Spectral Geometry in Non-standard Domains

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Abstract

This is a report on a project "Spectral geometry in non-standard domains" in the Fields Undergraduate Summer Research Program 2014. A group conducted research under the supervision of Prof.Masoud Khalkhali. This paper is contained some interesting things that we studied.

Contents

1 Heat Equation and Weyl's Law

Our goal in this first part is to introduce *Weyl's Law: one can hear the volume (and dimension) of a drum.* This is arguably the first, and still perhaps the most important, result of *spectral geometry.* In its original form it was first proved by Hermann Weyl in 1911, with many refinements and generalizations that appear later. First we need to introduce several important concepts of mathematics and physics.

1.1 The Spectrum of a Bounded Domain

1. When we play a drum, we can hear different modes of vibrations with different frequencies. It is a mathematical theorem that fundamental frequencies of any object/shape form a sequence

$$\nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \to \infty$$

2. These frequencies are easily obtained from the spectrum of the Laplacian operator that we define below. The spectrum of a shape contains a huge amount of information about its geometry.

   Laplacian in $\mathbb{R}^n$:

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

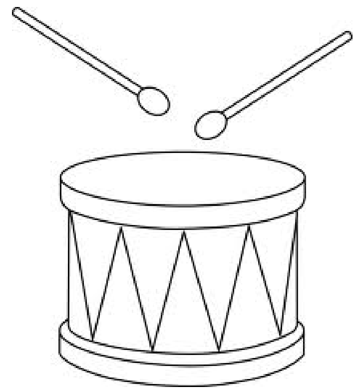


Fig. 1: Is there a relation between the shape of a drum and its frequencies?

3. An eigenvalue problem for a bounded domain $M \subset \mathbb{R}^n$ with piecewise smooth boundary:
$$\begin{cases} \Delta u = \lambda u \\ u|\partial M = 0, \ u \neq 0. \end{cases}$$
This type of boundary condition (vanishing on the boundary) is called *Dirichlet boundary condition.* There are other types of boundary conditions (Neumann, mixed, etc.), but we shall mostly focus on Dirichlet boundary conditions.

4. The *spectrum*: It is a remarkable fact that the above problem has a non-trivial solution only for a discrete set of non-negative numbers
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$
This set of values is called the *Dirichlet spectrum* of the domain M :
$$Spec(M) = \{\lambda_1, \lambda_2, \cdots\}$$
For each eigenvalue λ, the corresponding function u is called an *eigenfunction.* It is also known that each eigenvalue λ has *finite multiplicity.* They can have simple (or non-degenerate) eigenvalues or degenerate eigenvalues.

5. Isospectral versus Isometric: Domains M_1 and M_2 are called isospectral if they have the same spectrum (including multiplicities), and isometric if we can move one to other by an Euclidean motion. Isometric domains are isospectral (why ?), but the converse is not true (we shall see two 16 dimensional spaces that are isospectral, but not isometric! See Figure 2 for a two dimensional example.).

6. What is spectral geometry? It is mathematics/physics discipline that allows one to extract information about the geometry of a space from the knowledge of its spectrum.

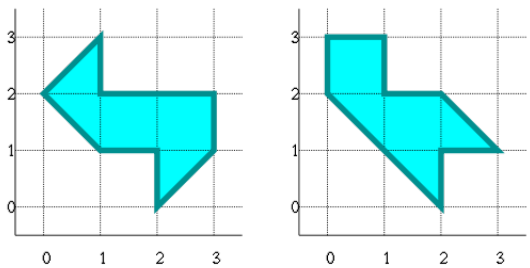
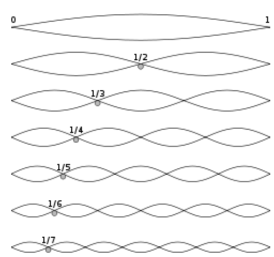
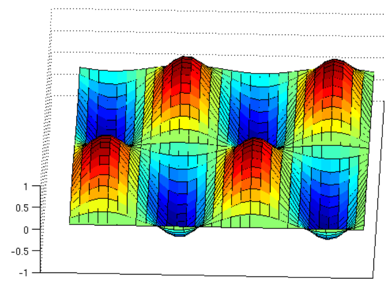


Fig. 2: Isospectral but not isometric

7. Example: The spectrum of a violin string. Here $M = [0, a]$ is a closed interval of length a.
$$\begin{cases} -u'' = \lambda u \\ u(0) = u(a) = 0 \end{cases}$$
The solutions are
$$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi n x}{a}, \qquad \lambda_n = (\frac{\pi n}{a})^2, \quad n = 1, 2, 3, \cdots$$
Note that all eigenvalues are simple (non-degenerate). The fact that $u_n, n \geq 1$ form an orthonormal basis for $L^2(M)$ can be independently proved using Fourier theory.



8. Example: Spectrum of a rectangular drum: $M = [0, a] \times [0, b]$. Eigenfunctions:
$$u_{m,n}(x, y) = \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b}, m, n = 1, 2, 3, \cdots$$
Eigenvalues
$$\lambda_{m,n} = (\frac{\pi m}{a})^2 + (\frac{\pi n}{a})^2$$



$$\lambda = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m, n \geq 1.$$

9. Example: spectrum of a Circular Drum: Laplacian in polar coordinates

$$\Delta = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \;\; 0 \leq r < a,\, 0 \leq \theta \leq 2\pi$$

Radially symmetric solutions

$$u(r, \theta) = u(r)$$

Sub in and get

$$-\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \lambda u$$

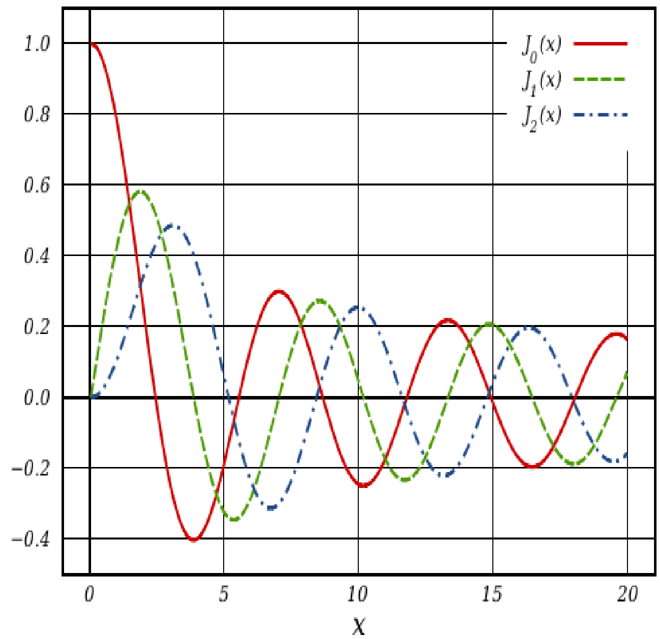
or

$$r^2 u" + r u' + \lambda r^2 u = 0$$

Recall: Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2) y = 0.$$

Let $\alpha = 0$. The only Bounded solutions: Bessel functions $J_{0n}(x)$



The Bessel function J_0 has an infinite number of positive roots,

$$0 < \alpha_{01} < \alpha_{02} < \cdots$$

So:

$$u(r) = J_0\left(\frac{\alpha_{0n}}{a}r\right)$$

is an eigenfunction with eigenvalue $\frac{\alpha_{0n}}{a}$.
For non radially symmetric solutions we put

$$u(r) = J_m(\lambda_{mn}r),\ m = 0, 1, \ldots, n = 1, 2, \ldots,$$

where $\lambda_{mn} = \alpha_{mn}/a$, with α_{mn} the n-th positive root of J_m. So

$$Spec(\text{circular drum}) = \{\lambda_{mn}\}$$

The spectrum of a circular drum is the set of positive roots of Bessel functions.

The smallest eigenvalue: the fundamental frequency is

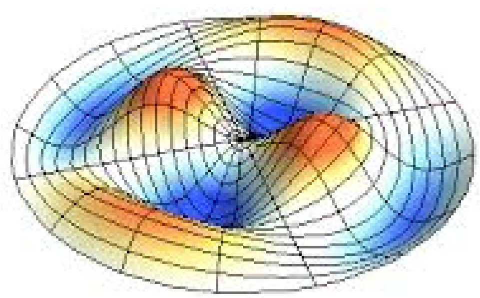
$$\lambda_1 = \frac{\alpha_{01}}{a}$$

with $\alpha_{01} = 2.405$. Compare this with the smallest frequency of a string of length a which is

$$\lambda_1 = \frac{\pi}{a},$$

with $\pi = 3.142$.

Another big difference between a circular drum and a string: eigenvalues of a string are in arithmetic progression; eigenvalues of of a drum are so random and out of proportion!



1.2 The Wave Equation

1. The spectrum of a domain is closely related to *frequencies* of its fundamental modes of vibrations. These vibrations satisfy the wave equation.

2. Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \Delta u,$$

   where c is the wave speed. We assume $c = 1$.

3. Assume the drumhead is clamped at its boundary. Look for a solution

$$u(x,t) = u(x)e^{i\omega t}$$

   we get

$$\Delta u = \omega^2 u$$

   So

$$\omega = \sqrt{\lambda}$$

   that is the frequencies of a drum are square roots of its spectrum.

1.3 Weyl's Law

1. Idea: the area of M can be obtained from its spectrum.

2. Eigenvalue counting function: total number of eigenvalues less than or equal to a given λ
$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

3. For a bounded domain M with piecewise smooth boundary in $\mathbb{R}^n$
$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(M)}{(2\pi)^n} \lambda^{n/2} \qquad \lambda \to \infty$$
where ω_n is the volume of the unit ball in $\mathbb{R}^n$

4. Eigenvalue growth: how fast λ_k grows as $k \to \infty$? Weyl's law is equivalent to
$$\lambda_k \sim C k^{\frac{2}{n}} \quad k \to \infty,$$
with
$$C = \frac{4\pi^2}{(\omega_n \mathrm{Vol}(M))^{\frac{2}{n}}}$$

1.4 The Heat Equation

1. Let M be a domain as before . The heat equation
$$\begin{cases} \frac{\partial \varphi}{\partial t} = -\Delta \varphi \\ \varphi(x,0) = \varphi_0(x) \\ \varphi(x,t) = 0, \ \forall x \in \partial M, t \geq 0. \end{cases}$$
is the evolution equation for distribution of temperature on M given the initial $(t = 0)$ distribution by φ_0.

2. It has a formal solution given by
$$\varphi(x,t) = e^{-t\Delta}\varphi_0, \qquad t > 0$$

1.5 The Heat Kernel

1. We give several equivalent definitions of the heat kernel:
   - As the fundamental solution: this is a function
   $$K(t,x,y) : \mathbb{R}^{>0} \times M \times M \to \mathbb{R}$$
   such that
   $$\frac{\partial K}{\partial t}(t,x,y) = -\Delta K(t,x,y) \ \text{ for all t} > 0 \text{ and x, y} \in \text{M},$$

$$\lim_{t\to 0} K(t,x,y) = \delta_x(y) \quad \text{for all x, y} \in \text{M},$$

in the sense of distributions, that is for all $f \in C_c^\infty(M)$ we have

$$\int_M K(t,x,y) f(y) dVol_y \to f(x) \qquad as\ t \to 0^+$$

$$K(t,x,y) = 0, \quad x \in \partial M \text{ or y} \in \partial\text{M}.$$

- The heat kernel is the kernel of the integral operator $e^{-t\Delta}$:

$$e^{-t\Delta} f(x) = \int_M K(t,x,y) f(y) dy$$

- A formula for K in terms of eigenvalues and eigenfunctions of Δ.

$$K(t,x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).$$

2. What is the physical meaning/significance of $K(t,x,y)$? Assume we have fixed x in M and assume that the initial (i.e. at $t = 0$) temperature distribution is a delta function at x, $\delta(x)$, for $x \in M$. So roughly we can imagine an initial temperature distribution which is equal to 0 away from x and is 1 at x. Then $K(t,x,y)$ gives the distribution of temperature at all times $t > 0$ and for all points $y \in M$.

3. Example: Heat Kernel for $\mathbb{R}$. We check that

$$K(t,x,y) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}}$$

is the fundamental solution of the Laplacian on $\mathbb{R}$. We need just one fact:

$$\int_{\mathbb{R}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}} dy = 1, \quad \forall x,\ \forall t > 0$$

To show that for all x and all f with compact support we have

$$\int_{\mathbb{R}} K(t,x,y) f(y) dy \to f(x) \qquad t \to 0$$

we write, for a given x

$$f(y) = f(y) - f(x) + f(x)$$

and estimate, using Taylor's expansion near x:

$$|f(x) - f(y)| \le M|x-y| \qquad for\ |x-y| \le \delta$$

Then we write

$$\int_{\mathbb{R}} K(t,x,y)f(y)dy = \int_{\mathbb{R}} K(t,x,y)(f(y)-f(x)+f(x))dy$$

$$\int_{\mathbb{R}} K(t,x,y)(f(y)-f(x))dy + \int_{\mathbb{R}} K(t,x,y)f(x)dy$$

The second integral is equal to $f(x)$ for all $t>0$ and to estimate the first we write it as

$$\int_{|x-y|\geq\delta} + \int_{|x-y|\leq|\delta|}$$

The first integral goes to 0 as $t \to 0$ (for any f in fact of polynomial growth) and the second integral can be estimated by

$$\int_{\mathbb{R}}$$

4. Heat kernel for a circle. Let $S^1 = \mathbb{R}/\mathbb{Z}$, denote a circle of radius $\frac{1}{2\pi}$. We give *two* expression for the fundamental solution of the heat equation on a circle. First by averaging the fundamental solution for $\mathbb{R}$ over $\mathbb{Z}$. This gives

$$K(t,x,y) = \sum_{n\in\mathbb{Z}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-n)^2}{4t}}$$

It is easy to see that this is indeed the fundamental solution of the heat equation for the circle.

On the other hand we can use the general formula

$$K(t,x,y) = \sum e^{-\lambda_n t}\varphi_n(x)\bar{\varphi_n}(y)$$

to find the fundamental solution. For circle we have

$$\lambda_n = 4\pi^2 n^2, \qquad \varphi_n(x) = e^{2\pi i n x}, n \in \mathbb{Z}$$

which will give us the formula

$$K(t,x,y) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n(x-y)}$$

Now the equality of two fundamental solutions (uniqueness) give us the identity

$$\sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-n)^2}{4t}}$$

5. Heat kernel for flat tori. Let $\Gamma \subset \mathbb{R}^n$ be a lattice and

$$M = \mathbb{R}^n/\Gamma$$

the flat torus defined by Γ. In a similar fashion we give two formula for the fundamental solution

$$K(t,x,y) = \sum_{n\in\Gamma} (4\pi t)^{-\frac{n}{2}} e^{-\frac{(x-y-n)^2}{4t}}$$

which is obtained by averaging the fundamental solution for $\mathbb{R}^n$ over Γ (method of images). Alternatively we have

$$K(t,x,y) = \sum_{n\in\Gamma^*} e^{-4\pi^2 n^2 t} e^{2\pi i n\cdot(x-y)}$$

The fact that these two are the same is equivalent to Jacobi's inversion formula for lattice theta series.

1.6 Eigenvalues and Eigenfunctions of Flat Tori and Weyl's Law

1. We give *two* proofs of Weyl's law for flat tori: one is technical and based on a trace formula and a Tauberian theorem, while the second is completely elementary and is based on estimating integral lattice points in a Euclidean ball and showing that as the radius of the ball goes to infinity the number of lattice points is proportional to the volume of the ball.

2. Square tori. Let $\Gamma = \mathbb{Z}^d$ be the standard lattice and

$$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$$

Then eigenvalues of Laplacian are labelled by points of $\mathbb{Z}^d$ and are

$$\lambda_m = 4\pi^2(m_1^2 + \cdots + m_d^2) \qquad m \in \mathbb{Z}^d$$

Alternatively, let $r(n)$ denote the number of ways one can represent n as a sum of d squares. Then eigenvalues of Δ are the numbers $4\pi^2 n, n \geq 0$, with multiplicity $r(n)$.

Weyl's Law in this case: Let

$$N(\lambda) = \text{ number of eigenvalues, with multiplicity, smaller than or equal to } \lambda$$

So: we are looking for the number of points with integral coordinates inside the ball of radius

$$r = \frac{\sqrt{\lambda}}{2\pi}$$

An elementary argument shows that this number is asymptotically given by the volume of the all of radius $2\pi r$ (error is of the order r^{d-1}

$$N(\lambda) \sim \frac{\beta(d)}{(2\pi)^d}\lambda^{d/2}$$

Note: the best formula for the second term is still missing! (Gauss circle problem).

Eigenfunctions of Δ are also labeled by $\mathbb{Z}^d$ and are

$$e_m(x) = e^{2\pi i x\cdot m}, \qquad m \in \mathbb{Z}^d.$$

By Fourier theory, we know that e_m, $m \in \mathbb{Z}^d$ form an orthonormal basis for $L^2(\mathbb{T}^d)$.

3. Tori in general. Let us start with dimension one. Let

$$S^1 = \mathbb{R}/L\mathbb{Z}$$

denote a circle of length $L > 0$ with its Riemannian metric induced from the standard metric on $\mathbb{R}$. Its Laplacian is

$$\Delta = -\frac{d^2}{dx^2}$$

Eigenvalues and eigenfunctions of Δ are labeled by the *dual lattice* $L^{-1}\mathbb{Z}$:

$$\lambda_n = 4\pi^2(L^{-1}n)^2, \qquad n \in \mathbb{Z},$$
$$e_n(x) = e^{2\pi i L^{-1}nx}.$$

The heat kernel can be computed explicitly as

$$k(t,x,y) = \sum_{n\in\mathbb{Z}} e^{-\lambda_n t} e_n(x)\bar{e_n}(y) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2(L^{-1}n)^2(x-y)}$$

Then the heat trace is

$$\mathrm{Tr} e^{-t\Delta} = \sum_{n\in\mathbb{Z}} e^{-\lambda_n t} = \sum_{n\in\mathbb{Z}} e^{-4\pi^2/L^2 n^2 t}.$$

To find its asymptotic expansion near 0, we use the Poisson summation formula and obtain an exact trace formula

$$\frac{L}{(4\pi t)^{1/2}} \sum_{\gamma\in\Gamma} e^{-4\pi^2 L^2 n^2/4t} \qquad (t \to 0)$$

Next, we look at flat n-dimensional tori. Let $\Gamma \subset \mathbb{R}^d$ be a compact lattice. The Laplacian for the *flat torus* $M = \mathbb{R}^d/\Gamma$ is

$$\Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

Let Γ^* denote the *dual lattice*

$$\Gamma^* = \{x \in \mathbb{R}^d : \langle x, y \rangle \in \mathbb{Z}\ forally \in \Gamma\}$$

For $\Gamma = (L\mathbb{Z}^d$, we have $\Gamma^* = (L^{-1}\mathbb{Z})^d$.

Eigenvalues and eigenfunctions of Δ are labeled by points $\gamma^* \in \Gamma^*$ of the dual lattice:

$$\lambda_{\gamma^*} = 4\pi^2 ||\gamma^*||^2,$$
$$e_{\gamma^*} = e^{2\pi i \langle \gamma^*, x \rangle}.$$

Notice that the *length spectrum* of M, that is the *lengths of closed geodesics representing free homotopy class of closed loops* in M are parametrized by elements of Γ, and are given by

$$||\gamma||, \quad \gamma \in \Gamma.$$

(first think about circles and then move up)

Using Poisson summation formula, we obtain a *trace formula* for the heat trace of Δ, and a relation between the two spectra:

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 ||\gamma^*||^2 t} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{d/2}} \sum_{\gamma \in \Gamma} e^{-||\gamma||^2/4t}$$

for all $t > 0$. And from this we obtain the asymptotic expansion of the heat trace $\mathrm{Tr} e^{-t\Delta}$:

$$\mathrm{Tr} \sum e^{-t\lambda_i} \sim \frac{\mathrm{Vol}(M)}{(4\pi)^{d/2}} t^{-d/2}, \qquad (t \to 0).$$

Using Karamata's Tauberian Theorem, we can relate $N(\lambda)$ to Vol (M) (Weyl's Law):

$$N(\lambda) \sim \frac{\beta(d)\mathrm{Vol}(M)}{(2\pi)^d} \lambda^{d/2} \qquad \lambda \to \infty$$

We see that: one can hear the volume of a flat torus (Weyl's Law)

Remark 1.1. *This proof of Weyl's law for tori, based on heat trace asymptotics, is kind of complicated: we used a trace formula (Poisson summation formula), and Karamat's Tauberian theorem. As we saw before one can instead give a purely elementary proof, based on counting lattice points inside balls.*

1.7 Problems

1. Prove the formula of Laplacian in polar coordinates we used in the first lecture.

2. Derive the formula of 3-d Laplacian in spherical coordinates.

3. Can you say why the spectrum is a real number? why it cannot be negative, and why it cannot be zero? (Hint: compare with a situation in linear algebra where we have a matrix of the form $A = DD^*$. Can you answer these questions for this matrix, assuming D has a trivial kernel. You need to define a vector space with an inner product and interpret the Laplacian as an operator on that space).

4. What is the importance of the first eigenvalue? We shall introduce the energy of a function on a domain and will see that the first eigenvalue is the smallest possible energy among all nonzero functions.

2 Laplacian

Our goal in this part is to show that the Laplacian on a bounded domain with Dirichlet boundary conditions is an *essentially self-adjoint operator* and present the *spectral decomposition theorem* for Laplacians. The self-adjoint property of the Laplacian is a consequence of the divergence theorem of multivariable calculus. Let us recall this result first.

2.1 Divergence Theorem

1. Gradient and Divergence: Let $M \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary, and let $f : M \to \mathbb{R}$ be a smooth function, i.e. $f \in C^\infty(M)$. The *gradient* of the function f is the vector field

$$\text{grad } f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Let $\text{Vect}(M)$ denote the space of all smooth vector fields on M. It is an (infinite dimensional) vector space. Now

$$\text{grad} : C^\infty(M) \to \text{Vect}(M)$$

is a linear operator since

$$\text{grad } (\alpha f + \beta g) = \alpha \text{ grad } f + \beta \text{ grad } g.$$

for all $\alpha, \beta \in \mathbb{R},\ f, g \in C^\infty(M)$.

Let X be a smooth vector field on M. The *divergence* of the vector field X is the function

$$\text{div } X = \sum \frac{\partial X}{\partial x_i}.$$

Notice that the Laplacian is equal to

$$\Delta = -\text{ div } \cdot \text{ grad}$$

Proof. On the one hand

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial^2 x_i}.$$

And on the other hand

$$-\operatorname{div} \cdot \operatorname{grad} f = -\operatorname{div} \cdot \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) = -\left(\frac{\partial^2 f}{\partial^2 x_1} + \ldots + \frac{\partial^2 f}{\partial^2 x_n} \right).$$

□

Theorem 2.1 (Divergence theorem)**.** *Let $M \subset \mathbb{R}^n$ be a domain with piecewise smooth boundary. Let X be a smooth vector field on M and $\vec{n}$ the unit outward normal vector. Then*

$$\int_M div\ X dv = \int_{\partial M} X \cdot \vec{n}\ dA.$$

Here is a nice Corollary of the divergence theorem. Let $C_0^\infty(M)$ denote the space of smooth functions on M that vanish on the boundary.

Proposition 2.1. *For all $f \in C_0^\infty(M)$, $X \in Vect(M)$*

$$\langle\ grad\ f, X \rangle = \langle f, -div\ X \rangle .$$

Notice that the inner product on $C^\infty(M)$ is $\langle \cdot, \cdot \rangle : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$ such that $\langle f, g \rangle = \int_M fg\ dv$. And the inner product on Vect (M) is

$$\langle X, Y \rangle = \int_M \sum_{i=1}^{n} X_i Y_i\ dv = \sum_{i=1}^{n} \langle X_i, Y_i \rangle .$$

Proof. Apply theorem 2.1 to the vector field fX.

$$\int_M \operatorname{div}\ fX dv = \int_{\partial M} fX \cdot \vec{n}\ dv = 0.$$

Since

$$f|_{\partial M} = 0, \text{ and therefore } fX|_{\partial M} = 0.$$

It is easy to see that

$$\int_M \operatorname{div}\ fX dv = \int_M f \operatorname{div}\ X dv + \int_M \operatorname{grad}\ f\ X dv.$$

Hence

$$\langle f, \operatorname{div}\ X \rangle + \langle \operatorname{grad}\ f, X \rangle = 0.$$

□

2. Properties of the Laplacian:

   Let $T : V \to W$ be a linear operator. We define an adjoint operator T^* to be the operator such that far all $x \in V, y \in W$

   $$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

   Using proposition 2.1 we can say that $(\text{grad})^* = -\text{div}$. But we know that $\Delta = -\text{div} \circ \text{grad} = (\text{grad})^* \circ \text{grad}$. Let $\nabla =$ grad then $\Delta = \nabla^* \nabla$.

   (a) The Laplacian Δ is a self-adjoint operator, i.e. for all $f, g \in C_0^\infty(M)$

   $$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle .$$

   (b) 2^{nd} Green's identity:

   $$\langle \Delta f, g \rangle = \langle \nabla f, \nabla g \rangle = \langle f, \Delta g \rangle .$$

   Proof. Apply Divergence theorem to the vector field $f \nabla g$, for $f, g \in C_0^\infty(M)$ we can obtain

   $$\int_M \text{div } (f \nabla g) dv = \int_M \nabla f \ \nabla g dv - \int_M f \Delta g dv = \int_{\partial M} f \nabla g \ \vec{n} dA = 0.$$

   Hence

   $$\langle \nabla f, \nabla g \rangle = \langle f, \Delta g \rangle .$$

   And we can apply Divergence theorem to the vector field $g \nabla f$ then

   $$\int_M \text{div } (g \nabla f) dv = \int_M \nabla f \ \nabla g dv - \int_M g \Delta f dv = \int_{\partial M} g \nabla f \ \vec{n} dA = 0.$$

   Since

   $$\langle \nabla f, \nabla g \rangle = \langle g, \Delta f \rangle .$$

   □

2.2 Spectrum of Laplacian

Corollary 2.1.

$$Spec(\Delta) \subset (0, \infty).$$

Proof. Let $\Delta f = \lambda f$. We will prove that $\lambda \in (0, \infty)$. From 2^{nd} Green's identity

$$\begin{aligned} \langle \nabla f, \nabla f \rangle &= \langle \Delta f, f \rangle \\ &= \langle \lambda f, f \rangle \\ &= \lambda \langle f, f \rangle \geqslant 0 \end{aligned}$$

So $\lambda \geqslant 0$. Assume M is connected. The Laplacian $\Delta f = 0$, then

$$0 =< \Delta f, f >=< \nabla f, \nabla f > .$$

Therefore $\nabla f = 0$, i.e. $\frac{\partial f}{\partial x_i} = 0, \forall i$ which means that f is constant on M.

Since $f \mid_{\partial M} = 0$, then $f = 0$ on M. □

Corollary 2.2. *For $\lambda_1 \neq \lambda_2$, $u_1 \perp u_2$, i.e. $< u_1, u_2 >= 0$.*

$$\begin{cases} \Delta u_1 = \lambda_1 u_1 \\ \triangle\ u_2 = \lambda_2 u_2 \end{cases}$$

Proof. Using 2^{nd} Green's identity for u_1, u_2

$$\langle \Delta u_1, u_2 \rangle = \langle u_1, \Delta u_2 \rangle = \langle \nabla u_1, \nabla u_2 \rangle$$

we can get

$$< \lambda_1 u_1, u_2 >=< u_1, \lambda_2 u_2 >$$
$$\Rightarrow (\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0.$$

Since $\lambda_1 \neq \lambda_2$, then $< u_1, u_2 >= 0$. □

1. For the Laplacian Δ on M we have

$$\begin{cases} \Delta u = \lambda u \\ u \mid_{\partial M} = 0 \\ u \neq 0 \end{cases}$$

   From this, we get: $\text{spec}(\Delta) = \{0 < \lambda_1 \leqslant \lambda_2 \leqslant ... \to \infty\}$
   Since $\langle u, u \rangle \neq 0$, we can normalize u, such that $< u, u >= 1$. Eigenspace of λ is

$$E_\lambda = \{u \in C_0^\infty(M) \mid \Delta u = \lambda u\}$$

2. Fact: $\dim E_\lambda < \infty$, $\forall \lambda \in \text{spec}(\Delta)$.
   simple (non-degenerate) eigenvalue: $\dim E_\lambda = 1$,
   degenerate eigenvalue: $\dim E_\lambda \geqslant 1$.

2.3 Orthogonal Decomposition Theorem

Theorem 2.2. *Let $u_1, u_2, ...$ be normalized eigenfunctions for eigenvalues $\lambda_1, \lambda_2, ...$, then $\{u_n\}_{n=1}^\infty$ is an orthonormal basis for $H = L^2(M)$, i.e. $\forall f \in L^2(M)$, we can write*

$$f = \sum_{n=1}^\infty a_n u_n, a_n \in \mathbb{R}$$

Note: L^2- convergence:

$$||f - \sum_{n=1}^N a_n u_n|| \to 0, N \to \infty$$

2.4 Spectral Decomposition of Laplacian S^1

1. Spectral decomposition of Laplacian is a main statement of Fourier series $\{e^{2\pi inx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(S^1)$.

2. Example: Circular Drums

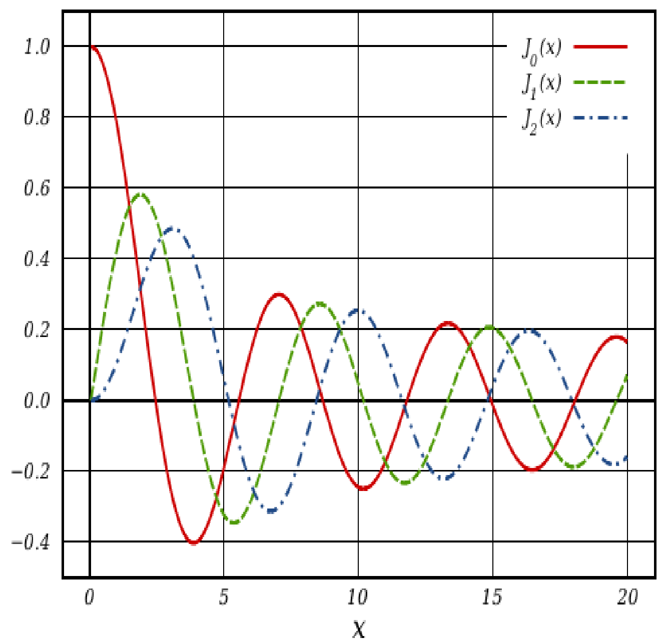
   $spec(\Delta) =$

$$J_0 : \alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{04}, \cdots$$

$$J_1 : \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \cdots$$

$$J_2 : \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \cdots$$

$$J_3 : \alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}, \cdots$$



   We can see that as n increases, the first root of J_n gets larger.

   Note that for a circular drum, all eigenvalues are simple, and the corresponding functions are $u_{nm} = J_n\left(\frac{\alpha_{nm}}{a}x\right)$.

2.5 A Nice Formula for Heat Kernel

1. Here is a nice formula for Heat Kernel:

$$K(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_n t} u_n(x) u_n(y)$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \leq \infty$ is the spectrum of Δ, and $u_1, u_2, u_3, \cdots$ are the orthonormal eigenfunctions. (Note that we assume real-valued functions here.)

Proof. Forgetting about the convergence.

(a) wts: $\forall x,\ \frac{\partial K}{\partial t} = -\Delta_y K$

$$\begin{aligned}\frac{\partial K}{\partial t} &= \sum \frac{\partial e^{-\lambda_n t} u_n(x) u_n(y)}{\partial t} \\ &= \sum -\lambda_n e^{-\lambda_n t} u_n(x) u_n(y) \\ &= \sum -e^{-\lambda_n t} u_n(x) \Delta u_n(y) (\text{ since } \lambda_n u_n(y) = \Delta_y u_n(y)) \\ &= -\Delta \left(\sum e^{-\lambda_n t} u_n(x) u_n(y) \right) \\ &= -\Delta_y K(t, x, y)\end{aligned}$$

(since Δ is a linear operator: $\Delta(f+g) = \Delta f + \Delta g,\ \Delta cf = c\Delta f,\ \Delta\left(\sum f_n\right) = \sum \Delta f_n$)

(b) Boundary conditions: $K(t, x, y) = 0$ if $x \in \partial M$ or $y \in \partial M$. Ok, since u_n satisfies boundary conditions.

(c) wts: $K(t, x, y) \to \delta_x(y)$ as $t \to 0$, i.e. $\int_M K(t, x, y) f(y) dy \to f(x)$ as $t \to 0$.
Expand $f(y) = \sum_{n=1}^{\infty} a_n u_n(y)$ (since $\{u_n\}_{n=1}^{\infty}$ form an orthonormal basis).

$$\begin{aligned}\int_M (\sum_{n=1}^{\infty} e^{-\lambda_n t} u_n(x) u_n(y) \sum_{m=1}^{\infty} a_m u_m(y)) dy = \\ &= \sum_{n,m=1}^{\infty} e^{-\lambda_n t} u_n(x) \int_M a_m u_n(y) u_m(y) dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n u_n(x).\end{aligned}$$

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n u_n(x) = \sum_{n=1}^{\infty} \lim_{t\to\infty} e^{-\lambda_n t} a_n u_n(x) = \sum_{n=1}^{\infty} a_n u_n(x) = f(x).$$

□

3 Eigenvalue Inequalities

Our goal in this part is to: 1) Define the Dirichlet energy of a map and establish some important eigenvalue inequalities, in particular the Max-Min principle, and 2) review Mark Kac's proof of Weyl's law based on Wiener measure and a Tauberian theorem. We also review the Poisson Summation Formula in this section.

3.1 The Dirichlet Energy

1. Let $M \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary. The Dirichlet Energy of $u : M \to \mathbb{R}$ (smooth) is defined as

$$E(u) = \int_M \langle \bigtriangledown u(x), \bigtriangledown u(x) \rangle \, dx$$
$$= \int_M \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 dx$$
$$= \langle \bigtriangledown u, \bigtriangledown u \rangle$$

   ($\bigtriangledown u \in Vect(M)$) Note that $E(u) \geq 0$.

2. Here are some examples in dimensions 1 and 2.

$$E(u) = \int_a^b u'(x)^2 dx,$$
$$E(u) = \int \int_M \left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \right] dxdy$$

3. We need to know what is the relation between this notion of energy and the eigenvalues and eigenfunctions of Laplacian? The following variational principle, known as the Dirichlet principle, shows that critical values of the Dirichlet energy correspond to eigenvalues and eigenfunctions of the Laplacian:

   Proposition 3.1. *(The Variational Principle; Dirichlet Principle) Let $u \in C_0^\infty(M)$ be a critical point of E. Then*

$$\Delta u = \lambda u$$

   for a suitable λ.

4. Using the orthogonal decomposition theorem, we can state a much precise relation between minima of the Dirichlet energy and the eigenvalues of the Laplacian. In particular the first eigenvalue is the minimum of E restricted to functions with norm one:

Proposition 3.2. *(Minimum Energy Principle) For the first eigenvalue of the Laplacian we have*

$$\lambda_1 = \min\{E(u)|\ \langle u, u\rangle = 1\} = \min\{\frac{E(u)}{\langle u, u\rangle}|u \neq 0\}$$

Proof. Let u_1, u_2, u_3, ... be o.n. basis for $L^2(M)$ of eigenfunction s.t. $\Delta u_i = \lambda_i u_i$, i=1,2,3,...
$\forall u \in L^2(M)$, $u = \sum_{i=1}^{\infty} a_i u_i$, $a_i = \langle u, u_i\rangle$, $\langle u_i, u_j\rangle = \delta_{ij}$, $\langle u, u\rangle = \sum_{i=1}^{\infty} a_i^2$
So

$$\begin{aligned}
E(u) &= \langle \bigtriangledown u, \bigtriangledown u\rangle \\
&= \left\langle \bigtriangledown \sum_{i=1}^{\infty} a_i u_i, \bigtriangledown \sum_{i=1}^{\infty} a_i u_i \right\rangle \\
&= \sum_{i,j=1}^{\infty} a_i a_j = \langle \bigtriangledown u_i, \bigtriangledown u_j\rangle \\
&= \sum_{i,j=1}^{\infty} a_i a_j = \langle \bigtriangledown * \bigtriangledown u_i, u_j\rangle \\
&= \sum_{i,j=1}^{\infty} a_i a_j = \langle \Delta u_i, u_j\rangle \\
&= \sum_{i,j=1}^{\infty} a_i a_j = \langle \lambda_i u_i, u_j\rangle \\
&= \sum_{i,j=1}^{\infty} a_i a_j \lambda_i \delta_{ij} \\
&= \sum_{i=1}^{\infty} a_i^2 \lambda_i \geq \sum_{i=1}^{\infty} a_i^2 \lambda_1 (since \lambda_i \leq \lambda_2 \leq \lambda_3 \leq ...) \\
&= \lambda_1 \sum_{i=1}^{\infty} a_i^2 \\
&= \lambda_1 \langle u, u\rangle
\end{aligned}$$

So $\langle \bigtriangledown u, \bigtriangledown u\rangle \geq \lambda_1 \langle u, u\rangle$
So $\frac{\langle \bigtriangledown u, \bigtriangledown u\rangle}{\langle u, u\rangle} \geq \lambda_1$
So $\frac{E(u)}{\langle u, u\rangle} \geq \lambda_1$

But

$$\begin{aligned}\frac{E(u_1)}{\langle u_1, u_1\rangle} &= \frac{\langle \bigtriangledown u_1, \bigtriangledown u_1\rangle}{\langle u_1, u_1\rangle} \\ &= \frac{\langle \Delta u_1, u_1\rangle}{\langle u_1, u_1\rangle} \\ &= \frac{\lambda_1 \langle u_1, u_1\rangle}{\langle u_1, u_1\rangle} \\ &= \lambda_1\end{aligned}$$

□

Corollary 3.1. *If $M_1 \subset M_2$, then*

$$\lambda_1(M_2) \leqslant \lambda_1(M_1),$$

i.e. the bigger the domain, the smaller the first eigenvalue.

Proof.

$$\lambda_1(M_1) = \min\{E(u), u \in C^\infty(M_1)\}$$

But then, $u \in C^\infty(M_2), u = 0$ on M_2, and

$$\lambda_1(M_2) = \min\{E(u), u \in C^\infty(M_2)\}$$

□

5. Here are a couple of examples:

Example 1) Let $M_1 = [0,a] \subset [0,b] = M_2, and a < b.$ then

$$\lambda_1(M_1) = (\frac{\pi}{a})^2 > (\frac{\pi}{b})^2 = \lambda_1(M_2).$$

Example 2) The minimun of $\int_0^1 u'^2(x)dx$, among all u, with $\int_0^1 u^2 dx = 1$ is π^2.

Proof. By Minimum Energy Principle,

$$E(u) = \int_0^1 u'^2(x)dx \geqslant \lambda_1[0,1] = \pi^2$$

□

Question: Prove this directly (use Fourier series) with $u_1(x) = \frac{1}{\sqrt{\pi}} sin(\pi x)$.

Lemma 3.1. *In general*

$$\lambda_k = \min\{E(u) : u \perp u_1, \ldots, u \perp u_{k-1}\} \text{ for } k \geq 2$$

and $\langle u, u\rangle = 1$.

Proof. For all u such that $u \perp u_i,\ i = 1, \dots, k-1$ we can write

$$u = \sum_{n=k}^{\infty} a_n u_n.$$

And we know that $E(u) \le \lambda_k \langle u, u \rangle$, because $\lambda_k \le \lambda_{k+1} \le \dots$. Notice that $E(u_k) = \lambda_k$. □

3.2 Max-Min Principle

Theorem 3.1 (Max-Min Principle)**.**

$$\lambda_k = \max_{dim V = k-1} \left\{ \min_{u \in V^{\perp}, u \neq 0} \frac{E(u)}{\langle u, u \rangle} \right\}.$$

Corollary 3.2 (Corollary of Max-Min Principle)**.** *If* $M_1 \subset M_2$ *then* $\lambda_k(M_2) \ge \lambda_k(M_1)$ *for all* $k = 1, 2, \dots$.

Proof. Use Max-Min Principle to reduce it to the fact that if $A \subset B \subset \mathbb{R}$ then $\inf B \le \inf A$. Notice that $L^2(M_1) \subset L^2(M_2)$.

Let $V \subset L^2(M_1)$ and $dim\ V = k - 1$. And let $u \in L^2(M_1)$. If $u \perp V$ in $L^2(M_1)$ then same is true in $L^2(M_2)$. Since inner product for $u, v \in L^2(M_1)$ has a property

$$\int_{M_1} \langle u, v \rangle = \int_{M_2} \langle u, v \rangle.$$

□

3.3 Proof of Weyl's Law

Here we reproduce, with some details, Kac's heat equation proof of Weyl's Law. His crucial heat kernel estimates can be understood and justified using Wiener measure and Brownian motion.

1. The estimate $K_{M_1}(t, x, y) \le K_{M_2}(t, x, y)$ if $M_1 \subset M_2$. Now, what is $K(t, x, y)$? It is the amount of diffused stuff(heat,...) at y at time $t > 0$, knowing that at $t = 0$,

$$K(0, x, y) = \begin{cases} 1; \text{if } y = x \\ 0; \text{if } y \neq x \end{cases} \tag{1}$$

   and knowing that on ∂M, $temperature = 0$. (i.e.$K(t, x, y) = 0$ if $y \in \partial M$)

2. If $M_1 \subset M_2 \subset \mathbb{R}^n$, then $K_{M_1}(t, x, y) \le K_{M_2}(t, x, y) \le K_0(t, x, y)$, where $x, y \in M_1 \cap M_2$, $K_0(t, x, y)$ is the Heat Kernel for $\mathbb{R}^n$ $K_0(t, x, y) = (4\pi t)^{\frac{-n}{2}} e^{\frac{-(x-y)^2}{4t}}$

3. Understand the result by Brownie Motion

$$K_{M_1}(t,x,y) = \int_{paths x \mapsto y in M_1} ?,$$

$$K_{M_2}(t,x,y) = \int_{paths x \mapsto y in M_2} ?.$$

If $M_1 \subset M_2$, there are more paths $x \mapsto y$ in M_2 than M_1.

Let $M \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary. Note that same proof works in $n-$dim. We know that heat kernel is

$$K(t,x,y) = \sum_{n=1}^{\infty} e^{-t\lambda_n} u_n(x) u_n(y).$$

First step is a principle of non feeling the boundary at short time. So approximation

$$K(t,x,y) \sim \frac{1}{4\pi t} e^{-\frac{(x-y)^2}{4t}},\ 0 < t \ll 1.$$

Now let $x = y$. Hence $K(t,x,x) = \frac{1}{4\pi t}$. We integrating this over M and get

$$\int_M K(t,x,x)dx \sim \int_M \frac{dx}{4\pi t} = \frac{Area(M)}{4\pi t}.$$

$$LHS = \sum_{n=1}^{\infty} e^{-t\lambda_n} \int_M u^2(x)dx = \sum_{n=1}^{\infty} e^{-t\lambda_n}.$$

So we get a very useful expression

$$\sum_{n=1}^{\infty} e^{-t\lambda_n} \sim \frac{Area(M)}{4\pi t}. \tag{2}$$

Apply Tauberian theorem for 2 and get

$$N(\lambda) \sim \frac{Area(M)}{4\pi} \lambda,$$

where $N(\lambda) = \sharp\{\lambda_i \leq \lambda\}$.

Recall that trace of matrix A is

$$Tr(A) = \sum_{i=1}^{\infty} a_{ii}.$$

And we have a *trace formula*

$$Tr(A) = \sum_{i=1}^{\infty} \lambda_i$$

where λ_i is an eigenvalue of matrix A.

Let $A : H \to H$ be a operator (and $dim\ H = \infty$). We know that in some basis

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \ddots & \end{pmatrix}.$$

Apply trace formula to the operator $A = e^{-t\Delta}$. We get

$$Tr(A) = Tr\left(e^{-t\Delta}\right) = \sum_{n=1}^{\infty} e^{-t\lambda_n}.$$

3.4 More Precise Argument

1. Now we need to estimate:

   For $Q \subset M$,

   $$K_Q(t, x, y) \leqslant K_M(t, x, y) \leqslant K_{\mathbb{R}^2}(t, x, y) = (4\pi t)^{-1} e^{-\frac{(x-y)^2}{4t}}$$

   We know that if $Q =$ square with side length a, then

   $$\begin{cases} \frac{4}{a^2} \sum_{m,nodd} e^{-\frac{(m^2+n^2)\pi^2}{4a^2}t} \sim \frac{1}{4\pi t} \end{cases} \tag{3}$$

   $$\text{LHS} = \sum_{n,m}^{\infty} e^{-t\mu_{m,n}}$$

   where $\mu_{m,n}$ are the eigenvalues of square Q.

   $$\begin{cases} \lambda_{m,n} = \pi^2(\frac{m^2}{a^2} + \frac{n^2}{a^2}), m, n \geqslant 1 \\ \mu_{m,n} = sin(\frac{\pi ma}{x}) sin(\frac{\pi na}{y}) \end{cases} \tag{4}$$

   $$\Rightarrow K(t, (x, y), (x', y')) = \sum e^{-t\mu_{m,n}} u_{mn}(x, y) u_{mn}(x', y')$$

   Hence,

   $$\text{LHS} = \sum e^{-t\mu_{m,n}} u_{mn}(x) u_{mn}(x)$$

   But $K_0(t, x, x) = \frac{1}{4\pi t}$.

2. Why does formula (2) hold?

   Proof. Use Poisson Summation Formula (PSF).

   $\forall f : \mathbb{R} \to \mathbb{R}$ (rapidly increasing at ∞),

   $$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

where Fourier transform $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)dx$

□

3. e.g: $f(x) = e^{-ax^2}, a > 0, \hat{f}(y) =?$

$$
\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-2\pi ixy} e^{-ax^2} dx \\
&= \int_{-\infty}^{\infty} e^{-2\pi ixy - ax^2} dx \\
&= \int_{-\infty}^{\infty} e^{-a(x^2 - \frac{2\pi i}{a} xy)} dx \\
&= \int_{-\infty}^{\infty} e^{-a((x - \frac{\pi i}{a} y)^2 + \frac{\pi^2}{a^2} y^2)} dx \\
&= \int_{-\infty}^{\infty} e^{-a((x - \frac{\pi i}{a} y)^2 + \frac{\pi^2}{a^2} y^2)} dx \\
&= \int_{-\infty}^{\infty} e^{-a(x - \frac{\pi i}{a} y)^2} e^{-\frac{\pi^2}{a} y^2} dx \\
&= e^{-\frac{\pi^2}{a} y^2} \int_{-\infty}^{\infty} e^{-a(x - \frac{\pi i}{a} y)^2} dx \\
&= e^{-\frac{\pi^2}{a} y^2} \int_{-\infty}^{\infty} e^{-ax^2} dx \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2}{a} y^2}, because \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}
\end{aligned}
$$

4. Now apply PSF to $f(x) = e^{-\pi x^2 t}, a = \pi t, t > 0,$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum e^{-\frac{\pi n^2}{t}}$$

This is Jacobi's Inversion Formula.

For Theta Function:

$$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, t > 0$$

$$\Theta(t) = \frac{1}{\sqrt{t}} \Theta(\frac{1}{t})$$

From Jacobi's Inversion Formula, we can see

$$\Theta(t) \sim \frac{1}{\sqrt{t}}, t \to 0$$

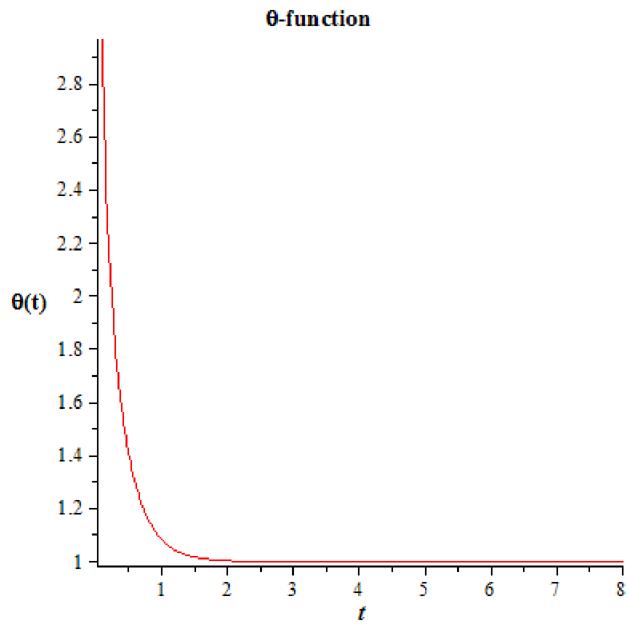


Fig. 3: Theta-function

so,

$$\lim_{t\to 0}\frac{\Theta(t)}{\frac{1}{\sqrt{t}}}=1$$

5. e.g. For $\Theta(t)=\sum_{m,n\in\mathbb{Z}}e^{-\pi(n^2+m^2)t}$, then $\Theta(t)\sim(\frac{?}{\sqrt{t}}), t\to 0$.
   Therefore, Jacobi's Inversion Formula:
   $$\Theta(t)\sim\frac{c}{t}, t\to 0$$
6. **Question**: How many terms need to add the series
   $$\Theta(t)=\sum_{n=\mathbb{Z}}e^{-\pi n^2 t}$$
   in order to find $\Theta(\frac{1}{10})$ in a 2-decimal digits? Note: Jacobi noticed that when using $\Theta(t)=\frac{1}{\sqrt{t}}\Theta(\frac{1}{t})$, if adding 2 terms on RHS, you get $\Theta(t)$ in 500 digits.

3.5 The Poisson Summation Formula

The *Poisson Summation Formula* (PSF) is a *trace formula* and is a feature of Fourier theory and harmonic analysis on *abelian groups*.

The prototypical trace formula is a well known linear algebra fact. All trace formulae, including the PSF, can be regarded as a generalization (typically to infinite dimensional spaces where some analysis is required) of the following fact: for any n by n matrix A we have

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

The two sides are of very different nature. The LHS is related to geometry, while the RHS is spectral.

(a) Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz class function. In fact, it suffices to assume that f'' is continuous and $|f|, |f'|, |f''|$ are bounded by $(1+x^2)^{-1}$. The *Poisson summation formula* states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(p) = \int_{\mathbb{R}} e^{-2\pi i p x} f(x) dx.$$

We shall give two proof of this. Here is our first proof:

Proof. Let

$$g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

This is a periodic function and by Fourier inversion formula for periodic functions we know that

$$g(x) = \sum_{n\in\mathbb{Z}} \hat{g}_n e^{2\pi i n x}$$

and hence

$$g(0) = \sum_{n\in\mathbb{Z}} \hat{g}_n$$

But the Fourier coefficient

$$\hat{g}_n = \int_0^1 g(x) e^{-2\pi i n x} dx$$

can be computed as

$$\int_0^1 g(x)e^{-2\pi i n x}dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i n x}dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi i n x}dx$$

$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(y)e^{-2\pi i n(y-m)}dx = \int_{\mathbb{R}} e^{-2\pi i n y} f(y) dy = \hat{f}(n)$$

$\square$

(b) (PSF as a trace formula). And here is a better proof of PSF that makes it evident that it is a trace formula. A key idea here is that

Characters are eigenfunctions of convolution operators

Proof. The function f defines a convolution operator

$$T : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z}), \quad T(g) = f * g$$

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

One checks that $f * g$ is a periodic smooth function, hence T is well defined. Now we claim that the operator T is an integral operator with a kernel $k(x, y)$ defined by

$$k(x, y) = \sum_{n\in\mathbb{Z}} f(x - y + n)$$

Suffices to check that

$$(Tg)(x) = \int_0^1 k(x, y)g(y)dy, \qquad \forall g \in L^2(\mathbb{R}/\mathbb{Z})$$

To see this, we compute the RHS

$$\int_0^1 k(x, y)g(y)dy = \int_0^1 \sum_{n\in\mathbb{Z}} f(x-y+n)g(y)dy = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x-y)g(y)dy$$

$$= \int_{-\infty}^{\infty} f(x-y)g(y)dy = f * g(x)$$

On the other hand we can show that this compact operator T is diagonalizable in the orthonormal basis

$$e_n = e^{2\pi ix}, \qquad n \in \mathbb{Z}$$

In fact we have

$$T(e_n) = f * e_n == \int_{-\infty}^{\infty} f(x-y)e^{2\pi iny}dy =$$

(This is incomplete) □

(c) If instead of $\mathbb{Z}$ we work over the lattice $L\mathbb{Z}$, the the right hand side will turn into a sum over the *dual lattice* $L^* = L^{-1}\mathbb{Z}$ and we get

$$\sum_{x\in L} f(x) = \frac{1}{L} \sum_{y\in L^*} \hat{f}(y)$$

The Fourier transform of the Gaussian $f(x) = e^{-\pi x^2}$, is equal to itself

$$\hat{f}(p) = e^{-\pi p^2}.$$

Using this and applying the PSF to the lattice $\sqrt{t}\mathbb{Z}$, we obtain Jacobi's inversion formula for theta series

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}.$$

Or,

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta(\frac{1}{t}).$$

In particular we get the remarkable asymptotic expansion for θ-series near 0:

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \sim \frac{1}{\sqrt{t}} \qquad (t \to 0)$$

Notice that for t small the left hand side is a slowly convergent series, while the right hand side gives a very good value for this series. For example, for $t = 0.01$ we need 21 terms of the LHS to compute it with one significant digit, while the right hand side gives its value to 130 digits! (we have to check this with Mathematica!)

The Fourier transform of $(t/\pi)(x^2 + t^2)^{-1}$ is $e^{-2\pi|\gamma|t}$, so

$$\sum_{n\in\mathbb{Z}} (n^2 + t^2)^{-1} = \frac{\pi}{t} \sum_{n\in\mathbb{Z}} e^{-2\pi|n|t} = \frac{\pi}{t}\frac{1 + e^{-2\pi t}}{1 - e^{-2\pi t}}$$

If we let $t \to 0$, we get

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3.6 In $\mathbb{R}^n$

Let $\Gamma \subset \mathbb{R}^d$ be a lattice and let Γ^* denote its *dual lattice* defined

$$\Gamma^* = \{x \in \mathbb{R}^d : \langle x, y\rangle \in \mathbb{Z} \;\; \forall y \in \Gamma\}$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz class function. The Poisson summation formula states that

$$\sum_{\gamma\in\Gamma} f(\gamma) = \frac{1}{\operatorname{Covol}(\Gamma)} \sum_{\gamma^*\in\Gamma^*} \hat{f}(\gamma^*),$$

(Example of Jacobi Inversion Formula for θ-series)
Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Its theta series is defined as

$$\theta_\Gamma(t) = \sum_{x\in\Gamma} e^{-\pi t x^2}, \qquad t > 0.$$

$$f(x) = e^{-\pi x^2}, \qquad x \in \mathbb{R}^n$$

denote the Gaussian. It is well known that its Fourier transform is given by

$$\hat{f}(x) = e^{-\pi x^2}, \qquad x \in \mathbb{R}^n.$$

Let us apply the PSF to the lattice

$$\Gamma_t = \sqrt{t}\,\Gamma, \qquad t > 0$$

We obtain the Jacobi Inversion Formula Theta Series

$$\sum_{x\in\Gamma} e^{-\pi t x^2} = \frac{1}{\operatorname{Covol}(\Gamma)} \sum_{y\in\Gamma^*} e^{-\frac{\pi x^2}{t}}, \qquad t > 0$$

3.7 A Tauberian Theorem

To derive the Weyl's Law from the asymptotic expansion of the heat trace we need a Tauberian theorem. Given a sequence of non-negative numbers

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

Assume the Dirichlet series

$$f(t) = \sum e^{-t\lambda_i}$$

is convergent for all $t > 0$. The *eigenvalue counting function*:

$$N(\lambda) = \#\{\lambda_i \le \lambda\}$$

can be related to the asymptotic behaviour of $f(t)$ near 0, thanks to Karamata's Tauberian

Theorem 3.2. *Let $d\mu(\lambda)$ be a positive measure on $\mathbb{R}_+$ such that the integral*

$$f(t) = \int_0^\infty e^{-t\lambda} d\mu(\lambda)$$

converges for $t > 0$, and such that

$$\lim_{t\to 0} t^\alpha f(t)$$

4 Heat Trace

4.1 (Asymptotic) Heat Kernel Expansion Theorem

Let M be a closed space, with no boundary, then we have an expansion of the form

$$K(t,x,y) \sim (4\pi t)^{-\frac{n}{2}}(a_0(x) + a_1(x)t + a_2(x)t^2 + ...)$$

as $t \to 0$.
Additionally we have that

$$a_0(x) = 1$$
$$a_1(x) = \frac{1}{6}s(x)$$

where $s(x)$ is the scalar curvature of M and

$$s(x) = 2K(x)$$

Here, $K(x)$ denotes Gaussian curvature and is a measure of intrinsic bending of our space or surface at x.
In the case where our surface has a boundary, we get the following expansion,

$$K(t,x,y) \sim (4\pi t)^{-\frac{n}{2}}(b_0(x) + b_1(x)t^{1/2} + b_2(x)t + ...)$$

as $t \to 0$.

4.2 Trace of the Heat Operator

$$Z(t) := tr(e^{-t\Delta}) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

$$Z(t) = tr(e^{-t\Delta}) = \int_M K(t,x,x)dx$$
$$\sim (4\pi t)^{-n/2}(a_0(x) + a_1(x)t + a_2(x)t^2 + ...)$$

In the case with no boundary, and

$$\sim (4\pi t)^{-n/2}(b_0(x) + b_1(x)t^{1/2} + b_2(x)t + ...)$$

in the case with a boundary.

Consider $Z(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$ is a trace of the heat operator. Our goal is to find short-time expansion of $Z(t)$ in four examples: circle, violin string, flat torus and rectangular.

1. Consider a circle with length 2π (i.e. S^1). In this case the trace of the heat operator is
$$Z(t) = \sum_{n=1}^{\infty} e^{-n^2 t}.$$

Find an asymptotic expansion of $Z(t)$. Recall that theta-function is $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$. And from Poisson summation formula we can get

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

This formula implies that

$$\theta(t) \sim \frac{1}{\sqrt{t}} \text{ as } t \to 0$$

Hence

$$Z(t) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} = \theta\left(\frac{t}{\pi}\right).$$

Since

$$\lim_{t\to 0} \frac{\theta\left(\frac{t}{\pi}\right)}{\frac{1}{\sqrt{\frac{t}{\pi}}}} = 1.$$

We can get that $Z(t) \sim \frac{1}{\sqrt{t}}\sqrt{\pi}$. So $Z(t) = \frac{1}{\sqrt{4\pi t}} 2\pi$. But from heat kernel expansion theorem $\int_M a_0 = Vol(M) = 2\pi$.

2. Let consider a string with a length 2π. We know that eigenvalues are $\lambda_n = \frac{n^2}{4},\ n = 1, 2, \ldots$. Notice that these are simple eigenvalues. So

$$Z(t) = \sum_{n=1}^{\infty} e^{\frac{-n^2 t}{4}} = \frac{\theta(\frac{t}{4\pi}) - 1}{2} = \frac{\tilde{Z}(\frac{t}{4}) - 1}{2}$$

where $\tilde{Z}(t) = \theta\left(\frac{t}{\pi}\right)$. Analogously, $\tilde{Z}(\frac{t}{4}) \sim \frac{2\sqrt{\pi}}{\sqrt{t}}$. Using this expansion we can get

$$Z(t) \sim \frac{\sqrt{\pi}}{\sqrt{t}} - \frac{1}{2} = (4\pi t)^{-\frac{1}{2}}\left(2\pi - \sqrt{\pi} t^{\frac{1}{2}} + \ldots\right).$$

From kernel expansion theorem it is easy to see that $\int_M a_0 = 2\pi$ since length of a string is equal to 2π.

3. We now consider a rectangular domain, M, with length a and width b. From a previous calculation we found that

$$spec(\Delta) = \pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2}), m, n = 1, 2, ...$$

with corresponding eigenfunctions

$$u_{m,n} = sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}x)$$

Therefore,

$$\begin{aligned} Z(t) &= \sum_{m,n=1}^{\infty} e^{-\pi^2(\frac{m^2}{a^2}+\frac{n^2}{b^2})t} \\ &= \sum_{m,n=1}^{\infty} e^{-\pi^2(\frac{m^2}{a^2})t} e^{-\pi^2(\frac{n^2}{b^2})t} \\ &= \sum_{m=1}^{\infty} e^{-\pi^2(\frac{m^2}{a^2})t} \sum_{n=1}^{\infty} e^{-\pi^2(\frac{n^2}{b^2})t} \end{aligned}$$

Now, using the fact that $\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}}\Theta(\frac{1}{t})$ the above give,

$$\begin{aligned} &= \left(\frac{\Theta\left(\frac{\pi t}{a^2}\right)-1}{2}\right)\left(\frac{\Theta\left(\frac{\pi t}{b^2}\right)-1}{2}\right) \\ &= \frac{1}{4}\left(\frac{1}{\sqrt{\frac{\pi t}{a^2}}}\Theta\left(\frac{1}{\frac{\pi t}{a^2}}\right)-1\right)\left(\frac{1}{\sqrt{\frac{\pi t}{b^2}}}\Theta\left(\frac{1}{\frac{\pi t}{b^2}}\right)-1\right) \end{aligned}$$

and if we use the fact that as $t \to 0$ $\Theta\left(\frac{1}{\frac{\pi t}{a^2}}\right) \to 1$ we see,

$$\begin{aligned} &\sim \frac{1}{4}\left(\frac{a}{\sqrt{\pi t}}-1\right)\left(\frac{b}{\sqrt{\pi t}}-1\right), t \to 0 \\ &= \frac{1}{4}\left(\frac{ab}{\pi t}-\frac{a+b}{\sqrt{\pi t}}+1\right), t \to 0 \\ &= \frac{1}{4\pi t}\left(Area(M)-\sqrt{\pi}\left(\frac{1}{2}Perimeter(M)\right)\sqrt{t}+\pi t\right), t \to 0 \end{aligned}$$

The above gives the first 3 terms in the expansion.

Corollary: One can hear the area and the perimeter of a rectangle.

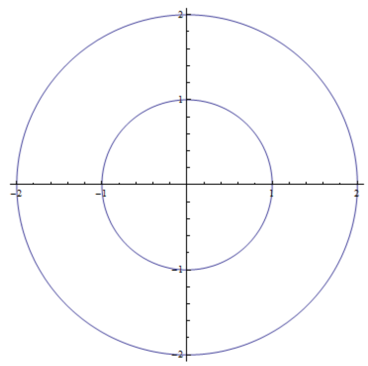
Further, Kac (put citation here**) proves that we can hear the area and perimeter for any bounded domain, M, with smooth boundary and r holes.

$$Z(t) \sim \frac{A}{4\pi t} - \frac{L}{8\sqrt{nt}} + \frac{1}{6}(1-r), t \to 0$$

where A and L are the area and perimeter of M, respectively, and r is the number of holes in M.

Question: Find the eigenvalues and eigenfunctions for concentric circles.

Proof. Let M be a domain $\{(x,y) : r^2 \leq x^2+y^2 \leq R^2\}$ or in polar coordinates $M = \{(\rho,\theta) : r \leq \rho \leq R,\ 0 \leq \theta < 2\pi\}$.



We know that Laplacian in polar coordinates

$$\Delta = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right).$$

By analogy with circular we can get that solution of Bessel equation in this case is

$$u(\rho) = C_0 J_0(\sqrt{\lambda}\rho) + C_1 I_0(\sqrt{\lambda}\rho),$$

where $J,\ I$ are the Bessel function.

But solution satisfies boundary conditions

$$C_0 J_0(\sqrt{\lambda}R) + C_1 I_0(\sqrt{\lambda}R); C_0 J_0(\sqrt{\lambda}r) + C_1 I_0(\sqrt{\lambda}r).$$

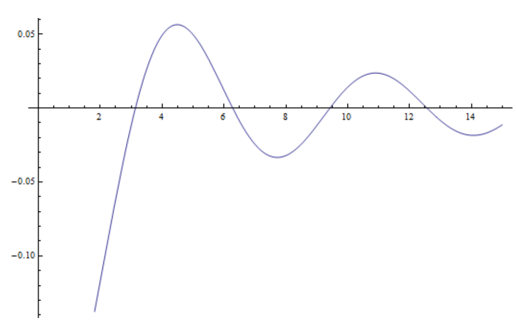
This is a linear system of equation. It's has nontrivial solution when determinant of system is equal to 0. So

$$\begin{vmatrix} J_0(\sqrt{\lambda}R) & I_0(\sqrt{\lambda}R) \\ J_0(\sqrt{\lambda}r) & I_0(\sqrt{\lambda}r) \end{vmatrix} = 0.$$

Therefore we get equality

$$J_0(\sqrt{\lambda}R) I_0(\sqrt{\lambda}r) = J_0(\sqrt{\lambda}r) I_0(\sqrt{\lambda}R). \tag{5}$$

This is the transcendental equation with infinite numbers of positive roots.



Let $\nu_n,\ n = 0, 1, 2, \dots$ are these roots. Then $\sqrt{\lambda} = \nu_n,\ n = 0, 1, 2, \dots$ are eigenvalues. So

$$u_n(\rho) = -I_0(R\nu_n)J_0(\nu_n\rho) + J_0(R\nu_n)I_0(\nu_n\rho)$$

is an eigenfunction with eigenvalue ν_n^2, where ν_n the n-th positive root of (5).

□

4. Now we consider the flat torus $\mathbb{T}^2 = \mathbb{R}/a\mathbb{Z} \times \mathbb{R}/b\mathbb{Z}$. We see above that eigenvalues in this case are $\lambda_{n,m} = 4\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),\ m, n \in \mathbb{Z}$. Therefore,

$$Z(t) = \sum_{m,n\in\mathbb{Z}}^{\infty} e^{-4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)t} = \sum_{m\in\mathbb{Z}}^{\infty} e^{-4\pi^2\frac{m^2}{a^2}t} \sum_{n\in\mathbb{Z}}^{\infty} e^{-4\pi^2\frac{n^2}{b^2}t} = \theta(\frac{4\pi}{a^2}t)\theta(\frac{4\pi}{b^2}t)$$

$$= \frac{a}{\sqrt{4\pi t}}\theta(\frac{a^2}{4\pi}\frac{1}{t})\frac{b}{\sqrt{4\pi t}}\theta(\frac{b^2}{4\pi}\frac{1}{t}) = \frac{ab}{4\pi t} \sim \frac{Area(M)}{4\pi t} \text{ as } t \to 0.$$

It means that terms $a_1 = a_2 = \dots = 0$ in heat kernel expansion for flat torus. And we get that

$$a_1 = \int_{\mathbb{T}^2} \frac{1}{6}s(x)dx = 0,$$

where $s(x)$ is scalar curvature.

5 Sphere S^2

Our goal now to determine the eigenvalues, multiplicities, and eigenfunctions of the Laplacian on the 2-d round sphere. Then generalize this results to spheres in any dimension.

5.1 Spectrum of Spheres

1.
$$M = S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

Also, on S^2,

$$\begin{cases} \triangle\ u = \lambda u \\ u \neq 0 \end{cases} \tag{6}$$

Laplacian in $\mathbb{R}^3$,

$$\triangle = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

In spherical coordinates, we use (θ, ϕ, r), thus we have:

$$\begin{cases} x = rsin\phi cos\theta \\ y = rsin\phi sin\theta \\ z = rcos\phi \end{cases} \tag{7}$$

The corresponding Laplacian is

$$-\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{S^2}$$

with

$$\triangle_{S^2} = \frac{1}{sin\theta}\frac{\partial}{\partial\theta}sin\theta\frac{\partial}{\partial\theta} + \frac{1}{sin\theta^2}\frac{\partial^2}{\partial\phi^2}$$

2. Use the symmetry of the problem:

$$G = \text{group of rotation of } \mathbb{R}^3$$

$\forall g \in G, \forall f \in C^\infty(S^2)$,

$$(gf)(x) = f(g^{-1}x)$$

G acts on functions, and $\triangle$ acts on functions, also these two actions commute.

$$g\triangle = \triangle g, \forall g \in G$$

i.e.

$$g(\triangle f) = \triangle (gf)$$

3. Let P^k = homogeneous polynomials of degree k in x, y, z, $k = 0, 1, 2, ...$, then

$$\begin{aligned} &P^0 = \{1\} \\ &P^1 = \{x, y, z\} \\ &P^3 = \{x^2, y^2, z^2, xy, xz, yz\} \\ &P^4 = \{x^3, y^3, z^3, x^2y, ...\} \\ &... \end{aligned}$$

$g(P^k) \subset P^k$, e.g. $gP^2 \subset P^2$, and g is a 3×3 matrix. Then

$$\begin{aligned} g(f) &= g(x^2) = ? \\ g(f)(x) &= f(g^{-1}x) \\ &= \text{get a polynomial of order 2 again} \end{aligned}$$

$$f(x, y, z) \in P^k, f(x, y, z) = \sum_{i=0}^{k} f_i(x, y) z^{k-i},$$

and $f_i(x, y)$ is a homogeneous polynomial of degree i.

We then know,

$$\dim\{f_i; \deg f_i = i\} = \dim\{x^m y^n | m + n = i\} = i + 1$$

$$\begin{aligned} \Rightarrow \dim P^k &= \sum_{i=0}^{k}(i+1) \\ &= 1 + 2 + 3 + \cdots + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

e.g. $\dim P^3 = 10$

4. Back to $\triangle$: $P^k \to P^{k-2}$,

$$-(\partial_x^2 + \partial_y^2 + \partial_z^2) f(x, y, z) \in P^{k-2}$$

Spherical Harmonics of Degree k:

$$\begin{aligned} H^k &= \text{Ker } \triangle \\ &= \{f \in P^k | \triangle f = 0\} \end{aligned}$$

Here are some examples:

$$\begin{aligned} &H^0 = P^0 = \{1\} \\ &H^1 = P^1 = \{x, y, z\} \\ &H^2 = \{ax^2 + by^2 + cz^2 + dxy + exz + fyz | a + b + c = 0\} \end{aligned}$$

5. e.g. Why $\triangle$: $P^4 \to P^2$ is surjective?

e.g. Why $\exists f \in P^4, \text{s.t.} \ \triangle (f) = x^2$?

Consider the example $f = -\frac{1}{3\times 4}x^4$. And same with y^2, z^2.

$$\triangle (f) = xy \to f = -\frac{1}{6}x^3 y$$

Corollary 5.1. *(rank + nullity Theorem)*

$$dimKer \triangle = dimP^k - dimP^{k-2}$$

6. From Linear Algebra:

$$T : V \to W$$

$$\text{rank}(T) + \text{null}(T) = \dim(V)$$

$$\text{dimim}(T) + \text{dimKer}(T) = \dim(V)$$

So,

$$\dim H^k = \frac{(k+1)(k+2)}{2} - \frac{(k-1)k}{2} = 2k+1$$

$$\Rightarrow \dim H^k = 2k+1$$

$$f \in H^k, \triangle\, f = 0$$

7. Claim:

$$\forall f \in H^k, \Delta_{S^2} f = k(k+1)f$$

(i.e. H^k is an eigenspace of Δ_{S^2} with eigenvalue $k(k+1)$)

Proof. Recall :

$$-\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^2}$$

and we call this **.

$\forall f \in H^k$ is homogeneous of degree k, $f = r^k \tilde{f}$, where $\tilde{f}$ is a function on S^2.

For example,

$$H^2 = \left\{x^2 + by^2 + cz^2 + dxy + exz + fyz | a+b+c = 0\right\}$$

$$k = 2, r = \sqrt{x^2+y^2+z^2},$$

so we have $r^2 = x^2 + y^2 + z^2$, and then

$$F = \left(x^2+y^2+z^2\right)\left(\frac{ax^2}{r^2} + \frac{by^2}{r^2} + \frac{cz^2}{r^2} + ...\right)$$

Let

$$\tilde{F} = \frac{ax^2}{r^2} + \frac{by^2}{r^2} + \frac{cz^2}{r^2} + ...,$$

we have

$$F = r^2 \tilde{F}.$$

Note that $\tilde{F}$ is a function of θ and ϕ only.

Now,

$$\forall f \in H^k \subset P^k, \Delta f = 0$$

If we write $f = r^k \tilde{f}(\phi, \theta)$, and we use ** to compute Δf,

then we get

$$-\Delta f = \frac{\partial^2}{\partial r^2}\left(r^k \tilde{f}(\phi,\theta)\right) + \frac{2}{r}\frac{\partial}{\partial r}\left(r^k \tilde{f}(\phi,\theta)\right) + \frac{1}{r^2}\Delta_{S^2}\left(r^k \tilde{f}(\phi,\theta)\right)$$

$$\begin{aligned}\Rightarrow 0 &= k(k-1)r^{k-2}\tilde{f}(\phi,\theta) + \frac{2}{r}kr^{k-1}\tilde{f}(\phi,\theta) - \frac{r^k}{r^2}\Delta_{S^2}\tilde{f}(\phi,\theta) \\ &= \left(k(k-1) + 2k - \Delta_{S^2}\right)\tilde{f}(\phi,\theta)\end{aligned}$$

then we have
$$\Delta_{S^2}\tilde{f}(\phi,\theta) = k(k+1)\tilde{f}(\phi,\theta)$$

i.e. $\forall f \in H^k$(spherical harmonic of degree k), by restricting $f|_{S^2} = \tilde{f}(\phi,\theta)$, we can get an eigenfunction, with eigenvalues $k(k+1), n = 0, 1, 2, 3, ...,$ and these are non-simple eigenvalues with multiplicity $2k+1$.

Eigenvalues grow quadratically with multiplicities. Acutally, theses are the only eigenvalues. □

8. Checking Weyl's Law for S^2

   Let $\#\{\lambda_i \leq \lambda\} = N(\lambda)$
$$N(\lambda) \sim \frac{w_n Vol(M)}{(2\pi)^n}\lambda^{\frac{n}{2}}, \text{as } t \to \infty$$
   where w_n is the volume of the unit ball in $\mathbb{R}^n$.

   When $n=2$, $M=S^2$, $w_2 = \pi$, we have
$$N(\lambda) \sim \frac{\pi 4\pi}{4\ pi^2}\lambda = \lambda.$$
   Check for S^2 that $N(\lambda) \sim \lambda$ as $\lambda \to \infty$:
$$\begin{aligned} \lambda &= \#\{\lambda_i \leq \lambda\} \\ \lambda_i &\leq \lambda \\ k(k+1) &\leq \lambda \\ k^2 + k &\leq \lambda \\ k^2 &\leq \lambda \\ k &\leq \sqrt{\lambda} \end{aligned}$$
   But how many eigenvalues within k?
$$\begin{aligned} N(\lambda) &\sim 1+3+5+...+(2k+1) \\ &= (k+1)^2 \\ &= k^2+2k+1 \\ &= \lambda + 2\sqrt{(}\lambda)+1 \end{aligned}$$
   So we have
$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda} = 1.$$

9. **Problem**: Find eigenvalues and multiplicities of Δ(Laplacian) on
$$S^n = \left\{(x_1, ..., x_n, x_{n+1}) \,|\, \sum_{i=1}^{n+1} x_i^2 = 1\right\}.$$

Idea:

P^k is homogeneous polynomials of degree k in $x_1, ..., x_{n+1}$.

$$H^k \subset P^k, H^k = Ker\Delta_{\mathbb{R}^{n=1}}$$

Compute dimP^k, dimH^k(using $\Delta : P^k \to P^{k-2}$ is surjective)
Write $f \in H^k$ as $f = r^k \tilde{f}$, and check that

$$\Delta_{\mathbb{R}^{n+1}} f = 0 \Rightarrow \Delta_{S^n} \tilde{f} = \lambda \tilde{f}.$$

(using expression of $\Delta_{R^{n+1}}$ in spherical coordinates in $\mathbb{R}^{n+1}$: $(r, \phi_1, \phi_2, ..., \phi_n)$)
and check Weyl's Law.

Theorem 5.1. *$P^k \mapsto P^{k-2}$ by Δ is a surjection.*

Proof. Clearly Δ is a linear map. Now we want to prove surjectivity of Δ.
We prove the n-th dimension case. $x_1^{a_1} x_2^{a_2} ... x_n^{a_n} \in P^{k-2}$ where
$a_1 + a_2 + .. + a_n = k-2$. We tentative guess the answer to be $x_1^{a_1+2} x_2^{a_2} ... x_n^{a_n}$.
But $\Delta(x_1^{a_1+2} x_2^{a_2} ... x_n^{a_n}) = (a_1+2) * (a_1+1) x_1^{a_1} x_2^{a_2} ... x_n^{a_n} + a_2 * (a_2-1) x_1^{a_1} x_2^{a_2-2} ... x_n^{a_n} + .. + a_n * (a_n-1) x_1^{a_1} x_2^{a_2} ... x_n^{a_n-2}$
By the linearity of Δ, if we can find the preimage of $a_n * (a_n-1) x_1^{a_1} x_2^{a_2} ... x_n^{a_n-2}$, we're done. This case can easily be achieved by induction on the minimum of $\{a_1, a_2, ... a_n\}$,where we increase a_1 by 2 every time but decrease all other terms to the power of 1 or 0.

The base case is $x_1^a x_2 ... x_m$ (Not all terms are required to be present). The preimage of this is $\frac{1}{(a+2)(a+1)} x_1^{a+2} x_2 ... x_m$. ◻

Find the dimension of H^k the kernel of $p^k \mapsto p^{k-2}$ by Δ with the space dimension m.

Proof. We know Δ is surjective. Hence it's reduced to finding the dimension of p^k where p^k is the homogeneous polynomial of k^{th} degree with variable $x_1, x_2, ... x_k$.

This is a classical combinatorics question of choosing some number of balls of a few red, blue, green balls. This is solved by using generating functions. For example, if you have 3 red balls, 5 blue balls and 6 green balls, what's combination of choosing 4 balls? If you expand $(1 + x + ..x^3)(1 + x + ..x^5)(1 + x + ..x^6)$, what's the coefficient of x^4? You can choose x from the first bracket and x^3 from the second, or choose x^2 from the first, 1 from the second, and x^2 from the third. You can easily say the two questions are equivalent.

Similarly, to find the dimension of p^k we can expand $(1+x+x^2+\ldots+x^k+\ldots)^m$ and find the coefficient of x^k. We can use Taylor's series. It is easy to see that $(1+x+x^2+\ldots+x^k+\ldots)^m = \frac{1}{(1-x)^m}$. Let $f(x)=(1-x)^{-m}$. Find Taylor's expansion for the function $f(x)$

$$f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(k)}(0)}{k!}x^k + \ldots.$$

So $k-th$ coefficient of $f(x)$ is equal to $\frac{f^{(k)}(0)}{k!}$. Compute this coefficent

$$\frac{f^{(k)}(0)}{k!} = \frac{m(m+1)\cdot\ldots\cdot(m+k-1)}{k!} = \frac{(m+k-1)!}{k!(m-1)!}.$$

But this is exactly $\binom{m+k-1}{k}$. Therefore dimension of P^k is $\binom{m+k-1}{k}$. Hence the dimension of H^k is

$$dim\ H^k = dim\ P^k - dim\ P^{k-2} = \frac{(m+k-1)!}{k!(m-1)!} - \frac{(m+k-3)!}{(k-2)!(m-1)!} = (m^2+2mk-2k-3m+2)\frac{(m+k-3)!}{k!(m-1)!}.$$

□

Now we can write $f = r^k\tilde{f}(\theta_1,\ldots,\theta_{n-1})$ where $\tilde{f}$ is a function on S^n. The Laplacian in spherical coordinates $r,\theta_1,\ldots,\theta_{n-1}$ in n dimensions is

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^n}.$$

On the one hand $\Delta_{\mathbb{R}^{n+1}}f = 0$. Then

$$0 = k(k-1)r^{k-2}\tilde{f} + k(n-1)r^{k-2}\tilde{f} + r^{k-2}\Delta_{S^n}\tilde{f}.$$

Hence

$$\Delta_{S^n}\tilde{f} = k(k+n-2)\tilde{f}$$

for any $f \in H^k$ by restricting $f|_{S^n} = \tilde{f}$. So we get eigenvalues $\lambda_k = k(k+n-2),\ k=0,1,\ldots$ with multiplicity $(n^2+2nk-2k-3n+2)\frac{(n+k-3)!}{k!(n-1)!}$.

10. Imagine we have $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \to \infty$

(1) How fast does λ_k grow?

Eg.

$$\lim_{k\to\infty}\frac{\lambda_k}{\sqrt{k}} = c$$

(2)$N(\lambda) = \#\{\lambda_i \le \lambda\}$, this is the eigenvalue counting function.

$$(2) \Rightarrow (1)$$

Weyl's Law gave us answer to (2) for $\{\lambda_i\} = spec(\Delta)$

$$N(\lambda) \sim \frac{w_n Vol(M)}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

$$N(\lambda) \sim c\lambda^{\frac{n}{2}} \text{as } \lambda \to \infty,$$

where $c = \frac{w_n Vol(M)}{(2\pi)^n}$.

What about (1)? What is the rate of the growth of λ_i's as $i \to \infty$

Answer:

$$\lambda_k \sim c^{-\frac{2}{n}} k^{\frac{2}{n}} \text{as } k \to \infty.$$

Proof. Let $\lambda = \lambda_k$

$$N(\lambda_k) = k = c\lambda_k^{\frac{n}{2}}$$

$$\lambda_k \sim c^{-\frac{2}{n}} k^{\frac{2}{n}} \text{as } k \to \infty$$

□

Conversely, (2) ⇒ (1)

$$\begin{aligned} \lambda_k &\sim c^{-\frac{2}{n}} k^{\frac{2}{n}}, \text{ as } k \to \infty \\ N(\lambda) &= \# \{\lambda_k \leq \lambda\} \\ &= \# \left\{ k | c^{-\frac{2}{n}} k^{\frac{2}{n}} \leq \lambda \right\} \\ &\sim c\lambda^{\frac{n}{2}}, \text{ as } \lambda \to \infty \end{aligned}$$

11. **Question:** According to Prime Number Theorem,

$$\# \{P \leq x\} \sim \frac{x}{\log x}.$$

Let P_k be the $k-th$ prime, find the rate of P_k in k.

Proof. We choose $x = P_k$, then $\# \{P_i \leq P_k\} = k$. Therefore

$$k \sim \frac{P_k}{\log P_k}.$$

The rate of P_k is

$$P_k \sim -kLambertW\left(-\frac{1}{k}\right),$$

where the LambertW function satisfies $x = LambertW(x)e^{LambertW(x)}$. So

$$P_k \sim e^{-Lambert\left(-\frac{1}{k}\right)}.$$

We can conclude that prime numbers have exponential growth.

□

6 ζ-function

Our goals: 1) define spectral *zeta*-function; 2) give the relation between *zeta*-function and the heat trace.

6.1 Asymptotic Behaviour of Eigenvalues

$dim(M) = 0 \Rightarrow \lambda_k \sim e^k, k \to \infty$
.
.
.
$dim(M) = \frac{1}{4} \Rightarrow \lambda_k \sim ck^8, k \to \infty$
$dim(M) = \frac{1}{2} \Rightarrow \lambda_k \sim ck^4, k \to \infty$
$dim(M) = 1 \Rightarrow \lambda_k \sim ck^2, k \to \infty$
$dim(M) = 2 \Rightarrow \lambda_k \sim ck, k \to \infty$
$dim(M) = 3 \Rightarrow \lambda_k \sim ck^{\frac{2}{3}}, k \to \infty$
$dim(M) = 4 \Rightarrow \lambda_k \sim c\sqrt{k}, k \to \infty$
.
.
.
$dim(M) = \infty \Rightarrow \lambda_k \sim c \log k, k \to \infty$ (For $\mathbb{R}^{\mathbb{Z}}$, we don't know if this is correct)
That is, $dim(M) = 2 \Rightarrow \lambda_k \sim ck^{\frac{2}{n}}, k \to \infty$. Therefore the dimension of M and the volume of M are related to the spectrum of Δ on M.

Question: What happens to Weyl's Law in the zero-dimensional case?

6.2 Spectral ζ-function

We know that spectrum of Laplacian is $spec(\Delta) = \{0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\}$. Define the spectral ζ-function

$$\zeta(s) = \frac{1}{\lambda_1^s} + \frac{1}{\lambda_2^s} + \ldots = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}.$$

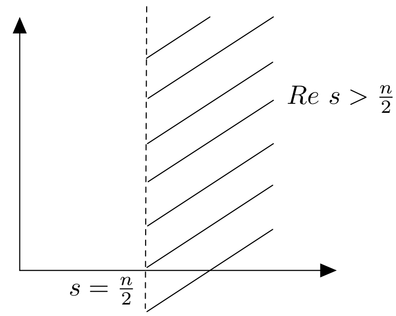
Recall that $\lambda_k \sim Ck^{\frac{2}{n}}$ and $dim\ M = n$. It is easy to see that from this $\zeta(s)$ is convergent for $s > \frac{n}{2}$.

In fact, $\zeta(s)$ has analytical continuation to $\mathbb{C}\backslash\{\frac{n}{2}-j\}$, i.e. $\zeta(s) : \mathbb{C}\backslash\{\frac{n}{2}-j\} \to \mathbb{C}$ where $j = 0, 1, \ldots$. As $\zeta(s)$ for all s such that $Re\ s > \frac{n}{2}$ then $\zeta(s)$ is original Riemann ζ-function.

E.g. let's consider domain $M = S^1$. We know that $spec(\Delta) = \{n^2$ with multiplicity 2, 0 simple eigenvalue $\}$. Therefore,

$$\zeta(s) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2)^s} = \sum_{n=1}^{\infty} \frac{2}{(n^2)^s} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2\zeta_R(2s)$$

where $\zeta_R(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$ is Riemann ζ-function. And it has analytical continuation for whole space $\mathbb{C} \setminus \{1\}$.



6.3 Gamma Function

For any $s \in \mathbb{C}$ with $Re(s) > 0$ we define,

$$\Gamma(s) = \int_0^\infty e^{-t} \frac{t^s}{t} dt$$

In fact, if $s \in \mathbb{Z}_+$, $\Gamma(s) = (s-1)!$. We can see this by noticing that $\Gamma(s+1) = s\Gamma(s)$.

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty e^{-t} \frac{t^{s+1}}{t} dt \\ &= \int_0^\infty e^{-t} t^s dt\end{aligned}$$

Now using integration by parts we get

$$\begin{aligned}\Gamma(s+1) &= t^s(-e^{-t})\Big|_0^\infty - \int_0^\infty -e^{-t} s t^{s-1} dt \\ &= s\int_0^\infty e^{-t} \frac{t^s}{t} dt \\ &= s\Gamma(s).\end{aligned}$$

We also know,

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

Which gives us,

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1! = (2-1)!$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = (3-1)!$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 = (4-1)!$$

$$\vdots$$

$$\Gamma(n) = (n-1)!, n \in \mathbb{Z}_+$$

6.4 Mellin Transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} \frac{t^s}{t} dt, Re(s) > 0$$

Proof. Want to show

$$\Gamma(s) = \lambda^s \int_0^\infty e^{-t\lambda} \frac{t^s}{t} dt$$

$$\int_0^\infty e^{-t\lambda} \frac{(t\lambda)^s}{t} dt = \int_0^\infty e^{-t\lambda} \frac{(t\lambda)^s}{\lambda t} d(\lambda t)$$

Now replace λt with t,

$$\begin{aligned} &= \int_0^\infty e^{-t} \frac{(t)^s}{t} dt \\ &= \int_0^\infty e^{-t\lambda} \frac{(t\lambda)^s}{t} dt \\ &= \Gamma(s) \end{aligned}$$

□

Finally, we can find the relation between ζ-function $\zeta(s) = \sum_{i=1}^\infty \frac{1}{\lambda_i^s}$ and heat trace $Z(t) = \sum_{i=1}^\infty e^{-\lambda_i t}$ for $t > 0$. From the Mellin transform we know that $\lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda_i} \frac{t^s}{t} dt,\ i = 1, \ldots$. Then we get

$$\sum_{i=1}^\infty \frac{1}{\lambda_i^s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{i=1}^\infty e^{-t\lambda_i} \frac{t^s}{t} dt.$$

So the relation between ζ-function and heat trace is

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty Z(t) \frac{t^s}{t} dt.$$

$$\left(\frac{1 - x^{m+1}}{1 - x} \right)^m$$

7 Asymptotic Expansion of the Heat Trace

7.1 A Tauberian Theorem

Theorem 7.1. *Let $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$ be such that:*

$$Z(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}$$

is convergent $\forall t > 0$.
Then $\forall \alpha > 0$, and $a \in \mathbb{R}$, TFAE:
(i)$Z(t) \sim a t^{-\alpha}, t \to 0$,
(ii)$N(\lambda) \sim \frac{a}{\Gamma(\alpha+1)} \lambda^{\alpha}, \lambda \to \infty$, where $N(\lambda) = \#\{\lambda_i \leqslant \lambda\}$.

Note: To prove Weyl's Law, we first need to prove (i) for $Z(t) = \mathrm{Tr}(e^{-t\Delta})$, then the Tuberian theorem.

In this case,

$$Z(t) \sim (4\pi t)^{-1} \mathrm{Area}(M), t \to 0$$
$$\alpha = 1, a = \frac{\mathrm{Area}(M)}{4\pi}$$

By Taubeiran Theorem,

$$N(\lambda) \sim \frac{\mathrm{Area}(M)}{4\pi} \frac{1}{\Gamma(2)} \lambda = \frac{\mathrm{Area}(M)}{4\pi} \lambda, \lambda \to \infty.$$

$$\begin{cases} Z(t) = \mathrm{Tr}(e^{-t\Delta}) = \sum_i e^{-\lambda_i t} \\ \zeta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}, \mathrm{Re}\ s > \frac{n}{2} \\ N(\lambda) = \#\{\lambda_i \leqslant \lambda\} \end{cases}$$

Recall: Relation between $Z(t)$ and $\zeta(s)$:

$$\lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t\lambda_i} \frac{t^s}{t} dt$$

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \sum e^{-t\lambda_i} \frac{t^s}{t} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} Z(t) \frac{t^s}{t} dt, \mathrm{Re}\ s > \frac{n}{2}$$

Assume $\dim M = 2$, then

$$Z(t) \sim (4\pi t)^{-1} (a_0 + a_1 t + a_2 t^2 + \cdots), t \to 0$$

For space without boundry, $\lambda_0 = 0$, then

$$Z(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} + 1, \lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

So,

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty (Z(t)-1)\frac{t^s}{t}dt \\ &= \frac{1}{\Gamma(s)}\left\{\int_0^1 + \int_1^\infty\right\}(Z(t)-1)\frac{t^s}{t}dt \\ &= \frac{1}{\Gamma(s)}\int_0^1 (Z(t)-1)\frac{t^s}{t}dt + \frac{1}{\Gamma(s)}\int_1^\infty (Z(t)-1)\frac{t^s}{t}dt \\ &= I_1 + I_2.\end{aligned}$$

Second integral approaches to 0 as $t \to \infty$, because $Z(t)-1$ has exponential decay and t^{s-1} is a polynomial. So I_2 is defined for $\forall s \in \mathbb{C}$. Now consider the first integral I_1.

$$I_1(s) = \frac{1}{\Gamma(s)}\int_0^1 Z(t)\frac{t^s}{t}\,dt - \frac{1}{\Gamma(s)}\int_0^1 \frac{t^s}{t}\,dt.$$

Second integral we can compute and get

$$\int_0^1 \frac{t^s}{t}\,dt = \int_\varepsilon^1 \frac{t^s}{t}\,dt = \frac{1}{s} - \frac{\varepsilon^s}{s} \to \frac{1}{s} \text{ as } \varepsilon \to 0.$$

But we now that $\Gamma(s)$ has a pole at $s=0$, so $\frac{1}{\Gamma(s)}$ has a zero at $s=0$. It means that $\frac{1}{\Gamma(s)}\frac{1}{s}$ is analytic in whole $\mathbb{C}$. For the $\frac{1}{\Gamma(s)}\int_0^1 Z(t)\frac{t^s}{t}\,dt$ we can use heat trace expansion as $t \to 0$. Assume that

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^1 (4\pi t)^{-1}(a_0 + a_1 t + \dots)\frac{t^s}{t}\,dt = \\ &\frac{1}{4\pi\Gamma(s)}\left(a_0\left.\frac{t^{s-1}}{s-1}\right|_0^1 + a_1\left.\frac{t^s}{s}\right|_0^1 + a_2\left.\frac{t^{s+1}}{s+1}\right|_0^1 + \dots\right) = \\ &\frac{1}{4\pi\Gamma(s)}\left(\frac{a_0}{s-1} + \frac{a_1}{s} + \frac{a_2}{s+1} + \dots\right).\end{aligned} \tag{8}$$

So $\frac{1}{4\pi\Gamma(s)}\frac{a_0}{s-1}$ has a pole at $s=1$ and $Res_{s=1} = \frac{a_0}{4\pi}$. All other terms are analytic in $\mathbb{C}$.

So spectral ζ-function for $dim\,M = 2$ has just one pole at $s = \frac{dim\,M}{2} = 1$ and has analytic continuation to all $\mathbb{C}\backslash\{1\}$. Finally $Res\,\zeta(s)|_{s=1} = \frac{a_0}{4\pi} = \frac{Area(M)}{4\pi}$.

Now we try to understand why importance of $\zeta(0)$.

Claim 7.1. $\zeta(0) = \frac{a_1}{4\pi} - 1$ *for* $n = 2$, *where* a_1 *is the second term in the heat trace expansion.*

Proof. How we can see above $\zeta(s) = I_1(s) + I_2(s)$. And $I_2(s)$ is analytic function for $\forall s \in \mathbb{C}$, so $I_2(0) = 0$. From (8) $I_1(0) = \frac{a_1}{4\pi} - 1$. Immediately $\zeta(0) = \frac{a_1}{4\pi} - 1$. But we know that $a_1 = \int_M a_1(x)\,dx = \frac{1}{6}$(Total scalar curvature). □

E.g. $M = S^2$ (round sphere). In this case scalar curvature is equal to $2K = 2$. So $a_1 = \frac{4\pi}{3}$.

7.2 Heat Kernel Expansion for S^2

We computed the spectrum of the Laplacian for 2-dimensional sphere. Recall that eigenvalues are equal $k(k+1)$ with multiplicity $2k+1,\ k=0,1,\ldots$. Now we want to compute all the terms of the expansion $(4\pi t)^{-1}\left(a_0+a_1t+a_2t^2+\ldots\right)$.

Let us form the heat trace

$$Z(t)=\sum_{k=0}^{\infty}(2k+1)e^{-(k^2+k)t}.$$

We can use Euler-Maclaurin summation formula

$$\sum_{k=a}^{b}f(k)=\int_a^b f(x)\,dx+\frac{f(b)+f(a)}{2}+\sum_{k=2}^{m}\frac{B_k}{k!}(f^{(k-1)}(b)-f^{(k-1)}(a))+R_m,$$

where $B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_{2i+1}=0,\ i=1,2,\ldots$ are the Bernoulli numbers and R_m is a remainder term. In our case let $f(k)=(2k+1)\,e^{-(k^2+k)t}, a=0,\ b=\infty$. So

$$Z(t)=\int\limits_0^{\infty}(2x+1)e^{-(x^2+x)t}\,dx+\frac{1}{2}+\frac{1}{6}\left(2e^{-(x^2+x)t}-t(2x+1)^2e^{-(x^2+x)t}\right)\Big|_0^{\infty}+\ldots$$

At first compute the integral

$$\int\limits_0^{\infty}(2x+1)e^{-(x^2+x)t}dx=-\frac{1}{t}e^{-(x^2+x)t}\Big|_0^{\infty}=\frac{1}{t}.$$

Therefore

$$Z(t)=\frac{1}{t}+\frac{1}{2}+\frac{1}{12}\left(-2+t\right)+\ldots=\frac{1}{4\pi t}\left(4\pi+\frac{4}{3}\pi t+\ldots\right).$$

But we know that $a_0=Area(S^2)=4\pi$ and $a_1=\frac{1}{3}\int\limits_{S^2}K\,dx=\frac{4\pi}{3}$, where K is Gaussian curvature. In this way we can compute all terms in the expansion.

7.3 $\zeta'(0)$ and Determinant

Let A is a finite operator such that

$$A=\begin{pmatrix}\lambda_1 & \ldots & 0\\ & \ddots & \\ 0 & \ldots & \lambda_n\end{pmatrix}.$$

We know that determinant $det\,A=\lambda_1\cdot\ldots\cdot\lambda_n$. But Laplacian is the infinite operator. So how we can define determinant for the Laplacian? One mathod is to use ζ-function. Notice that ζ-function is

$$\zeta(s)=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^s}.$$

As we just saw that $\zeta(s)$ is analytic at $s = 0$. Let us define $det\,\Delta = e^{-\zeta'(0)}$. We give the justification for this definition. Differentiate $\zeta(s)$ and get

$$\zeta'(s) = -\sum_{i=1}^{\infty} \log \lambda_i \frac{1}{\lambda_i^s}$$

for $Re\, s > 1$. Now we put $s = 0$

$$\zeta'(0) = -\sum_{i=1}^{\infty} \log \lambda_i = -\log \prod_{i=1}^{\infty} \lambda_i.$$

So $det\,\Delta = e^{-\zeta'(0)}$.

E.g. for $dim\, M = 2$. We can get that

$$\zeta(0) = -1 + \frac{a_1}{4\pi} = -1 + \frac{1}{4\pi} \text{ (Total scalar curvature).}$$

Now we have three important expressions for ζ-function

$$\begin{aligned} -\zeta'(0) &= \log det\,\Delta; \\ \zeta(0) &= -1 + \frac{1}{4\pi} \text{ (Total scalar curvature)} \\ Res\ \ \zeta(s)|_{s=1} &= \frac{Area(M)}{4\pi}. \end{aligned}$$

So ζ-function carries a lot of information about M. Same we can say about heat trace $Z(t)$.

The interesting **question:** what is $\zeta'(0)$ for S^2 and $det\,\Delta_{S^2}$?

8 Domain with Dimension Zero

8.1 Fractals

1. Example 1: Cantor Set (See Fig. 4)

2. Example 2: Diamond Fractals (See Fig. 5 and Fig. 6)

   Figure 5 shows the first two iterations of the diamonds $D_{4,2}$, $D_{6,2}$ and $D_{6,3}$. The further iteration of $D_{4,2}$ is shown in Figure 6.

3. Some fractal basic definitions

   - **Hausdorff-Besicovitch dimension** d_h:
     d_h refers to the spatial scaling properties of the fractal:

     $$d_h = \lim_{r\to 0} \frac{\ln V(r)}{\ln r}$$

     where $V(r)$ is the volume of the fractal at length scale r.

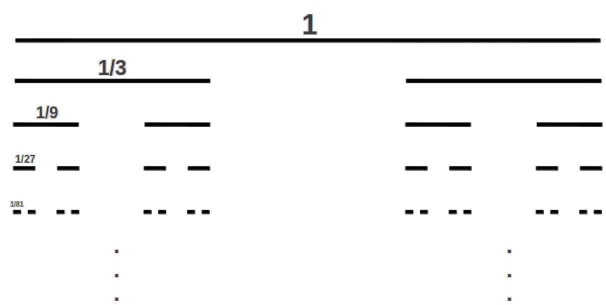


Fig. 4: Construction of the triadic Cantor set. In each iteration, the middle third of all the intervals is removed.

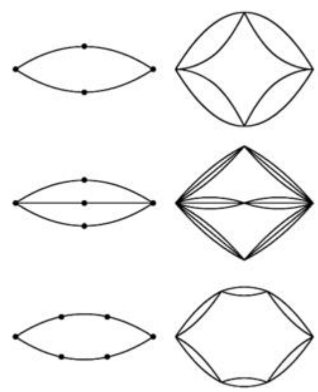


Fig. 5: The first 2 iterations for diamond fractals $D_{4,2},\ D_{6,2},\ D_{6,3}$

- **Spectral dimension** d_s:
  d_s refers to the scaling properties of the eigenvalues of the Laplacian defined on the fractal. A simple way to introduce it is through the small t asymptotics of the heat trace:
  $$d_s = -2 \lim_{t \to 0} \frac{\frac{d}{dt} \ln Z(t)}{\frac{d}{dt} \ln t}$$
  where $Z(t)$ is the heat trace.
- **Walk dimension** d_w:
  d_w is a ratio of dimensions and has the physical meaning of a diffusion index:
  $$d_w = \frac{2d_h}{d_s}$$
- For example, for the Sierpinski gasket shown in Fig. 7 $d_h = \frac{ln3}{ln2}$, $d_s = 2\frac{ln3}{ln5}$ and $d_w = \frac{ln5}{ln2}$.

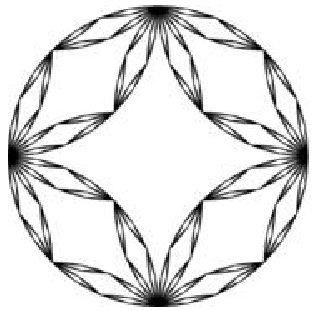


Fig. 6: Further iterations of the diamond fractal $D_{4,\,2}$

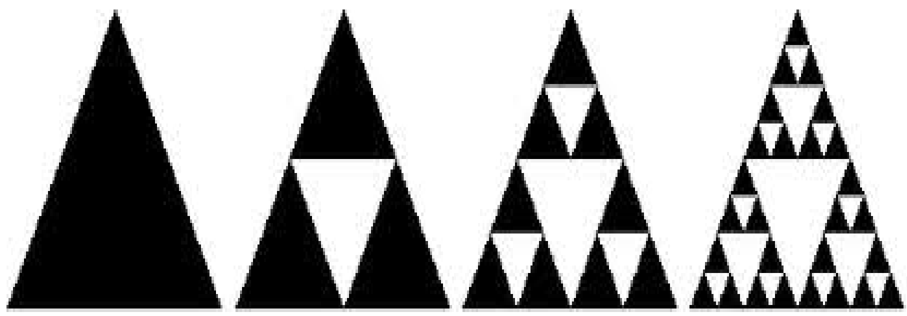


Fig. 7: Sierpinski gasket embedded in two dimensions. This fractal has Hausdorff dimension $d_h = \frac{ln3}{ln2}$, spectral dimension $d_s = 2\frac{ln3}{ln5}$ and walk dimension $d_w = \frac{ln5}{ln2}$.

4. **Eigenvalues and Multiplicities for Laplacian of Diamond Fractals**

   The spectrum of a diamond fractal is the union of two sets of eigenvalues. One set is composed of the non degenerate eigenvalues: $\pi^2 k^2$, for $k = 1, 2, 3, ...$

   The second set contains the degenerate eigenvalues (iterated eigenvalues): $\lambda_k = \pi^2 k^2 L_n^{d_w}$ obtained by rescaling dimensionless length L_n and time T_n at each iteration n according to $L_n^{d_w} = T_n$. These iterated eigenvalues have an exponentially large degeneracy, at each step, by $BL_n^{d_h} = B(l^{d_h})^n$, where $B = l^{d_h - 1} - 1$ is the branching factor of the fractal and l^{d_h} is the number of links into which a given link is divided.

   Note that we use the explicit scaling of the length $L_n = l^n$ upon iteration.

5. **Heat Trace of Diamond Fractals**

   The heat trace for a diamond fractal is:

$$Z(t) = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} + (l^{d_h - 1} - 1) \sum_{n=0}^{\infty} l^{n\, d_h} \sum_{k=1}^{\infty} e^{-k^2\pi^2 t l^{n\, d_\omega}}.$$

Log periodic oscillations of the heat trace (See Fig. 8)

$$Z(t) \sim \frac{c}{t^{d_s/2}} \left(1 + \alpha \cos\left(\frac{2\pi}{d_\omega \log l} \log t + \varphi\right) + \dots\right)$$

for some real constants $c, \alpha$and φ. Recall that

$$d_\omega = \frac{2d_h}{d_s}$$

where d_s is spectral dimension and d_h is Hausdorff dimension.

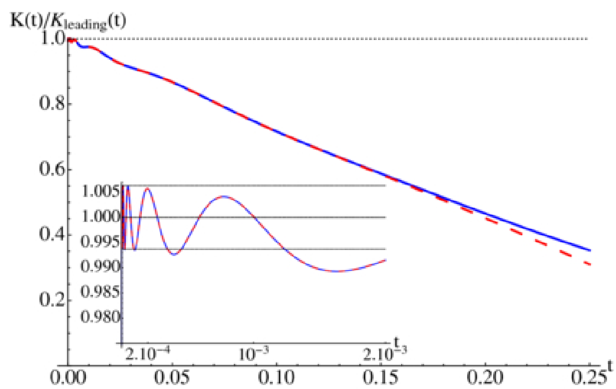


Fig. 8: The log periodic oscillations, at small t, for the heat trace $Z(t)$ on a fractal, normalized relative to the leading term $Z_{leading}(t) = c/t^{d_s/2}$. The solid curve is exact; the dashed curve is the first two terms in the approximate expression.

6. **ζ-function of Diamond fractals**

The spectral ζ-function of the fractals is

$$\zeta_D(s) = \frac{\zeta_R(2s)}{\pi^{2s}} l^{d_h-1} \left(\frac{1 - l^{1-d_\omega s}}{1 - l^{d_h - d_\omega s}}\right).$$

It has simple complex poles at

$$s = \frac{d_s}{2} + i\frac{2\pi m}{d_\omega \log l},\ m \in \mathbb{Z}$$

with a spectral dimension d_s, a walk dimension d_ω, Hausdorff dimension d_h and a spatial-decimation factor l.(See Fig. 9)

8.2 Quantum Sphere S_q^2

1. **Relations for quantum sphere S_q^2**

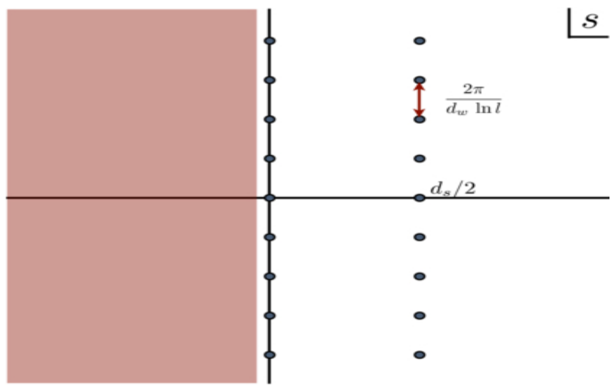


Fig. 9: Sketch of the complex pole structure of the zeta function for a fractal

$A,\, B \in M_n(\mathbb{C})$

$$\begin{aligned} AB &= q^2 BA \\ AB^* &= q^{-2} B^* A \\ BB^* &= q^{-2} A(1-A) \\ B^*B &= A(1-q^2 A) \end{aligned}$$

for some $0 < q < 1$

2. A quantum sphere has **non degenerate eigenvalues** for Dirac operator D such that $D^2 = \Delta$:

$$\lambda_k = \frac{q^{k+1/2} - q^{-(k+1/2)}}{q - q^{-1}}$$

where

$$k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

Note: As $q \to 1$, the above numbers approach to the spectrum of Dirac operator D for the sphere S^2

3. **Heat Trace of a Quantum Sphere**

The heat trace for quantum sphere is:

$$Z(t) = \sum_{k=1}^{\infty} e^{\frac{q^{-k} - q^k}{q - q^{-1}} t}$$

However, we don't know the heat trace expansion for quantum sphere.

4. **ζ-function of a Quantum Sphere**

The spectral ζ-function of the quantum sphere is

$$\zeta_q(s) = 4(1-q^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} \frac{q^{2k}}{(1-q^{s+2k})^2}$$

Note that $\Gamma(s)$ is an extension of factorials to $s \in \mathbb{C}$ with $Re(s) > 0$ and

$$\Gamma(s) = \int_0^{\infty} e^{-t} \frac{t^s}{t} \, dt$$

All poles of $\zeta_q(s)$ are complex of the second order:

$$-2k + i\frac{2\pi}{\log q} m$$

where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

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