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))}$$
$$+ t\Big(\|\mathbf{a}_{N+1}\cdot\nabla_x\langle \mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}$$
$$+ \beta\|\mathbf{G}_f\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}\Big)\,,$$

Rates for monotone moment sequences

$$\|\psi(t,\cdot,\cdot) - \mathcal{P}\psi(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))} \leq CN^{-q}\|\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^q(\mathbb{S}^2)))}$$

$$\|\mathbf{a}_{N+1}\cdot\nabla_x\langle \mathbf{m}_{N+1}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))}$$
$$\leq CN^{-(r+\frac{1}{2})}\|\nabla_x\psi\|_{C([0,T];L^2(\mathbb{R}^3;H^r(\mathbb{S}^2)))}$$

$$\|\mathbf{G}_f\langle \mathbf{m}\psi\rangle\|_{C([0,T];L^2(\mathbb{R}^3;\mathbb{R}^n))} \leq \begin{cases} CN^{-q}, & \alpha > q \\ CN^{-\alpha+\varepsilon}, & \alpha \leq q \end{cases}$$

NUMERICAL RESULTS

Numerical Results: General setup

- Use the code StaRMAP to compute the P_N and FP_N solutions for $N = 3, 5, 17, 33, (65)$.
- Apply the filter term after each sub-step to the updated components.
- Use the exponential filter of order $\alpha = 2, 4, 8, 16$

$$f(\eta) = exp(c\eta^{\alpha}), \text{ with } c = \log(\varepsilon_M)$$

  with ε_M being the machine precision. Set the effective filter opacity $f_{eff} = 10$ ($f_{eff} = \beta \log(f(\frac{N}{N+1})))$.
- Fix the spatial resolution, so that the space-time errors are negligibly small
- Compare to reference solution $P_{N_{\text{true}}}$
- Highest resolution P_{129} (8515 moments) on 500×500 grid (altogether 2.1×10^9 unknowns)

Numerical Results: Estimates

- Measure smoothness of true solution

$$B_N = \|\langle \mathbf{m}_N \psi \rangle\|_{L(\mathbb{R}^2,\mathbb{R}^n)} \sim N^{-q+\frac{1}{2}}$$

$$D_N = \|\langle \mathbf{m}_N \nabla_x \psi \rangle\|_{L(\mathbb{R}^2,\mathbb{R}^n)} \sim N^{-r+\frac{1}{2}}$$

- Compare to convergence estimate

$$E_N = \|\psi - \psi_N\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))}$$

$$R_N = \|\mathcal{P}\psi - \psi_N\|_{L^2(\mathbb{R}^3;L^2(\mathbb{S}^2))}$$

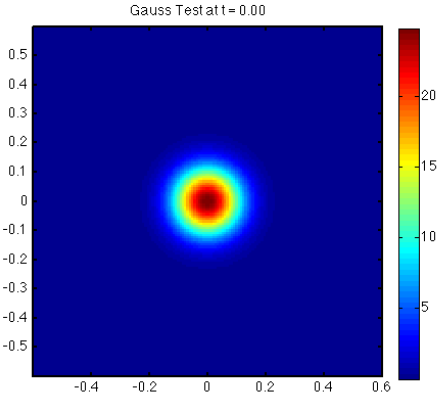
- Expectation

With filter: $E_N \sim R_N \sim N^{-\min\{q,r+\frac{1}{2},\alpha\}}$

Without filter: $E_N \sim N^{-\min\{q,r+\frac{1}{2}\}}, \ R_N \sim N^{-(r+\frac{1}{2})}$

- Central difference for ∇_x, trapezoidal rule for integration

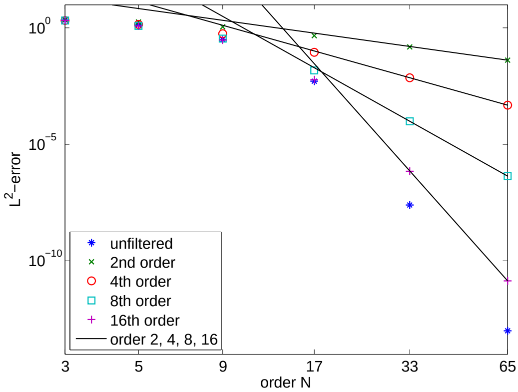
Gaussian Test: Setup



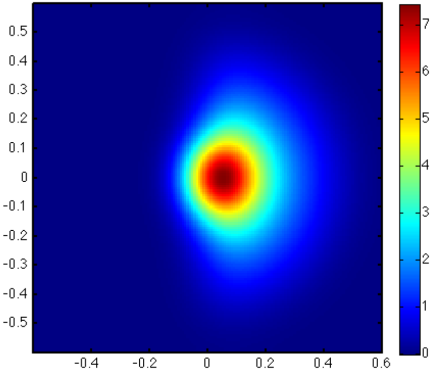
- Initial condition: $u_0^0 = \frac{1}{4\pi\times 10^{-3}} \exp\left(-\frac{x^2+y^2}{4\times 10^{-3}}\right)$, $u_\ell^k = 0$, for $k, \ell \neq 0$
- Purely scattering medium: $\sigma_t = \sigma_s = 1$

Gaussian Test: Results

- $q = r = \infty$
- With filter: $E_N \sim R_N \sim N^{-\min\{q, r+\frac{1}{2}, \alpha\}}$
- Without filter: $E_N \sim N^{-\min\{q, r+\frac{1}{2}\}}$, $R_N \sim N^{-(r+\frac{1}{2})}$

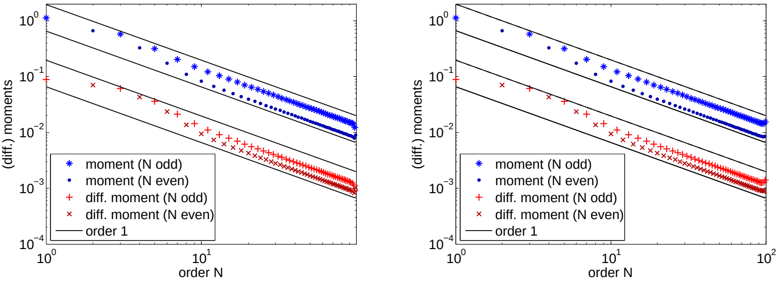


Hemisphere Test: Setup



- Source term: $S(t, x, \Omega) = \frac{1}{4\pi \times 10^{-3}} \exp\left(-\frac{x^2+y^2}{4\times 10^{-3}}\right) \chi_{\mathbb{R}^+}(\Omega_x)$
- Vacuum: $\sigma_t = 0$.

Hemisphere Test: Smoothness



(a) P_{98} (b) P_{99}

- From $B_N \sim N^{-q+\frac{1}{2}}$ and $D_N \sim N^{-r+\frac{1}{2}}$ we conclude

$$q \approx r \approx 0.5$$

Hemisphere Test: Results

Filter order	$\mathcal{E}_3^5$	$\mathcal{E}_5^9$	$\mathcal{E}_9^{17}$	$\mathcal{E}_{17}^{33}$	$\mathcal{R}_3^5$	$\mathcal{R}_5^9$	$\mathcal{R}_9^{17}$	$\mathcal{R}_{17}^{33}$
2	0.55	0.58	0.57	0.58	0.44	0.61	0.59	0.52
4	0.67	0.60	0.55	0.61	0.71	0.70	0.57	0.52
8	0.75	0.61	0.56	0.63	1.06	0.83	0.61	0.56
16	0.77	0.64	0.57	0.64	1.14	1.03	0.79	0.64
∞	0.71	0.59	0.56	0.65	1.33	1.26	0.99	0.96

- $q \approx r \approx 0.5$
- With filter:

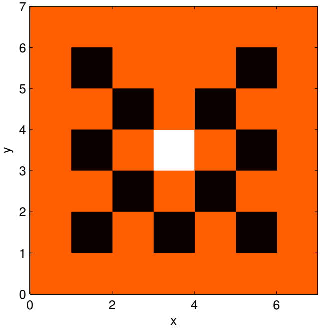
$$E_N \sim R_N \sim N^{-\min\{q, r+\frac{1}{2}, \alpha\}}$$

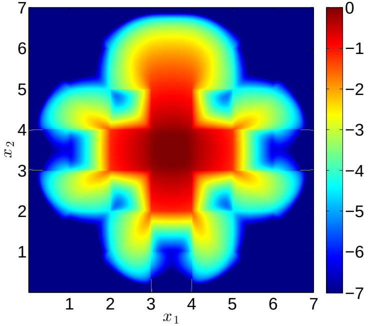
- Without filter:

$$E_N \sim N^{-\min\{q, r+\frac{1}{2}\}}$$

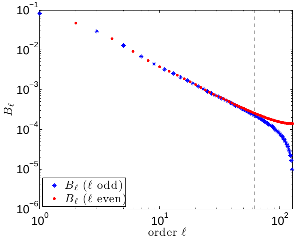
$$R_N \sim N^{-(r+\frac{1}{2})}$$

Checkerboard Test: Setup

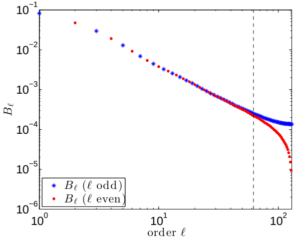




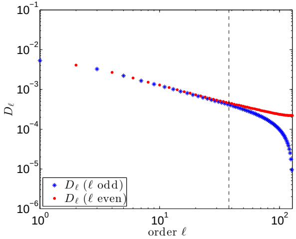
Checkerboard Test: Smoothness



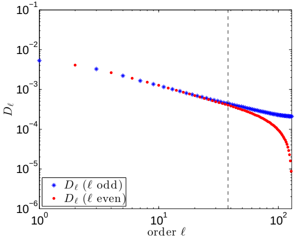
(c) B_ℓ computed with P_{128}



(d) B_ℓ computed with P_{129}



(e) D_ℓ computed with P_{128}



(f) D_ℓ computed with P_{129}

Checkerboard Test: More Smoothness

(N_1, N_2)	$B_{N_1}^{N_2}$	$D_{N_1}^{N_2}$
(2,4)	1.3188	0.6213
(4,8)	1.8212	0.8161
(8,16)	1.5208	0.8293
(16,32)	1.5782	0.8679

(a) even order moments

(N_1, N_2)	$B_{N_1}^{N_2}$	$D_{N_1}^{N_2}$
(3,5)	1.6167	0.7818
(5,9)	1.8371	0.8204
(9,17)	1.4901	0.7998
(17,33)	1.5511	0.7691

(b) odd order moments

- From $B_N \sim N^{-q+\frac{1}{2}}$ and $D_N \sim N^{-r+\frac{1}{2}}$ we conclude

$$q \approx 1.0 \quad \text{and} \quad r \approx 0.25$$

Checkerboard Test: Results

Filter order	$\mathcal{E}_3^5$	$\mathcal{E}_5^9$	$\mathcal{E}_9^{17}$	$\mathcal{E}_{17}^{33}$	$\mathcal{R}_3^5$	$\mathcal{R}_5^9$	$\mathcal{R}_9^{17}$	$\mathcal{R}_{17}^{33}$
2	0.89	0.80	0.94	1.05	0.86	0.78	0.93	1.05
4	1.02	1.15	1.13	1.05	0.98	1.21	1.21	1.06
8	1.20	1.22	1.04	1.06	1.32	1.55	1.14	1.16
16	1.61	1.31	1.03	1.04	2.10	2.12	1.23	1.20
∞	1.10	0.95	0.98	1.00	1.10	0.85	0.95	0.96

- $q = 1, \; r = 0.25$
- With filter:

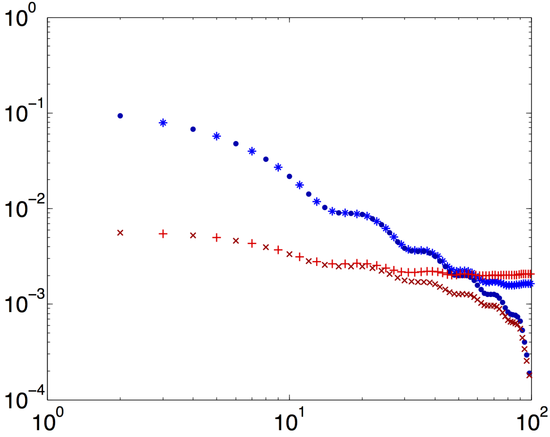
$$E_N \sim R_N \sim N^{-\min\{q, r+\frac{1}{2}, \alpha\}}$$

- Without filter:

$$E_N \sim N^{-\min\{q, r+\frac{1}{2}\}}$$

$$R_N \sim N^{-(r+\frac{1}{2})}$$

Things Aren't Always So Clear



Box source instead of Gaussian.

Summary & Outlook

Summary:

- Proof of global L^2 convergence rates for filtered spherical harmonic (FP_N) equations
- Depenence of the convergence rates on regularity of transport solution and order of the filter
- Highly resolved numerical experiments are pretty much in agreement with theoretical predictions

Outlook:

- Local analysis to show improvements by filtering
- Similar analysis for entropy or other non-linear closures