A *Selective* "Modern" History of the Boltzmann and Related Equations

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Synopsis

1. I have an ambivalent relation to surveys!
2. Key Words, Tools, People
3. Powerful Tools, I: Potentials for Interaction
4. An entertaining digression: The Digits of Π
5. Powerful Tools, II: Velocity Averaging
6. Powerful Tools, III: Functionals
7. % Powerful Tools, IV: Metrics on measures
8. Rest of the Digression

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- potentials for interaction
- velocity averaging
- functionals
- metrics on measures, with applications.

Let's begin!

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Soft potentials, and/or no angular cutoff [7]

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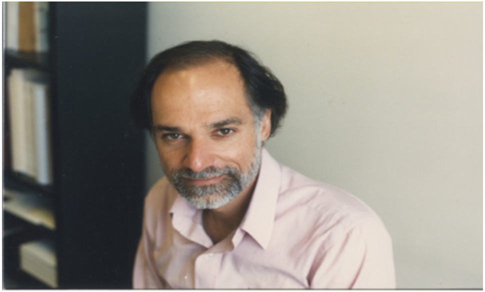
Cabannes (2), Toscani (2,6), Boblylev (1,2,4,6),
DiPerna, Lions (1),
Golse, Perthame, Degond, Wennberg (1,2,4,5,6)
Desvillettes, Villani, Carrillo (1,5,6)
Levermore (1,4,5), Gamba (3,4,5,6), St. Raymond (4).
Morimoto, Ukai, Yang (7).

If I have not listed (forgotten) you or one of your friends, forgive me...

A List of Tools

- BBGKY & Boltzmann hierarchies (Bogolyubov, Cercignani, Lanford)
- Perturbation Series as solutions (control of the hierarchies)
- Free Flow domination for rare clouds (I, Shinbrot)
- Velocity Averaging & renormalization (DiPerna, Lions)
- Potentials for Interaction (Varadhan, Bony, Beale for DVMs)
- Regularization by the collision operator (Yang, Morimoto, Ukai)



Tools, I: Potentials for Interaction

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Potential for interaction gives uniform global control of $\int_0^t \int u_i u_j dx\ dt$. This, combined with some other (older) tricks, produces global uniform boundedness and the existence of wave operators (in the absence of boundaries).

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All we need is $\sum F_i = 0 = \sum c_i F_i$ (mass and momentum conservation). Then the following *fantastic* calculation works:

Assume $c_i \neq c_j$ if $i \neq j$.
Let

$$I(t) = \sum_{i,j} \int_y \int_{x<y} (c_i - c_j) u_i(x) u_j(y) dx\ dy.$$

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Note: I(t) is bounded by mass conservation! One computes

$$\begin{aligned} \frac{dI}{dt} &= \sum_{i,j} \int_y \int_{x<y} \underbrace{(c_i - c_j)[F_i(y)u_j(x) + u_i(y)F_j(x)]}_{\text{sum to } 0\text{, by conservations}} dx\ dy \\ &+ \int \int_{-\infty}^{y} (c_i - c_j)(-c_i u_{i,x}) u_j(y) dx\ dy \\ &+ \int \int_{x}^{\infty} (c_i - c_j) u_i(x)(-c_j u_{j,y}) dy\ dx \end{aligned}$$

Do the inner integrals, collect terms....

So,

$$I(t) = \sum_{i,j} \int_y \int_{x<y} (c_i - c_j) u_i(x) u_j(y) dx\ dy$$

gives

$$\frac{dI}{dt} = -\sum_{ij} \int (c_i - c_j)^2 u_i(x) u_j(x) dx,$$

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or

$$I(t) - I(0) = -\int_0^t \int \sum_{ij} (c_i - c_j)^2 u_i(x) u_j(x) dx\ dt,$$

and $|I(t)| \leq C(mass)^2$,
so $\int_0^t \int u_i u_j dx\ dt \leq Cm^2$.

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But a potential for interaction exists in other, more fundamental mechanical contexts. Let me show you.

The “Mother” of all Kinetic Systems: N hard Spheres

masses $m_i > 0$, radii $d_i > 0$, $\quad i = 1 \dots N$
Positions $x_i(t) \in \mathbf{R}^3$, velocities $v_i(t) \in \mathbf{R}^3$.

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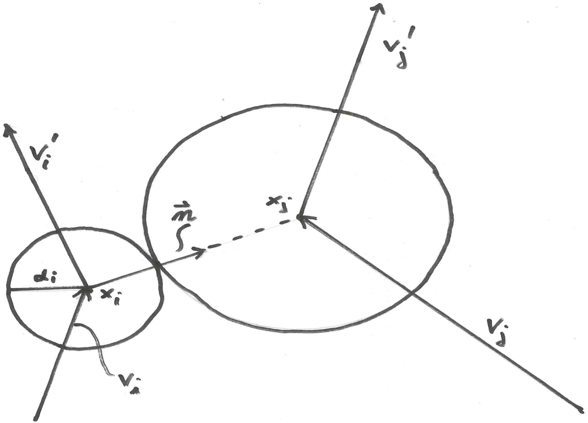
ingoing collision configuration

$$x_j = x_i + (d_i + d_j)n,$$

where $n \in S^2$ is such that

$$\begin{aligned} n \cdot (v_i - v_j) &> 0 \\ &= 0 \text{ (grazing)} \\ &< 0 \text{ (outgoing)} \end{aligned}$$

Picture:



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(energy conservation) $\Longrightarrow$

$$\begin{aligned} v_i' &= v_i - \frac{2m_j}{m_i+m_j}(n \cdot (v_i - v_j))n \\ v_j' &= v_j + \frac{2m_i}{m_i+m_j}(n \cdot (v_i - v_j))n \end{aligned}$$

This defines the collision transformation $J : (v_i, v_j) \to (v_i', v_j')$.

N Spheres in $\mathbf{R}^3$:

Abbreviate $x = (x_1, \ldots, x_N) \in \mathbf{R}^{3N}$, $\quad v = (v_1, \ldots, v_N) \in \mathbf{R}^{3N}$.
Define, in $\mathbf{R}^{3N}$,

$$\langle x, y \rangle_m = \sum_{i=1}^{N} m_i \langle x_i, y_i \rangle .$$

This is a useful inner product, for example, we have

$$\langle v(t), v(t) \rangle_m = \langle v(0), v(0) \rangle_m .$$

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If t is a collision instant, write $v^-(t)$ (ingoing) and $v^+(t)$ (outgoing). We will also write $x^0(t) = x(0) + tv(0)$ (free flow).

Assume $v(0) \neq 0$. Define
$u(t) := \frac{v(t)}{\|v(t)\|_m}, \quad e(t) := \frac{x(t)}{\|x(t)\|_m} \in S^{3N-1}$.

Theorem. There is
$e \in S^{3N-1}: \quad \lim_{t\to\infty} e(t) = e = \lim_{t\to\infty} u(t)$. The product $\langle u(t), e(t)\rangle_m$ is monotonically increasing to 1 (a potential for interaction; when it is equal to 1, there can be no more collisions).

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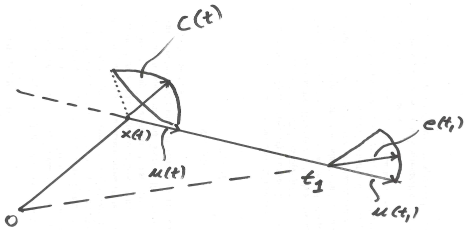
Proof.

STEP 1. By explicit calculation, if there are no collisions in $[t_1, t_2)$, we have for t in that interval

$$\langle e(t), u(t)\rangle_m = \langle e(t), u(t_1)\rangle_m \leq \langle e(t_1), u(t_1)\rangle_m .$$

Geometric meaning... picture:



A family of nested cone sections

STEP 2. Let $C(e(t)) := \{u \in S^{3N-1}; \langle u, e(t)\rangle_m \geq \langle u(t), e(t)\rangle_m\}$. This is a cone section.

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Lemma. If $t_2 \geq t_1$ then $C(e(t_2)) \subset C(e(t_1))$.

Proof: If there are no collisions between t_1 and t_2 then this follows from the calculation in STEP 1. Revisit the picture!
If there is a collision at a time t_2, one computes (this is where the *ingoing* configuration $(n \cdot (v_i^- - v_j^-) > 0))$ property enters!)

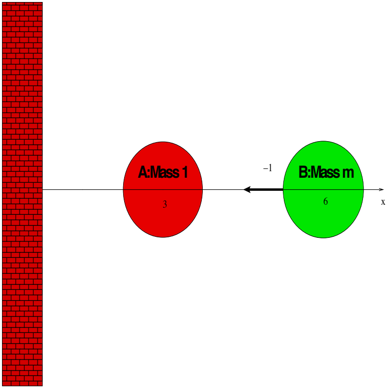
$$\left\langle u(t_2)^+, e(t_2)\right\rangle_m \geq \left\langle u(t_2)^-, e(t_2)\right\rangle_m .$$

This means that the cone C collapses around its axis $e(t) : C^+(e(t_2)) \subset C^-(e(t_2))$.

$\Longrightarrow$ **The product $\langle u(t), e(t)\rangle_m$ is a potential for interaction!**

An entertaining digression (Godunov, Sultanghazin, Galperin)

Consider:



The collision transformation takes the form

$$u_0' = u_0 - \frac{2m}{m+1}(u_0 - v_0) \quad (1)$$

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momentum, energy are conserved:

$$u_0' + mv_0' = u_0 + mv_0$$

$$(u_0')^2 + m(v_0')^2 = (u_0)^2 + m(v_0)^2$$

Ball A will bounce off the wall and head back right; it will collide again with ball B, but if ball B is heavier than ball A, this will not be the last collision:

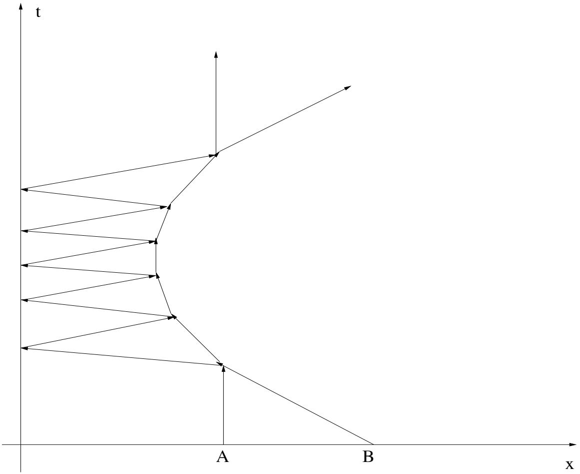

Figure : Many collisions in spacetime

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Then $u_1 = -u_0', \quad v_1 = v_0'$, or

$$u_1 = \frac{m-1}{m+1} u_0 - \frac{2m}{m+1} v_0 \qquad (3)$$

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The two particles were originally in a collision configuration because $v_0 - u_0 = -1 < 0$; if $v_1 - u_1 < 0$, they will collide again. We can then compute $(u_2, v_2), (u_3, v_3)$ etc., until we find a number k such that, for the first time, $v_k - u_k > 0$.

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Find the number k with little effort. The following table shows k as a function of m, the mass of particle B. Following Galperin's idea, we have taken $m = 100^n$, where $n = 0, 1, 2, 3, \ldots$.

m	N (total)	M (wall touches)
1	3	1
100	31	15
10,000	314	157
10^6	3142	...
10^8	31415	...

Table : Number of collisions: THE DIGITS OF π!

N and M are the numbers of total collisions and wall collisions, respectively. Remember: particle A is initially at rest, and particle B moves initially at $v_0 = -1$.

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Explanation? ... is another talk!

Powerful Tools, II: Velocity Averaging

Observed around 1987 (?) by Sentis, Golse, Lions, Perthame. DiPerna and Lions figured out how to use this for BE.

The Result For $f = f(x, v, t)$, let $Tf := (\partial_t + v \cdot \nabla_x) f$.

Lemma. (velocity averaging) Assume that $f \in L^2(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R})$, has compact support, and is such that $Tf \in L^2(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R})$. Then

$$\int f \; dv \in H^{1/2}(\mathbf{R}^3 \times \mathbf{R}).$$

(meaning $\int (\tau^2 + |z|^2)^{1/2} |\int \hat{f}(z, v, \tau) \; dv|^2 dz \; d\tau < \infty$.)

How this is used:

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Lemma. For a subsequence

i) $\int f_n dv \rightarrow \int f dv$ strongly in L^1

ii) $R_n(f_n) \rightarrow R(f)$ strongly in L^1

iii) ... convergence of the gain term... requires much hard work.

Powerful Tools (?), III: Functionals

A Case Study: Kinetic Granular Media Model
(Benedetto, Caglioti, Pulvirenti, in 1 D, 1997-1999).

Equation:

$$\partial_t f + v \cdot \nabla_x f = \lambda div_v[(\nabla W *_v f) f]$$

(think $W(v) = \frac{1}{3}|v|^3$.) General W such that $W(-v) = W(v)$.

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Formal properties: Mass and momentum conservation. Kinetic energy decrease:

$$K(t) := \frac{1}{2} \int \int |v|^2 f(x, v, t) dx\, dv \leq K(0)$$

(in general strict decrease).

"Derivation"

Consider a particle system

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= \epsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j)\nabla W(v_j - v_i) \\ &= \epsilon N \frac{1}{N} \sum \dots\end{aligned}$$

Define a measure $\mu_t^N = \frac{1}{N}\sum \delta_{(x_j,v_j)}$, $F_\alpha(x,v) = \eta_\alpha(x)\nabla W(v)$.

Then

$$(F_\alpha * \mu_t^N)(x,v) = -\frac{1}{N}\sum \eta_\alpha(x - x_j)\nabla W(v_j - v).$$

Formally, one takes the limit $N\epsilon \to \lambda$, in which

$$\mu_t^N(x,v) \to f(x,v,t)$$

and the system becomes $\dot{x} = v, \quad \dot{v} = -\lambda F_\alpha * f$.
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The Tools.

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1. Entropy: Let $U : [0.\infty) \to \mathbf{R}$, $U(0) = 0$, convex, and set $P_U(r) = rU'(r) - U(r) \geq 0$. Examples are $r^p, p > 1$, and $r \ln r$. Then, if f solves the model equation,

$$\frac{d}{dt} \int \int U(f) = ... = \lambda \int \int \int \Delta W(v-u) P_U(f)(x,v) f(x,u) du\, dv\, dx$$

r.h.s. is ≥ 0 because W is convex, so $\Delta W \geq 0$.
For $U = r \ln r$ one computes $P_U(f) = f$, and the r.h.s. is

$$\lambda \int \int \int \Delta W(v-u) f(x,v) f(x,u) du\ dv\ dx$$

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$$\begin{aligned}
\frac{d}{dt}J &= \int\int 2(x-tv)(-v)f + \int\int (x-tv)\partial_t f \\
&= \int\int \{-2xv + 2tv^2 + 2(x-tv)v)\} f \\
&\quad +\lambda \int\int (x-tv)^2 div_v[(\nabla W *_v f)f]\ dv\ dx \\
&= -2\lambda t^2 \int\int\int (v-u)\nabla W(v-u) f(x,v) f(x,u)\ du\ dv\ dx.
\end{aligned}$$

So,

$$J(f)(t) = J(f)(0) - \lambda \int_0^t s^2 \int\int\int (v-u)\nabla W(v-u) f \otimes f \; du \; dv \; dx \; ds$$

Compare with.

$$H(f)(t) = H(f)(0) + \lambda \int_0^t \int\int\int \Delta W(v-u) f \otimes f \;\; du \;\; dv \;\; dx \;\; ds$$

Note: in, say, one dimension, for $W(v) = \frac{1}{3}|v|^3$, we have

$$W''(v) = 2|v|, \quad \text{and} \quad vW'(v) = |v|^3.$$

This is the fundamental difference of the terms on the right. The production term on the right hand side in the second identity is uniformly bounded; however, this does not entail bounded entropy production, because of the different powers of $|v-u|$.

3. In 1 D: Can use potential for interaction:
Let $I(f)(t) = \int_v \int_u \int \int_{x<y} (v-u) f(x,v) f(y,u) \; dxdydudv$. Then, repeating the calculation done much earlier for DVMs, using only momentum and mass conservation,

$$\frac{d}{dt} J = - \int \int \int (v-u)^2 f(x,v) f(x,u) \; dxdvdu.$$

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Thank you

Revisit the Digression...

The explanation is hidden in the properties of the transformation (3,4). Things become simpler if one rescales the speeds v_0, v_1, v_2, etc. of ball B:

$$w_0 := \sqrt{m}v_0, w_1 := \sqrt{m}v_1,$$

etc.

Energy conservation then becomes the simpler equation

$$(u_0')^2 + (w_0')^2 = (u_0)^2 + (w_0)^2 \tag{5}$$

and the collision transformation (4) becomes

$$u_1 = \frac{m-1}{m+1}u_0 - \frac{2\sqrt{m}}{m+1}w_0 \tag{6}$$

$$w_1 = \frac{2\sqrt{m}}{m+1}u_0 + \frac{m-1}{m+1}w_0 \tag{7}$$

In this new coordinate system, the equations (6,7) are where the circle is hiding: set

$$\alpha = \frac{m-1}{m+1}, \quad \beta = \frac{2\sqrt{m}}{m+1} \quad \Longrightarrow$$

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there is an angle θ such that $\cos\theta = \alpha, \sin\theta = \beta$. Geometrically this means that in the $u - w$ plane, (6,7) is a rotation in the counterclockwise sense by the angle θ; in our setup we begin the rotation with the initial point $(0, -\sqrt{m})$. (u_j, w_j), computed from repeated application of (6, 7), arise from repeated rotations by θ in the $u - w$ plane for $j = 0, 1, 2, \ldots$, as shown in Figure 3, or as expressed by the transformation (rotation)

$$\begin{pmatrix} u_{j+1} \\ w_{j+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_j \\ w_j \end{pmatrix}$$

Energy conservation as stated in (5) is the key ingredient in this: the collision transformation must conserve the length of the vector (u_0, w_0), and only rotations or reflections do this.

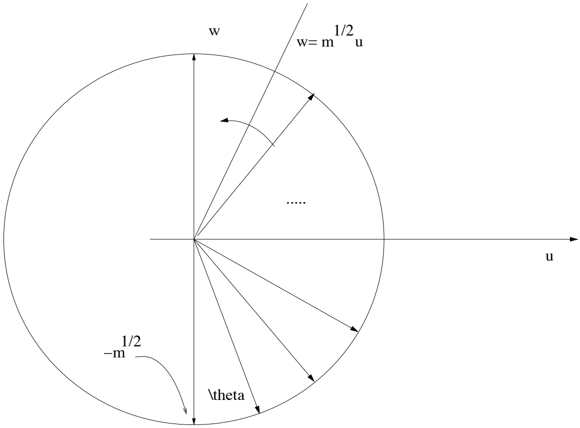

Figure : Collisions are rotations!

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We can also approximate θ in terms of m by observing that $\alpha = \cos\theta \approx 1 - \frac{\theta^2}{2}$, hence $\theta \approx \frac{2}{\sqrt{m+1}}$. Together: $k \approx \pi\frac{\sqrt{m+1}}{2}$, and this is an approximation of the expected number of wall touches:
For example, for $m = 10^4$, we find $2k \approx 100\pi \approx 314$.