Very Weak Solutions for Poisson-Nernst-Planck System

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Transport of Ionic Particles in Biological Environments

Outline

Poisson-Nernst-Planck System

$\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$

$$u_0, v_0 \in L^2_+(\Omega)$$

$$u, v \in L^\infty(0, T; L^2_+(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$$

$$\begin{aligned}
&u_t = \nabla \cdot (\nabla u - u \nabla \phi) \\
&v_t = \nabla \cdot (\nabla v + v \nabla \phi) \\
&\Delta \phi = u - v, \qquad x \in \Omega \\
&(\nabla u - u \nabla \phi) \cdot \nu = (\nabla v + v \nabla \phi) \cdot \nu = 0 \\
&\phi = 0, \qquad x \in \partial\Omega
\end{aligned}$$

Navier-Stokes-Poisson-Nernst-Planck System

$\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$

$$\mathbf{u}_0 \in L^2(\Omega)^n, \nabla \cdot \mathbf{u}_0 = 0, \quad u_0, v_0 \in L^2_+(\Omega)$$

$$\mathbf{u} \in L^\infty(0,T; L^2(\Omega)^n) \cap L^2(0,T; W^{1,2}(\Omega)^n)$$
$$u, v \in L^\infty(0,T; L^2_+(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \Delta \mathbf{u} + \Delta\phi \nabla\phi, \quad \nabla \cdot \mathbf{u} = 0$$
$$u_t + \mathbf{u} \cdot \nabla u = \nabla \cdot (\nabla u - u \nabla \phi)$$
$$v_t + \mathbf{u} \cdot \nabla v = \nabla \cdot (\nabla v + v \nabla \phi)$$
$$\Delta \phi = u - v, \qquad x \in \Omega$$
$$(\nabla u - u\nabla\phi) \cdot \nu = (\nabla v + v \nabla \phi) \cdot \nu = 0$$
$$\phi = 0, \mathbf{u} = 0, \qquad x \in \partial\Omega$$

Lyapunov Functional

$$W = \int_\Omega \frac{|\mathbf{u}|^2}{2} + u \log u + v \log v \, dx - \int_\Omega \frac{|\nabla \phi|^2}{2} + \phi(u - v) \, dx$$

$$= \int_\Omega \frac{|\mathbf{u}|^2}{2} + u \log u + v \log v \, dx + \int_\Omega \frac{|\nabla \phi|^2}{2} \, dx$$

$$\frac{d}{dt} W = - \int_\Omega |\nabla \mathbf{u}|^2 + u |\nabla(\log u - \phi)|^2 + v |\nabla(\log v + \phi)|^2 \, dx$$

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^n) \cap L^2(0, T; W^{1,2}(\Omega)^n)$$

$$u, v \in L^\infty(0, T; L \log L(\Omega))$$

$$\phi \in L^\infty(0, T; W_0^{1,2}(\Omega))$$

Energetic Variational Derivation

$$
\begin{aligned}
W &= \int_\Omega \frac{|\mathbf{u}|^2}{2} + u \log u + v \log v \, dx - \int_\Omega \frac{|\nabla \phi|^2}{2} + \phi(u-v)\, dx \\
&= \int_\Omega \frac{|\mathbf{u}|^2}{2} + u \log u + v \log v \, dx + \int_\Omega \frac{|\nabla \phi|^2}{2} \, dx
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\delta W}{\delta \phi} &= 0 && \Rightarrow \Delta \phi = u - v \\
-\frac{\delta^* W}{\delta^* u} &= {x_t}^{1} && \Rightarrow J = \nabla u - u \nabla \phi \\
-\frac{\delta^* W}{\delta^* v} &= y_t && \Rightarrow K = \nabla v + v \nabla \phi \\
-\frac{\delta W}{\delta x} &= F && \Rightarrow F = (u-v)\nabla \phi = \nabla \cdot (\nabla \phi \otimes \nabla \phi) - \nabla \frac{1}{2} |\nabla \phi|^2
\end{aligned}
$$

[1] $\delta u + \nabla \cdot (u \omega) = 0$

Mathematical Analysis in 2 and 3 Dimensions

Theorem (Biler, et al 1994)

For $\Omega \in \mathbb{R}^2$ *and* $u_0, v_0 \in L^2_+$ *there exists a unique solution* $u, v \in L^\infty(0,T;L^2_+) \cap L^2(0,T;W^{1,2})$ *of the PNP system and for* $\tau > 0$, $u, v \in C^{2,\alpha}((\tau,T)\times\Omega)$.

$$\|w\|^3_{L^3} \le \epsilon \|\nabla w\|^{3/2}_{L^2} + (1 + \|w\|^{3/2}_{L^1}) \exp(C\|w\|_{L\log L}\epsilon^{-1})$$

Theorem (Biler, et al 1994)

For $\Omega \in \mathbb{R}^3$ *and* $u_0, v_0 \in L^2_+$ *there exists a unique solution* $u, v \in L^\infty(0,T_0;L^2_+) \cap L^2(0,T_0;W^{1,2})$ *of the PNP system and for* $\tau > 0$, $u, v \in C^{2,\alpha}((\tau,T_0)\times\Omega)$.

$$T_0 = T_0(\Omega, \|u_0, v_0\|_{L^2})$$

Conjecture

The maximal interval of existence is $[0,\infty)$.

Recent Mathematical Analysis in 2 and 3 Dimensions

- Debye system: Biler, Hebisch, Hilhorst, Nadzieja (1992-1994)
- Asymptotics and long time behavior: Biler, Dolbeault (2000)
- Hydrodynamic coupling: Jerome, Sacco (2009), RJR (2009), Schmuck (2009), Deng, Zhao, Cui (2010), Jerome (2011)
- Diffusion limit of Vlasov-Poisson-Fokker-Planck system: Wu, Lin, Liu (2014)
- Quasilinear approximation with capacitance: Bothe, Fischer, Pierre, Rolland (2014)

Motivation

Lyapunov functional for a variety of boundary conditions places

ionic densities uniformly in $L^\infty(0,T;L^1(\Omega))$

potential in $L^\infty(0,T;W^{1,2}(\Omega))$

* Pose a very weak formulation with regularity informed by the a priori estimates – infer regularity from very weak formulation *

Very Weak Formulation : Derivation

Total charge system resembles Debye system for single anion.

$$
\begin{aligned}
&u_t - \nabla \cdot (\nabla u - u \nabla \phi) = f \\
&\Delta \phi = u
\end{aligned}
$$

Relationship between convection and Maxwell stress:

$$-u\nabla\phi = -\Delta\phi\nabla\phi = \nabla\tfrac{1}{2}|\nabla\phi|^2 - \nabla\cdot(\nabla\phi\otimes\nabla\phi)$$

$$
\boxed{
\begin{aligned}
&u_t - \Delta(u + \tfrac{1}{2}|\nabla\phi|^2) + \nabla\cdot\nabla\cdot(\nabla\phi\otimes\nabla\phi) = f \\
&\Delta\phi = u
\end{aligned}
}
$$

Very Weak Formulation

$$\boxed{\begin{array}{l} u_t - \Delta(u + \frac{1}{2}|\nabla\phi|^2) + \nabla\cdot\nabla\cdot(\nabla\phi\otimes\nabla\phi) = f \\ \Delta\phi = u \end{array}}$$

$$u \in L^\infty(0,T;L^1(\Omega)), \quad \phi \in L^\infty(0,T;W^{1,2}(\Omega))$$

$$\begin{aligned} \int_0^T \int_\Omega -uw_t - (u + \tfrac{1}{2}|\nabla\phi|^2)\Delta w + \nabla\phi\otimes\nabla\phi : \nabla^2 w \, dx\, dt \\ = \int_0^T \int_\Omega f w \, dx\, dt \end{aligned}$$

$$\int_0^T \int_\Omega -\nabla\phi\cdot\nabla\eta \, dx\, dt = \int_0^T \int_\Omega u\eta \, dx\, dt$$

Very Weak Solutions in Other Contexts

- Weyl's lemma

$$u \in L^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} u \Delta w \, dx = 0, \quad \forall w \in C_0^\infty(\mathbb{R}^n)$$

$$\Rightarrow u \text{ analytic, } \Delta u = 0$$

- Semi-linear 2nd order equations in L^1 : Brezis, Strauss (1973), Brezis, Cazenave, Martel (1996)
- p-Laplacian and heat flow thereof: Lewis (1993), Kinnunen, Lewis (2002)
- Semi-linear elliptic equations in boundary weighted spaces: Diaz, Rakotoson (2009, 2010)
- Divergence form elliptic equations: Jin, Mazya, Schaftingen (2009), Zhang, Bao (2012)
- Coarea flow: Hardt, Zhou (1994), Hardt, Tonegawa (1996)

Very Weak Formulation : Stationary Problem

$$\Delta(u + \tfrac{1}{2}|\nabla\phi|^2) - \nabla \cdot \nabla \cdot (\nabla\phi \otimes \nabla\phi) = -f$$
$$|\Delta\phi| \leq u$$

$$u \in L^1_{\mathrm{loc}}(\Omega), \quad \phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

$$\int_\Omega (u + \tfrac{1}{2}|\nabla\phi|^2)\Delta w - \nabla\phi \otimes \nabla\phi : \nabla^2 w \, dx = -\int_\Omega f w \, dx$$
$$\int_\Omega -\nabla\phi \cdot \nabla\eta = \int_\Omega \xi\eta \, dx, \quad |\xi|(x) \leq u(x)$$

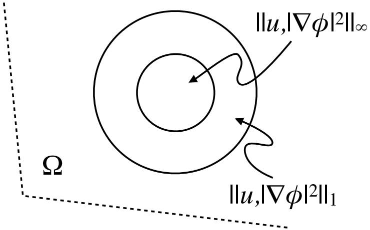
Main Result : Interior Regularity

Theorem (1)

If $(u, \phi) \in L^1_{\mathrm{loc},+}(\Omega) \times W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a very weak solution of*

$$\Delta(u + \tfrac{1}{2}|\nabla\phi|^2) - \nabla \cdot \nabla \cdot (\nabla\phi \otimes \nabla\phi) = -f, \quad |\Delta\phi| \leq u$$

with $\Delta^{-1} f \in L^\infty(\Omega)$, *then* $u, \nabla\phi \in L^\infty_{\mathrm{loc}}(\Omega)$.



Hineman, RJR (2014)

Interior Regularity : Multiple Species

Theorem (2)

If $(u_1, \ldots, u_q, \phi) \in (L^2_{\mathrm{loc},+}(\Omega))^q \times W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a weak solution of*

$$
\begin{aligned}
&\nabla \cdot (D_1(x)\nabla u_1 - z_1 u_1 \nabla\phi) = -f_1 \\
&\qquad \vdots \\
&\nabla \cdot (D_q(x)\nabla u_q - z_q u_q \nabla\phi) = -f_q \\
&\Delta\phi = z_1 u_1 + \cdots + z_q u_q + g
\end{aligned}
$$

then $u_1, u_2, \ldots, u_q, \nabla\phi \in C^\infty(\Omega)$.

Friedman, Tintarev (1987), RJR, Liu, Wang (2006), Lee et al (2011, 2014)

Elements of Proof : Monotonicity Formula

If $f \in L^1_{\text{loc}}(\Omega)$, then for almost every $x \in \Omega$,

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x)} f(z)\, dz = f(x)|B_1(0)|$$

and

$$\lim_{\rho \to 0} \frac{1}{\rho^p} \int_{B_\rho(x)} f(z)\, dz = 0, \quad p < n$$

Elements of Proof : Monotonicity Formula

$$\nabla \cdot (\nabla u + \tfrac{1}{2}|\nabla\phi|^2) - \nabla \cdot \nabla \cdot (\nabla\phi \otimes \nabla\phi) = -f$$

$$\Downarrow$$

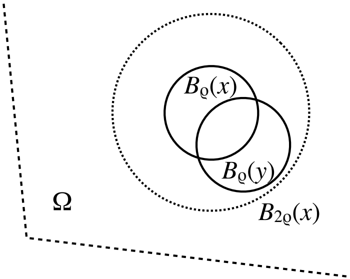
$$\nabla \cdot (\nabla u_\epsilon + \tfrac{1}{2}|\nabla\phi|_\epsilon^2) - \nabla \cdot \nabla \cdot (\nabla\phi \otimes \nabla\phi)_\epsilon = -f_\epsilon$$

$$\Downarrow$$

$$\boxed{\begin{aligned}&\int_{B_r(x)} (2-\beta) u \, dz - r \int_{\partial B_r(x)} (\nabla\phi \cdot \nu)^2 \, dS = r^{n-1+\beta} \frac{d}{dr} \frac{P_{x,r}}{r^{n-2+\beta}} \\ &+ \int_{B_r(x)} \frac{\beta}{2} |\nabla\phi|^2 \, dz + \int_{B_r(x)} f \tfrac{1}{2}(r^2 - |z-x|^2) \, dz\end{aligned}}$$

$$P_{x,r} = \int_{B_r(x)} u + \tfrac{1}{2}|\nabla\phi|^2 dz, \quad U_{x,\beta,\rho} = \int_0^\rho \frac{1}{r^{n-1+\beta}} \int_{B_r(x)} u \, dz \, dr$$

Riesz-like Function Space



$$U_{x,\beta,\rho} = \int_0^\rho \frac{1}{r^{n-1+\beta}} \int_{B_r(x)} u \, dz \, dr, \quad \mathcal{U}_{x,\beta,\rho} = \operatorname*{ess\,sup}_{y \in B_\rho(x)} U_{y,\beta,\rho}$$

$$L^\infty \subsetneq \{u : \mathcal{U}_{x,\beta,\rho} < \infty\} \subsetneq L^1, \quad \beta \in [0,2)$$

$$M^p \subsetneq \{u : \mathcal{U}_{x,\beta,\rho} < \infty\}, \quad \beta \in [0, 2 - n/p)$$

Riesz-like Function Space

$$(n-2+\beta)U_{x,\beta,\rho} + \frac{1}{\rho^{n-2+\beta}} \int_{B_\rho(x)} u\, dz = \int_{B_\rho(x)} \frac{u(z)}{|z-x|^{n-2+\beta}}\, dz$$

$$\begin{aligned}
|\nabla^k \phi(x) - \nabla^k \phi(y)| \quad & \leq \int\limits_{B_{r/2}(x)} u(z)|K_k(z,x) - K_k(z,y)|\, dz \\
& + \int\limits_{B_{r/2}(y)} u(z)|K_k(z,x) - K_k(z,y)|\, dz \\
& + \int\limits_{B_{2\rho}(x_0)\setminus(B_{r/2}(x)\cup B_{r/2}(y))} u(x)|K_k(z,x) - K_k(z,y)|\, dz \\
& + \int\limits_{B_{2\rho}(x_0)} |\phi|(x)|L_k(z,y) - L_k(z,y)|\, dz
\end{aligned}$$

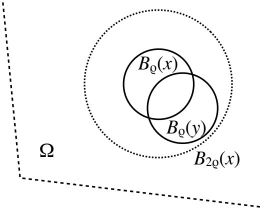
Riesz-like Function Space

Proposition

Let $\beta \in (0,1)$ *and* $\phi \in W^{1,2}_{\text{loc}}(\Omega)$ *satisfy* $|\Delta\phi| \leq u$

1. *If* $\mathcal{U}_{x,0,\rho} < \infty$, *then* $\phi \in L^\infty(B_\rho(x))$
2. *If* $\mathcal{U}_{x,\beta,\rho} < \infty$, *then* $\phi \in C^{0,\beta}(B_\rho(x))$
3. *If* $\mathcal{U}_{x,1+\beta,\rho} < \infty$, *then* $\phi \in C^{1,\beta}(B_\rho(x))$

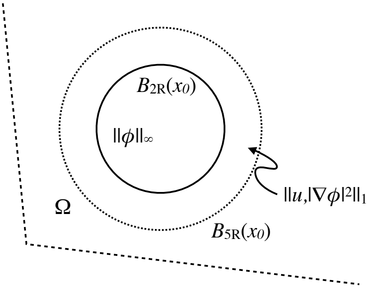
$$U_{x,\beta,\rho} = \int_0^\rho \frac{1}{r^{n-1+\beta}} \int_{B_r(x)} u \, dz \, dr, \quad \mathcal{U}_{x,\beta,\rho} = \operatorname*{ess\,sup}_{y \in B_\rho(x)} U_{y,\beta,\rho}$$



Elements of Proof : Zeroth Order Bounds

$\mathcal{U}_{x_0,0,\rho} < \infty$ and $\phi \in L^\infty_{\mathrm{loc}}(\Omega)$

$$\int_{B_r(x)} 2u\,dz = r^{n-1}\frac{d}{dr}\frac{P_{x,r}}{r^{n-2}} - r\int_{\partial B_r(x)} (\nabla\phi\cdot\nu)^2\,dS + \int_{B_r(x)} f\,\tfrac{1}{2}(r^2 - |z-x|^2)\,dz$$



Trace Inequality : Bounded to Hölder Continuity

$\phi \in L^\infty_{\mathrm{loc}}(\Omega)$ and $|\Delta\phi| \le u$

$$\int_{B_r(x)} |\nabla\phi|^2 \, dz \le C_0 \left(r \int_{\partial B_r(x)} (\nabla\phi \cdot \nu)^2 \, dS + \int_{B_r(x)} u \, dz \right)$$

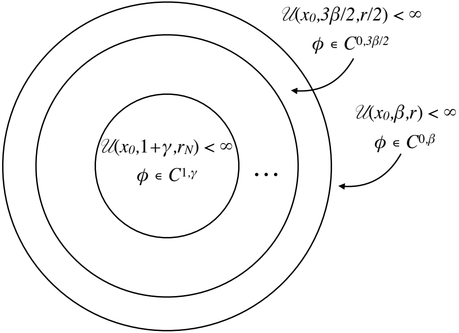
$$\Downarrow$$

$$\boxed{\begin{aligned} &(2 - \beta - C\beta) \int_{B_r(x)} u \, dz = r^{n-1+\beta} \frac{d}{dr} \frac{P_{x,r}}{r^{n-2+\beta}} \\ &+ \int_{B_r(x)} f \, \tfrac{1}{2}(r^2 - |z - x|^2) \, dz \end{aligned}}$$

Sequential Improvement

$$\mathcal{U}_{x_0,\beta,\rho} < \infty \text{ and } \phi \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$$

$$\int_{B_r(x)} |\nabla\phi|^2 \, dz \le \frac{2r}{\beta_{i+1}} \int_{\partial B_r(x)} (\nabla\phi\cdot\nu)^2 \, dS + C\left(\boxed{r^{n-2+2\beta}} + r^{\beta} \int_{B_r(x)} u \, dz \right)$$



Bounding the Excess

Monotonicity formula when $\beta = 2$:

$$\boxed{0 \leq \frac{d}{dr}\frac{P_{x,r}}{r^n} + \frac{E_{x,r}}{r^{n+1}} + \frac{1}{r^{n+1}}\int_{B_r(x)} f\,\tfrac{1}{2}(r^2 - |z-x|^2)\,dz.}$$

Mean value excess:

$$E_{x,r} = \int_{B_r(x)} |\nabla\phi|^2\,dz - r\int_{\partial B_r(x)} (\nabla\phi\cdot\nu)^2\,dS \leq Cr^{n+\gamma}$$

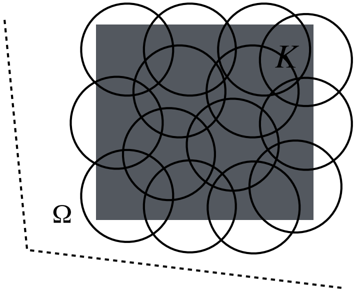
$$\begin{aligned}
&(u(x) + \tfrac{1}{2}|\nabla\phi|^2(x))|B_1(0)| \\
&\qquad \leq \frac{P_{x,\rho}}{\rho^d} + \int_0^\rho Cr^{-1+\gamma} + \frac{1}{r^{n-1}}\int_{B_r(x)} |f|\,dz\,dr
\end{aligned}$$

Inner Estimate from Outer Bound

Lemma

For $x_0 \in \Omega$ and $B_{5r_0}(x_0) \subset \Omega$, the number of levels N satisfies

$$N \leq C_1 + 3 \ln \left(C_T + P_{x_0,4r_0} + \|\phi\|_{L^1(B_{4r_0}(x_0))} + \mathcal{F}_{x_0,0,2r_0} \right)$$



Summary of Theorem and Proof

Theorem (1)

If $(u, \phi) \in L^1_{\mathrm{loc},+}(\Omega) \times W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a very weak solution of*

$$\Delta(u + \tfrac{1}{2}|\nabla\phi|^2) - \nabla\cdot\nabla\cdot(\nabla\phi \otimes \nabla\phi) = -f, \quad |\Delta\phi| \leq u$$

with $\Delta^{-1} f \in L^\infty(\Omega)$, *then* $u, \nabla\phi \in L^\infty_{\mathrm{loc}}(\Omega)$.

- Prove initial uniform and Hölder estimate
- Show sequential improvement of Hölder estimate
- Continue until gradient Hölder estimate reached
- Estimate mean value excess

Application I : Multiple Species

$$\nabla \cdot (D_1(x)\nabla u_1 - z_1 u_1 \nabla\phi) = 0$$

$$\vdots$$

$$\nabla \cdot (D_q(x)\nabla u_q - z_q u_q \nabla\phi) = 0$$

$$\Delta\phi = z_1 u_1 + \cdots + z_q u_q$$

$$u = u_1 + u_2 + \cdots + u_q$$

$$\Delta(u + \tfrac{1}{2}|\nabla\phi|^2) - \nabla\cdot\nabla\cdot(\nabla\phi\otimes\nabla\phi) = \sum_{i=1}^{q} \nabla\cdot((1 - D_i)\nabla u_i)$$

$$|\Delta\phi| = \left|\sum_{i=1}^{q} z_i u_i\right| \leq u$$

Application I : Multiple Species

$$f := -\sum_{i=1}^{q} \nabla \cdot ((1 - D_i)\nabla u_i)$$

$$\int_0^\rho \frac{1}{r^{n-1+\beta}} \int_{B_r(x)} f \, \tfrac{1}{2}(r^2 - |z-x|^2) \, dz \, dr$$
$$\leq \left(\max_{i=1,\ldots,q} \|D_i\|_{C^0(B_\rho(x))} + 1 \right) \frac{1}{\rho^{n-2+\beta}} \int_{B_\rho(x)} u \, dz \, dr$$
$$+ \max_{i=1,\ldots,q} \|\nabla D_i\|_{C^0(B_\rho(x))} U_{x_0,\beta-1,\rho}$$

Proposition

If $(u_1, \ldots, u_q, \phi) \in (L^2_{\text{loc},+}(\Omega))^q \times W^{1,2}_{\text{loc}}(\Omega)$ *is a weak solution, then* $u_1, u_2, \ldots, u_q, \nabla\phi \in L^\infty_{\text{loc}}(\Omega)$.

Further Applications

Poisson-Boltzmann with Natural Charge Boundary Conditions

$$\Delta\phi = \sum_{i=1}^{q} z_i e^{\lambda_i \phi_i}, \quad A\phi + B\nabla\phi \cdot \nu = \phi_0$$

Poisson-Boltzmann with Fixed Charge Boundary Conditions

$$\nabla \cdot (\nabla u_i - z_i u_i \nabla\phi) = f_i, \quad u_i = g_i$$

$$\Delta\phi = \sum_{i=1}^{q} z_i u_i, \quad A\phi + B\nabla\phi \cdot \nu = \phi_0$$

Friedman, Tintarev (1987), RJR, Liu, Wang (2006), Lee et al (2011, 2014)

Application II : Keller-Segel

Theorem (3)

If $(u,\phi) \in L^1_{\mathrm{loc},+}(\Omega) \times W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a very weak solution of*

$$\nabla \cdot (\nabla u + u \nabla \phi) = 0, \quad \Delta \phi = u$$

with $\Omega \subset \mathbb{R}^2$, *then* $u, \nabla \phi \in C^\infty_{\mathrm{loc}}(\Omega)$.

$$\int_{B_r(x)} nu - \frac{n-2}{2} |\nabla \phi|^2 \, dz = r \int_{\partial B_r(x)} u - \tfrac{1}{2} |\nabla \phi|^2 + (\nabla \phi \cdot \nu)^2 \, dS.$$

Application II : Keller-Segel

Theorem (3)

If $(u, \phi) \in L^1_{\text{loc},+}(\Omega) \times W^{1,2}_{\text{loc}}(\Omega)$ *is a very weak solution of*

$$\nabla \cdot (\nabla u + u \nabla \phi) = 0, \quad \Delta \phi = u$$

with $\Omega \subset \mathbb{R}^2$, *then* $u, \nabla \phi \in C^\infty_{\text{loc}}(\Omega)$.

Corollary

Stationary, two dimensional chemotaxis is singular measure valued or null (existence not claimed).

(see Perthame, *Transport Equations in Biology*)

Application II : Keller-Segel

Theorem (3)

If $(u,\phi) \in L^1_{\mathrm{loc},+}(\Omega) \times W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a very weak solution of*

$$\nabla \cdot (\nabla u + u \nabla \phi) = 0, \quad \Delta \phi = u$$

with $\Omega \subset \mathbb{R}^2$, *then* $u, \nabla \phi \in C^\infty_{\mathrm{loc}}(\Omega)$.

Counter example: $\Omega \subset \mathbb{R}^{n \geq 3}$:

$$u = 2(n-2)|x|^{-2} \in L^1_{\mathrm{loc},+}(\Omega), \quad \phi = 2 \log |x| \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

is a very weak solution of

$$\nabla \cdot (\nabla u + u \nabla \phi) = 0, \quad \Delta \phi = u$$

Time Dependent Formulation

$$\int_0^T \int_\Omega -uw_t - (u + \tfrac{1}{2}|\nabla\phi|^2)\Delta w + \nabla\phi \otimes \nabla\phi : \nabla^2 w \, dx\, dt$$
$$= \int_0^T \int_\Omega f w \, dx \, dt$$
$$\int_0^T \int_\Omega -\nabla\phi \cdot \nabla\eta \, dx \, dt = \int_0^T \int_\Omega \xi\eta \, dx \, dt, \quad |\xi| \le u$$

Conjecture

A very weak solution

$$u \in L^\infty(0,T; L^1(\Omega)), \quad \phi \in L^\infty(0,T; W^{1,2}(\Omega))$$

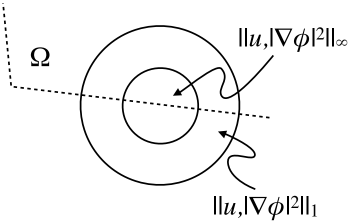
enjoys $u, |\nabla\phi|^2 \in L^\infty_{\text{loc}}((0,T) \times \Omega)$.

Summary

- Viewed as Maxwell stress, total charge convection converted into 2nd order term
- Formulated weak system with solutions in the space $L^1(\Omega) \times W^{1,2}(\Omega)$ appropriate to Lyapunov functional
- Proved interior L^∞ estimate based on L^1 data
- Inferred full interior regularity for Poisson-Boltzmann with various boundary conditions
- Applied argument to two dimensional, stationary chemotaxis
- Further applications to regularization and compactness

Conclusion

- Boundary regularity



- Variable dielectric $\nabla \cdot (\epsilon(x)\nabla\phi) = \sum_{i=1}^{q} z_i u_i + g$

$$
\begin{aligned}
\underline{\epsilon} \int_{B_r(x)} |\nabla\phi|^2 \, dz &\leq \int_{B_r(x)} \epsilon(x)|\nabla\phi|^2 \, dz \\
&\leq \int_{\partial B_r(x)} \epsilon(x)|\phi - \bar{\phi}||\nabla\phi \cdot \nu| dS \\
&+ \int_{B_r(x)} (u + |g|)|\phi - \bar{\phi}| \, dz
\end{aligned}
$$

Conclusion

- Challenges :

  derive mean value type monotonicity formula for time dependent problem – develop interior/boundary estimates
- Local monotonicity formulas in other contexts
  - Heat equation: Fulks (1971)
  - Semilinear diffusion equation: Giga, Kohn (1985)
  - Harmonic maps: Struwe (1988), Chen, Struwe (1989)
  - Mean curvature flow: Hamilton (1993), Ecker (2001)
- Thank you Chun and Maximilian!